Theorem 9.1 (Negative Cycle Optimality Conditions). *A feasible solution x^* is an optimal solution of the minimum cost flow problem if and only if it satisfies the negative cycle optimality conditions: namely, the residual network $G(x^*)$ contains no negative cost (directed) cycle.*

Proof. Suppose that x is a feasible flow and that $G(x)$ contains a negative cycle. Then x cannot be an optimal flow, since by augmenting positive flow along the cycle we can improve the objective function value. Therefore, if x^* is an optimal flow, then $G(x^*)$ cannot contain a negative cycle. Now suppose that x^* is a feasible flow

and that $G(x^*)$ contains no negative cycle. Let x° be an optimal flow and $x^* \neq x^\circ$. The augmenting cycle property stated in Theorem 3.7 shows that we can decompose the difference vector $x^\circ - x^*$ into at most m augmenting cycles with respect to the flow x^* and the sum of the costs of flows on these cycles equals $cx^\circ - cx^*$. Since the lengths of all the cycles in $G(x^*)$ are nonnegative, $cx^\circ - cx^* \geq 0$, or $cx^\circ \geq cx^*$. Moreover, since x° is an optimal flow, $cx^\circ \leq cx^*$. Thus $cx^\circ = cx^*$, and x^* is also an optimal flow. This argument shows that if $G(x^*)$ contains no negative cycle, then x^* must be optimal, and this conclusion completes the proof of the theorem.◆

```
algorithm cycle-canceling;
begin
    establish a feasible flow x in the network;
    while G(x) contains a negative cycle do
    begin
        use some algorithm to identify a negative cycle W;
        δ : = min{r_ij : (i, j) ∈ W};
        augment δ units of flow in the cycle W and update G(x);
    end;
end;
```

Figure 9.7 Cycle canceling algorithm.

Let us now consider the number of iterations that the algorithm performs. For the minimum cost flow problem, mCU is an upper bound on the initial flow cost

[since $c_{ij} \leq C$ and $x_{ij} \leq U$ for all $(i, j) \in A$] and $-mCU$ is a lower bound on the optimal flow cost [since $c_{ij} \geq -C$ and $x_{ij} \leq U$ for all $(i, j) \in A$]. Each iteration of the cycle-canceling algorithm changes the objective function value by an amount $(\sum_{(i,j) \in W} c_{ij})\delta$, which is strictly negative. Since we are assuming that all the data of the problem are integral, the algorithm terminates within $O(mCU)$ iterations and runs in $O(nm^2CU)$ time.

Theorem 9.4 (Complementary Slackness Optimality Conditions). *A feasible solution x^* is an optimal solution of the minimum cost flow problem if and only if for some set of node potentials π, the reduced costs and flow values satisfy the following complementary slackness optimality conditions for every arc $(i, j) \in A$:*

$$\text{If } c_{ij}^{\pi} > 0, \text{ then } x_{ij}^{*} = 0. \tag{9.8a}$$

$$\text{If } 0 < x_{ij}^{*} < u_{ij}, \text{ then } c_{ij}^{\pi} = 0. \tag{9.8b}$$

$$\text{If } c_{ij}^{\pi} < 0, \text{ then } x_{ij}^{*} = u_{ij}. \tag{9.8c}$$

Proof. We show that the reduced cost optimality conditions are equivalent to (9.8). To establish this result, we first prove that if the node potentials π and the flow vector x satisfy the reduced cost optimality conditions, then they must satisfy (9.8). Consider three possibilities for any arc $(i, j) \in A$.

Case 1. If $c_{ij}^{\pi} > 0$, the residual network cannot contain the arc (j, i) because $c_{ji}^{\pi} = -c_{ij}^{\pi} < 0$ for that arc, contradicting (9.7). Therefore, $x_{ij}^{*} = 0$.

Case 2. If $0 < x_{ij}^{*} < u_{ij}$, the residual network contains both the arcs (i, j) and (j, i). The reduced cost optimality conditions imply that $c_{ij}^{\pi} \geq 0$ and $c_{ji}^{\pi} \geq 0$. But since $c_{ji}^{\pi} = -c_{ij}^{\pi}$, these inequalities imply that $c_{ij}^{\pi} = c_{ji}^{\pi} = 0$.

Case 3. If $c_{ij}^{\pi} < 0$, the residual network cannot contain the arc (i, j) because $c_{ij}^{\pi} < 0$ for that arc, contradicting (9.7). Therefore, $x_{ij}^{*} = u_{ij}$.

We have thus shown that if the node potentials π and the flow vector x satisfy the reduced cost optimality conditions, they also satisfy the complementary slackness optimality conditions. In Exercise 9.28 we ask the reader to prove the converse result: If the pair (x, π) satisfies the complementary slackness optimality conditions, it also satisfies the reduced cost optimality conditions. ◆