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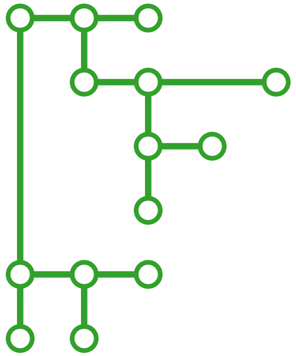
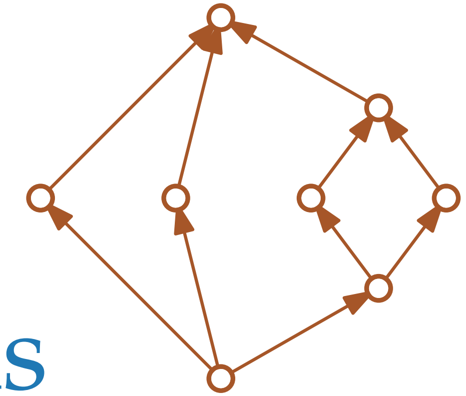
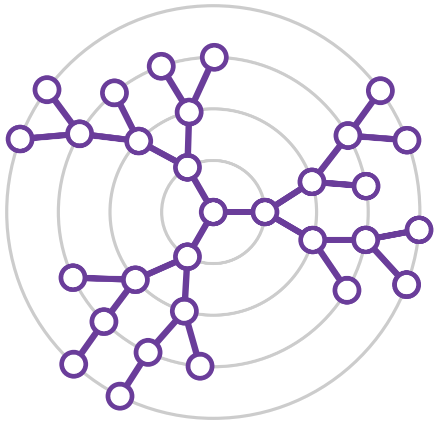


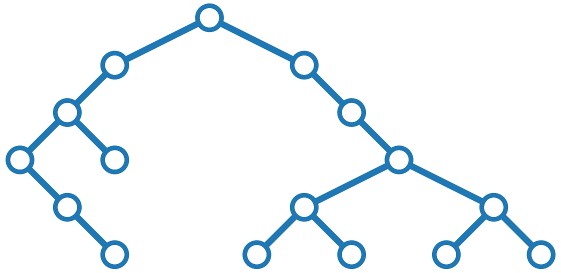
Visualization of Graphs



Lecture 2:
Drawing Trees and Series-Parallel Graphs



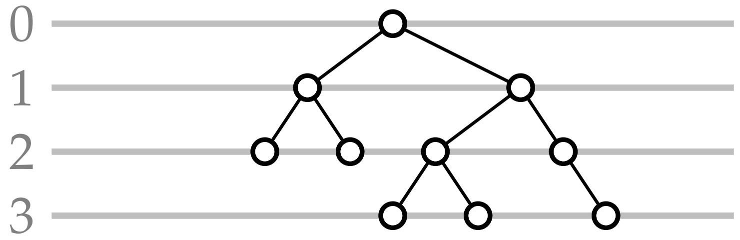
Part I:
Layered Drawings



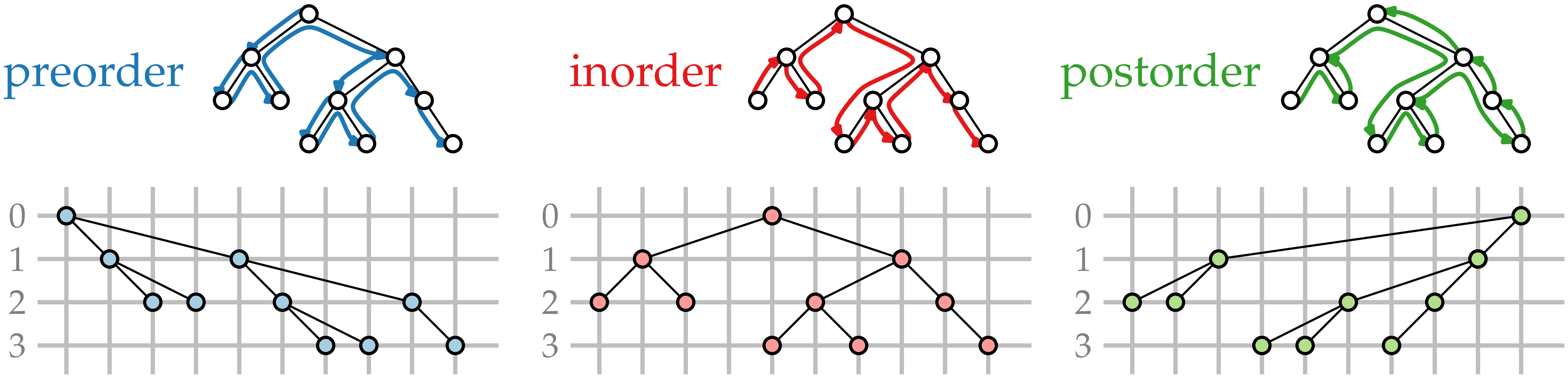
Philipp Kindermann

First Grid Layout of Binary Trees

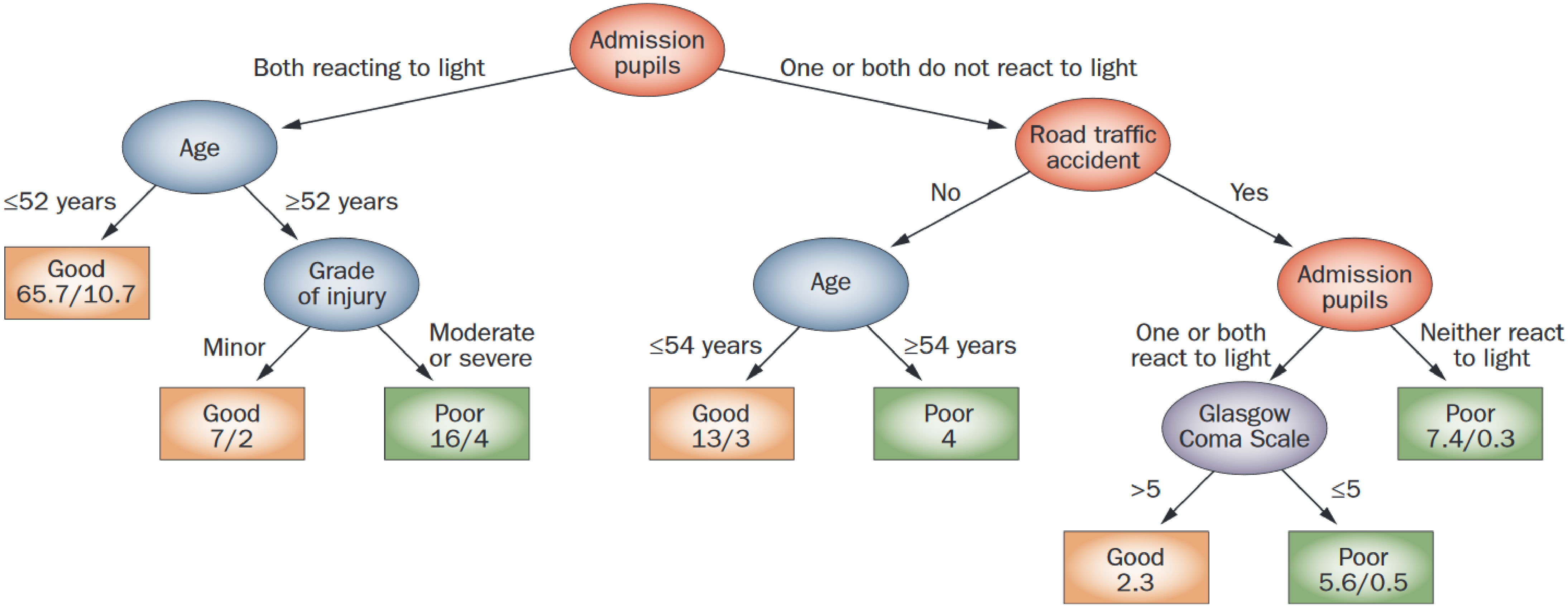
1. Choose y-coordinates: $y(u) = \text{depth}(u)$



2. Choose x-coordinates:



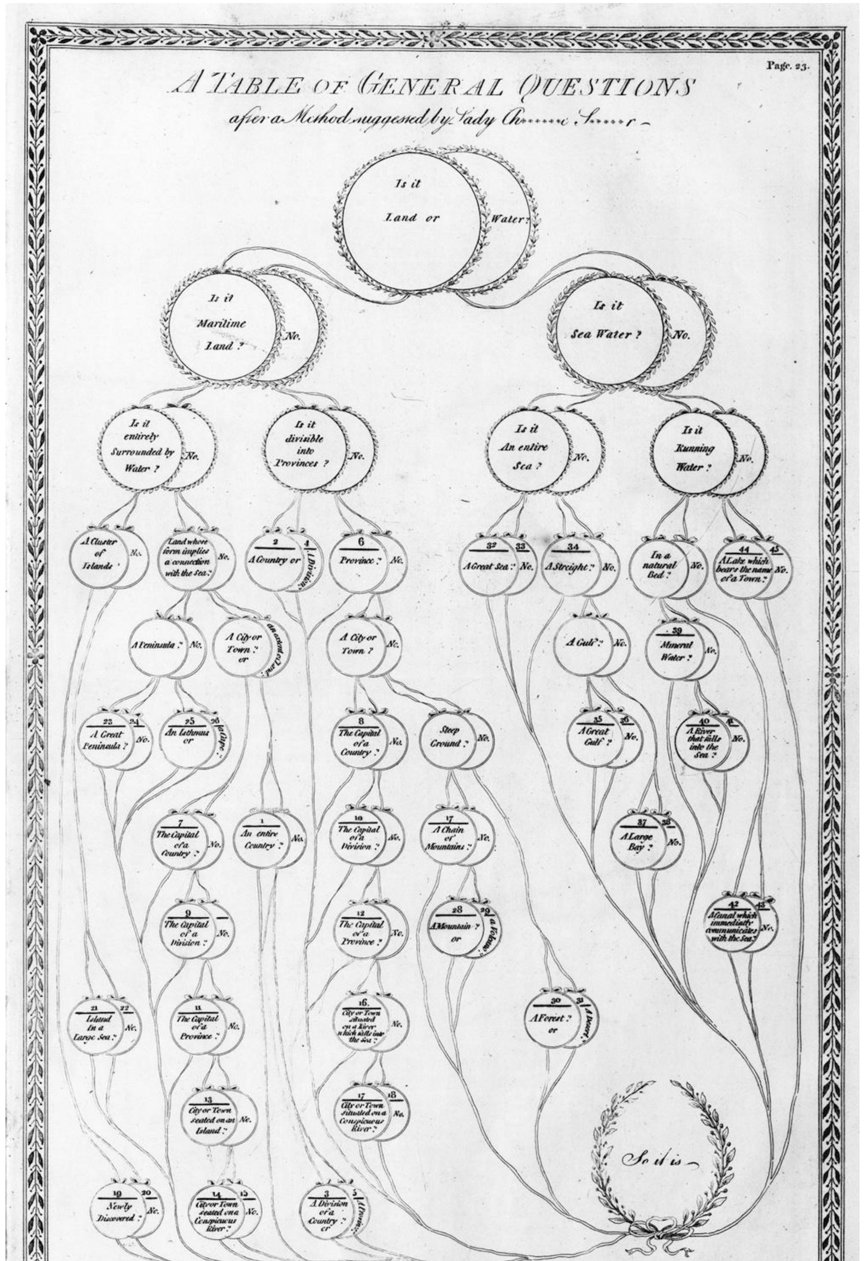
Layered Drawings – Applications



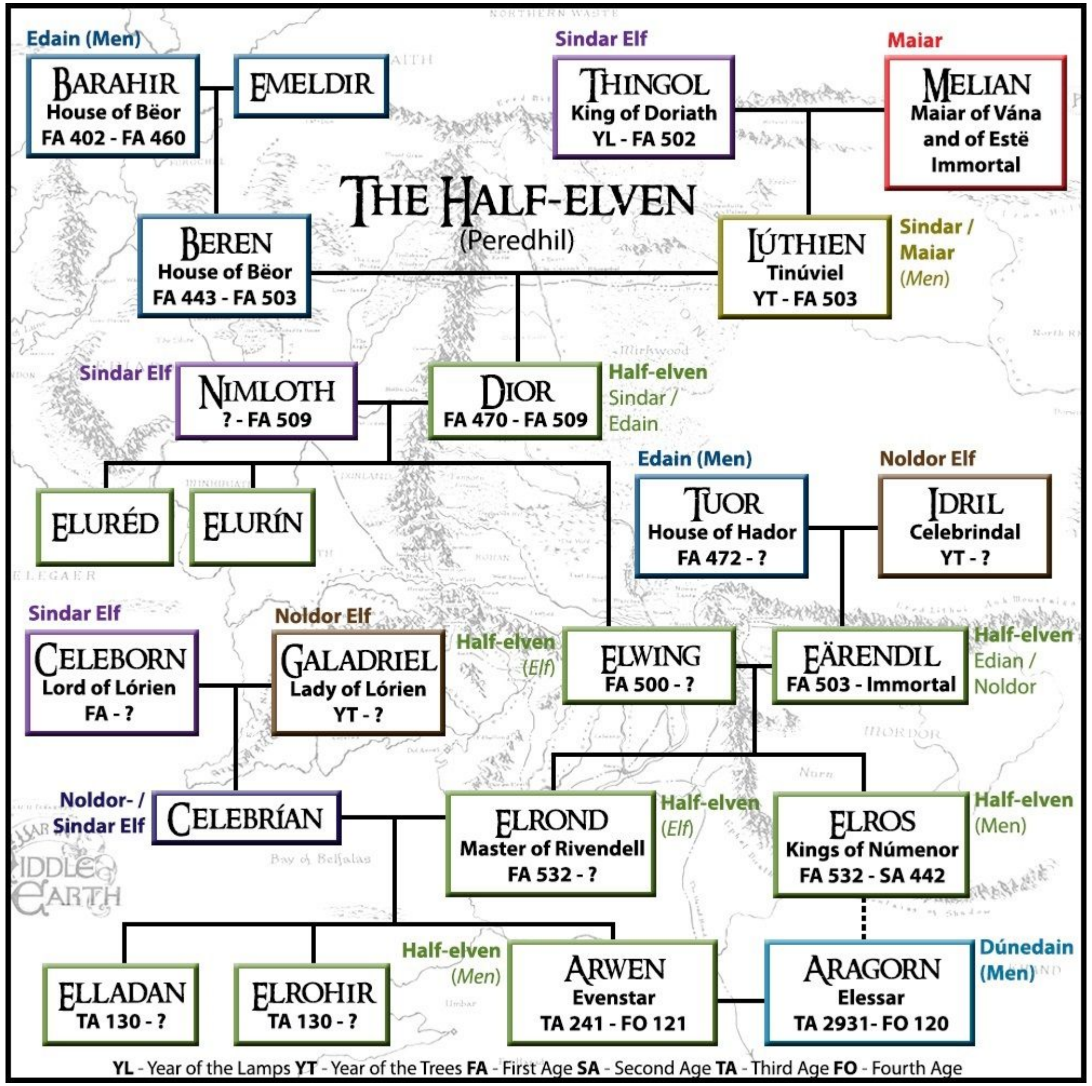
Decision tree for outcome prediction after traumatic brain injury

Source: Nature Reviews Neurology

Layered Drawings – Applications

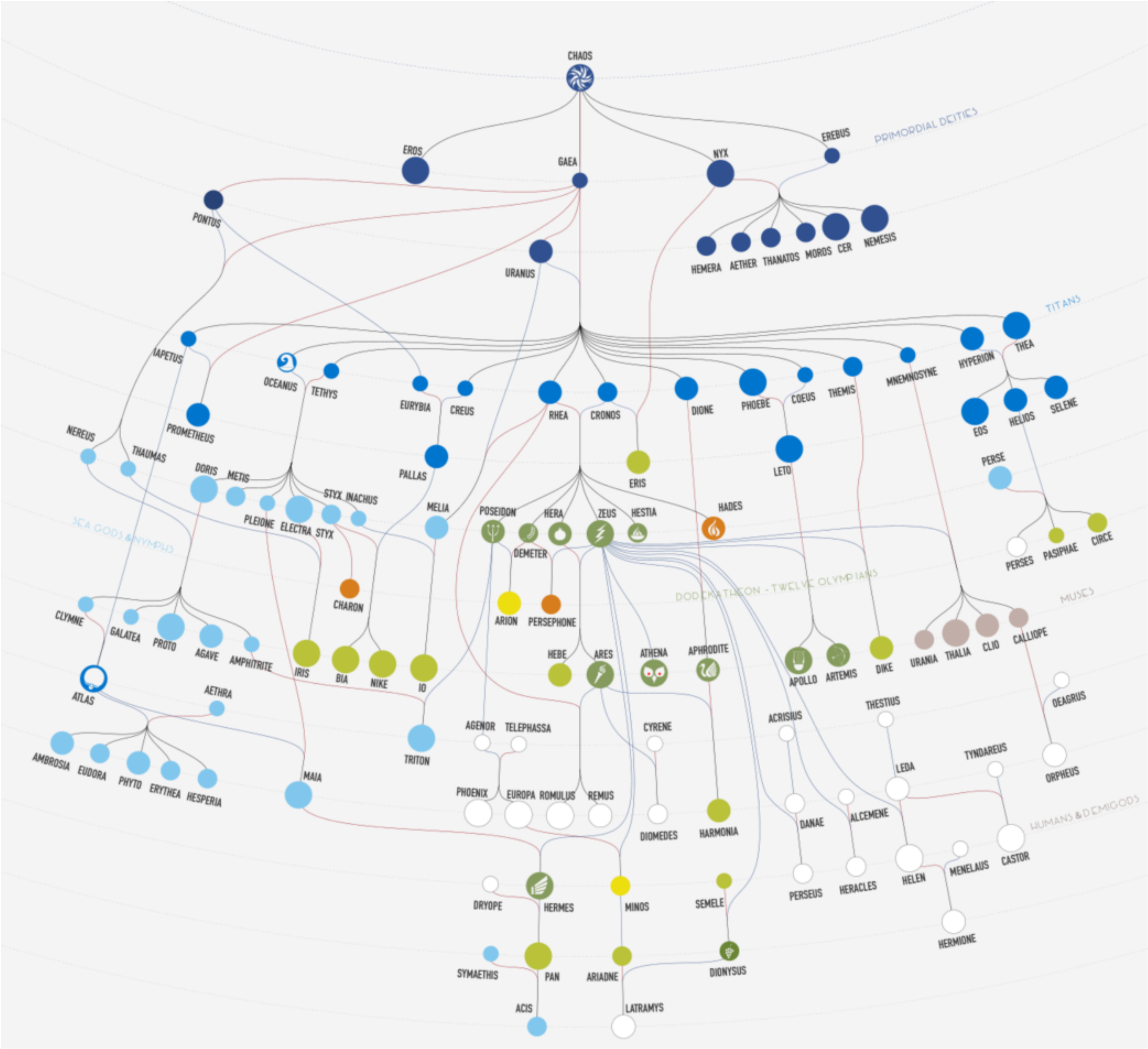


Aloisius Gaultier 1821



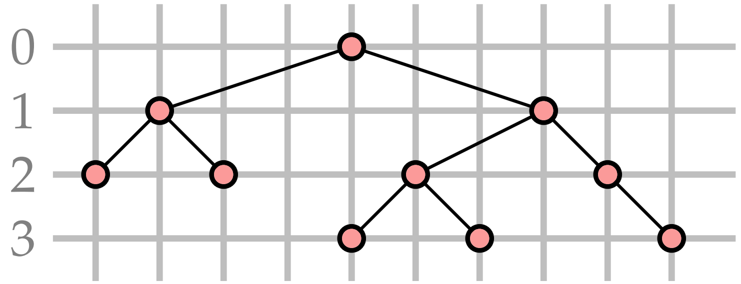
Family tree of LOTR elves and half-elves

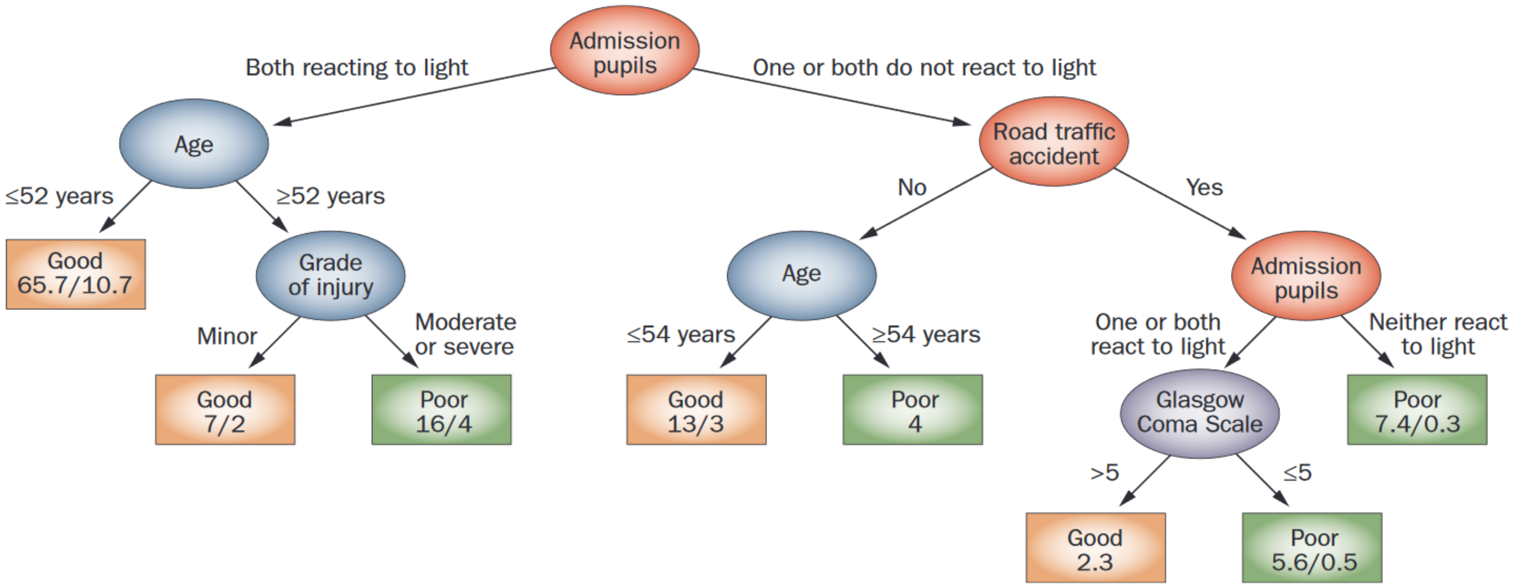
Layered Drawings – Applications



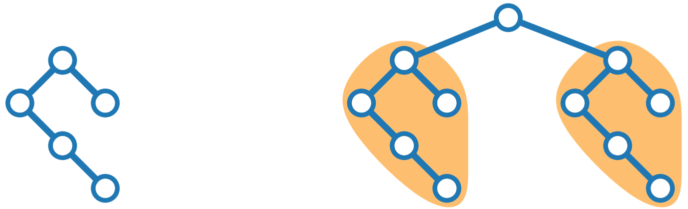
J. Klawitter, T. Mchedlidze, *Link:* go.uniwue.de/myth-poster

Layered Drawings – Drawing Style





- What are properties of the layout?
- What are the drawing conventions?
- What are aesthetics to optimize?



Drawing conventions

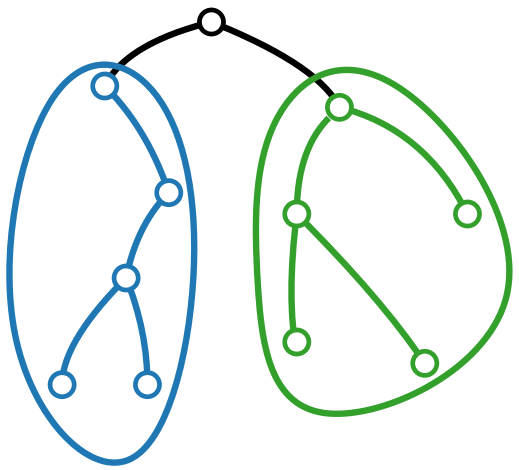
- Vertices lie on layers and have integer coordinates
- Parent centered above children
- Edges are straight-line segments
- Isomorphic subtrees have identical drawings

Drawing aesthetics

- Area
- Symmetries

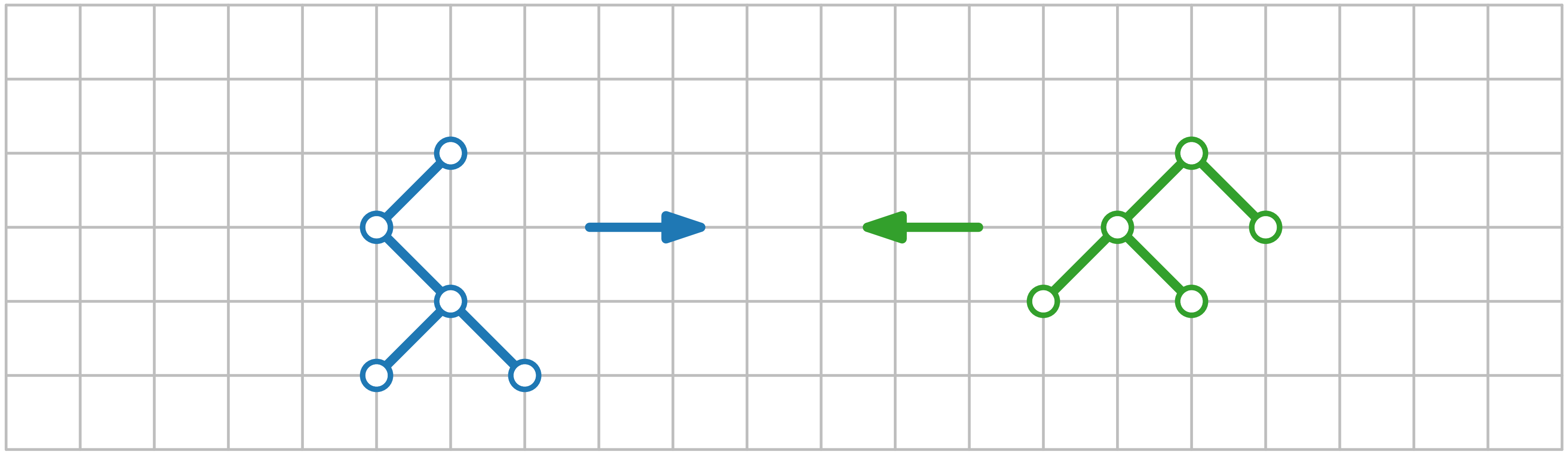
Layered Drawings – Algorithm

Input: A binary tree T
Output: A layered drawing of T



Base case: A single vertex 
Divide: Recursively apply the algorithm to draw the left and right subtrees

Conquer:



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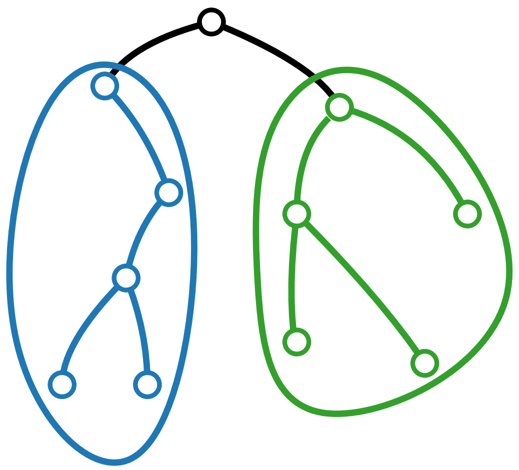

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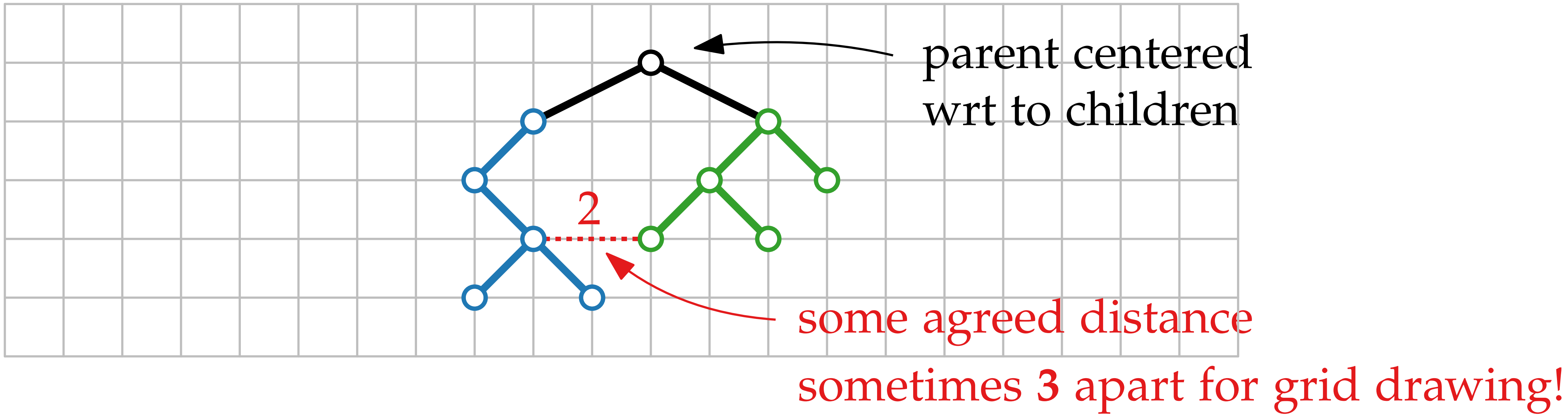

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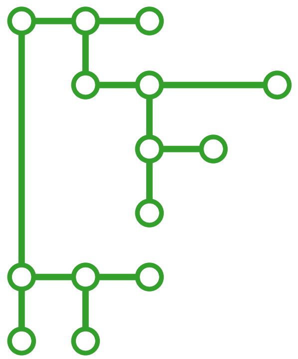
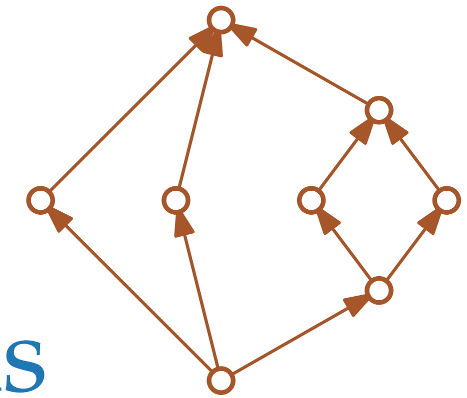
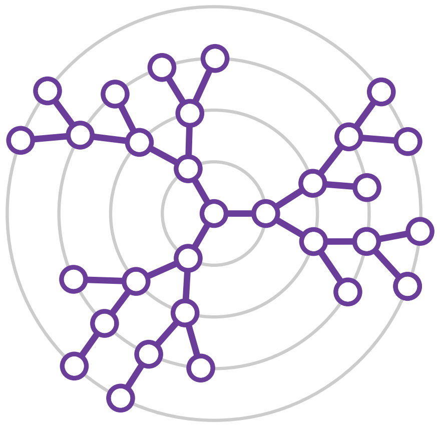


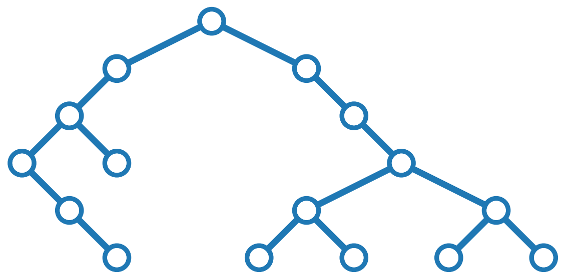
Visualization of Graphs



Lecture 2:
Drawing Trees and Series-Parallel Graphs



Part II:
Layered Drawings – Algorithmic Details



Philipp Kindermann

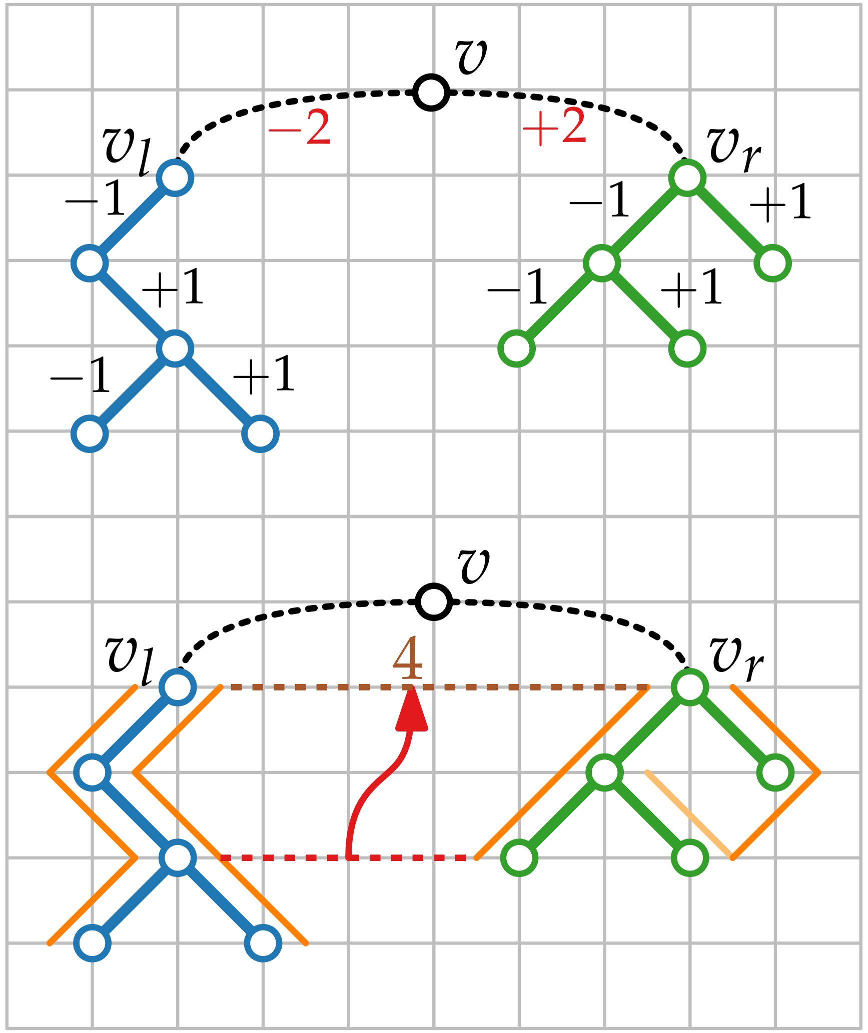
Layered Drawings – Algorithm Details

Phase 1 – postorder traversal:

- For each vertex compute horizontal displacement of left and right child
- x-offset$(v_l) = -\lceil \frac{d_v}{2} \rceil$, x-offset$(v_r) = \lceil \frac{d_v}{2} \rceil$
- At vertex u (below v) store left and right contour of subtree $T(u)$
- Contour is linked list of vertex coordinates/offsets
- Find d_v = min. horiz. distance between v_l and v_r

Phase 2 – preorder traversal:

- Compute x- and y-coordinates



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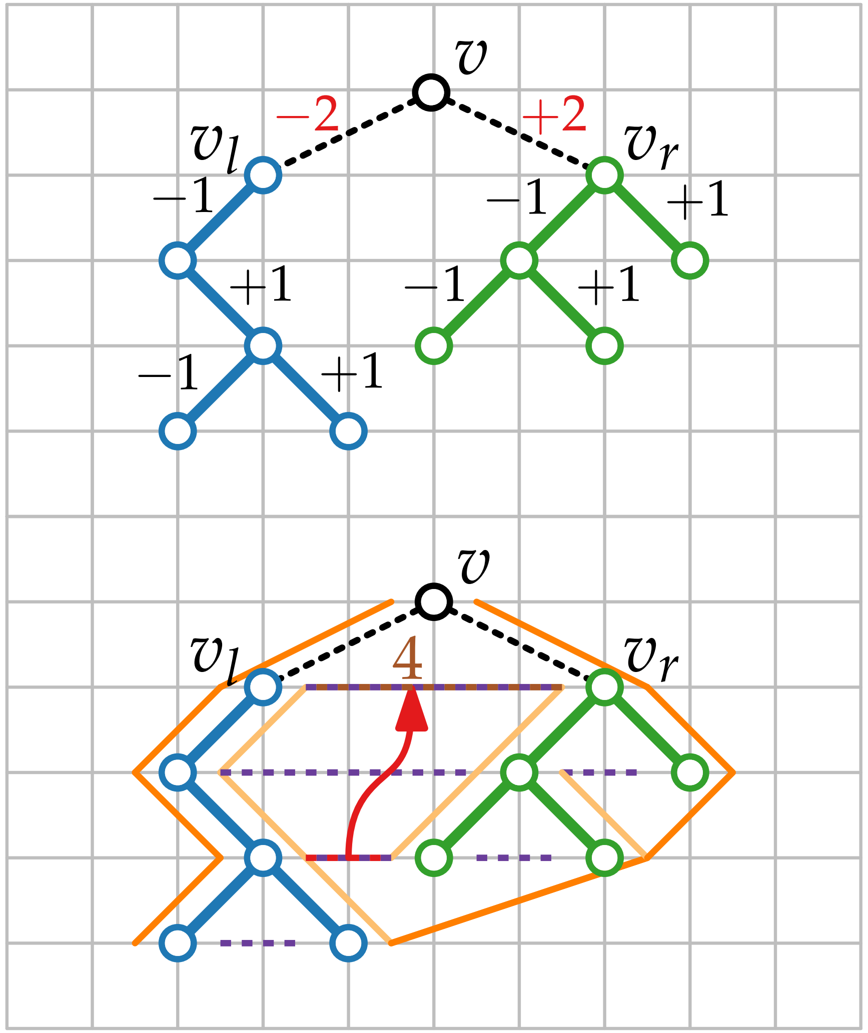


Phase 2 – preorder traversal:

- Compute x- and y-coordinates

Runtime?

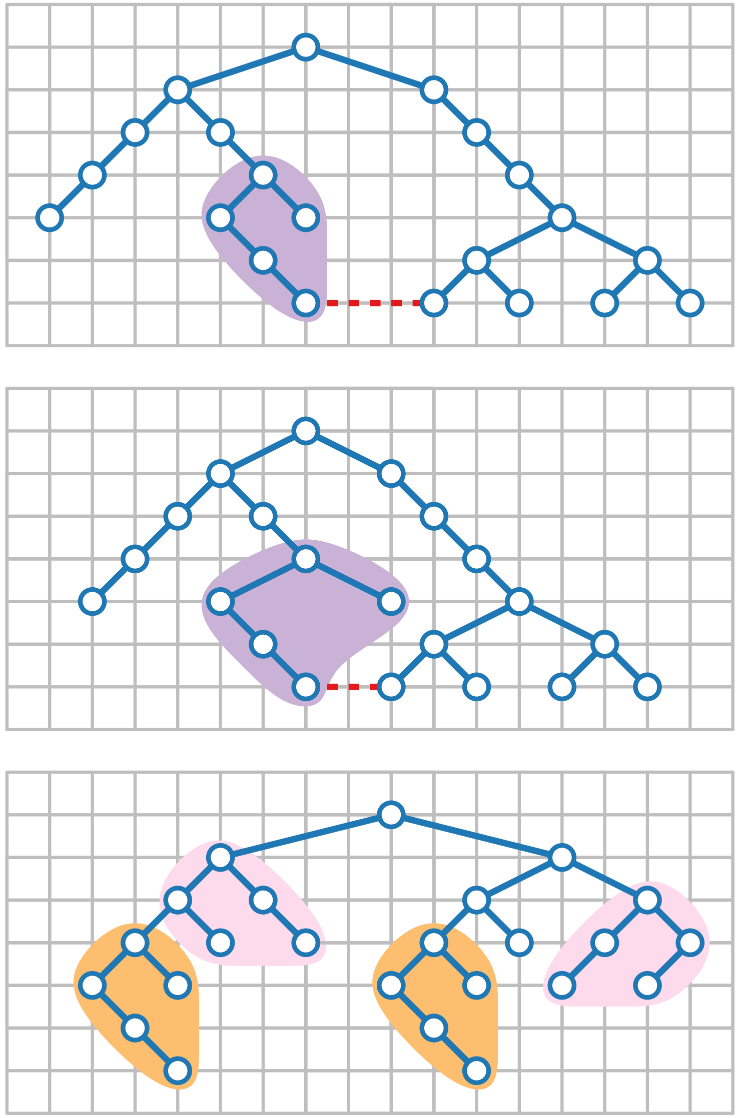
- How often do we have to walk along a contour? $\Rightarrow \mathcal{O}(n)$

Layered Drawings – Result

Theorem. [Reingold & Tilford '81]

Let T be a binary tree with n vertices. We can construct a drawing Γ of T in $\mathcal{O}(n)$ time, such that:

- Γ is planar, straight-line and strictly downward
- Γ is layered: y-coordinate of vertex v is $-\mathrm{depth}(v)$
- Horizontal and Vertical distances are at least 1
- Each vertex is centred wrt its children
- Area of Γ is in $\mathcal{O}(n^2)$ – but not optimal! ← NP-hard
- Simply isomorphic subtrees have congruent drawings, up to translation
- Axially isomorphic subtrees have congruent drawings, up to translation and reflection



Layered Drawings – Result

Theorem. [Reingold & Tilford '81]

Let T be a ~~binary~~ rooted tree with n vertices. We can construct a drawing Γ of T in $\mathcal{O}(n)$ time, such that:

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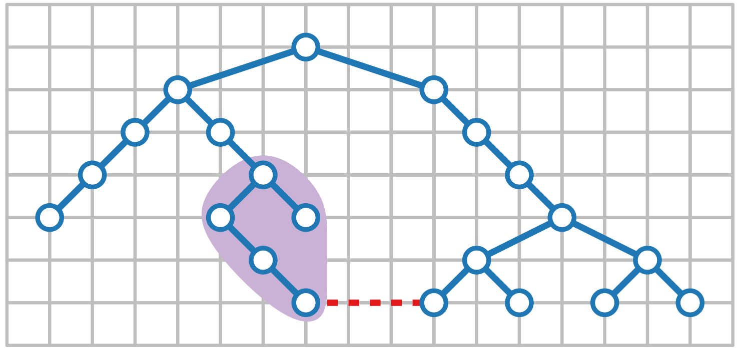

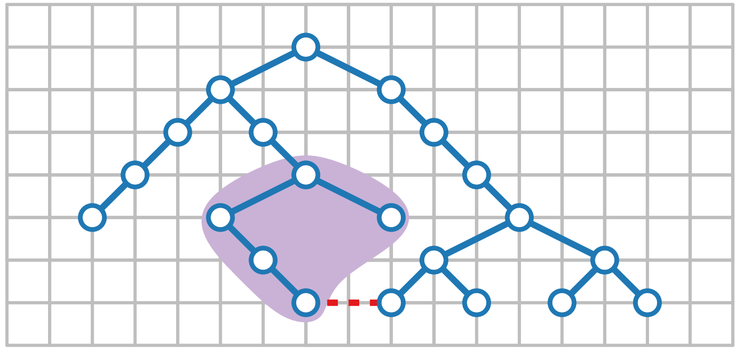

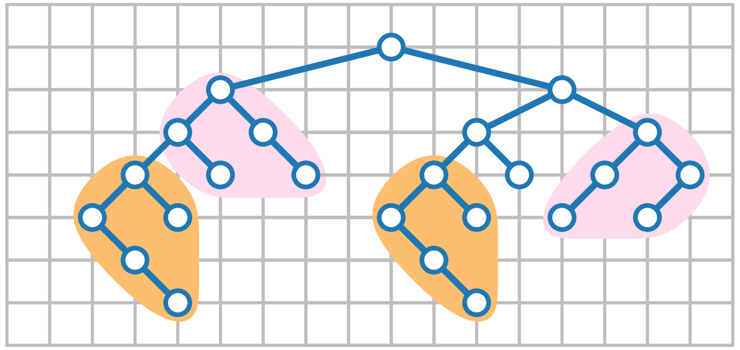

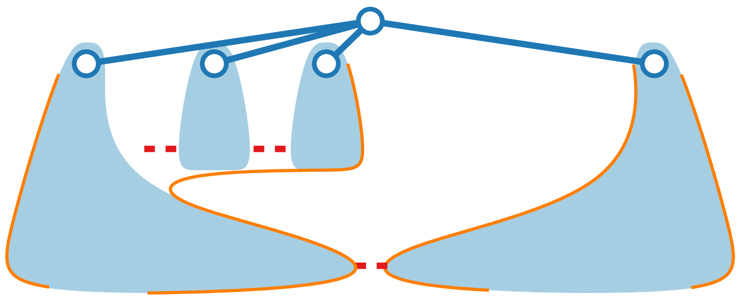

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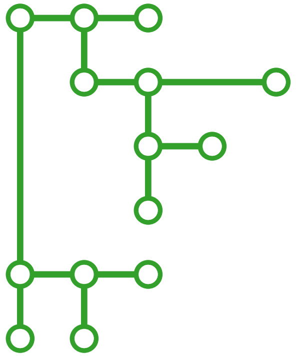
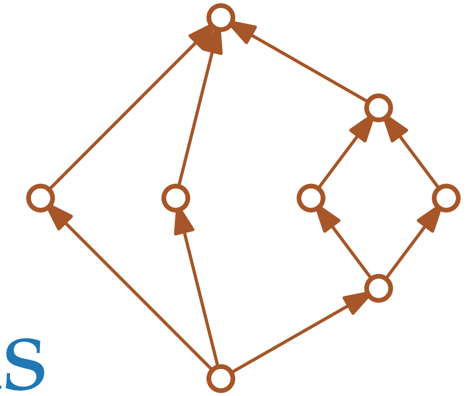
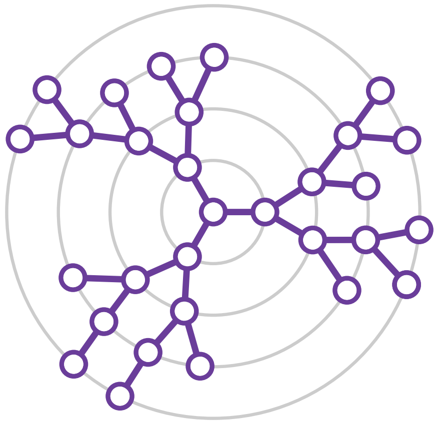


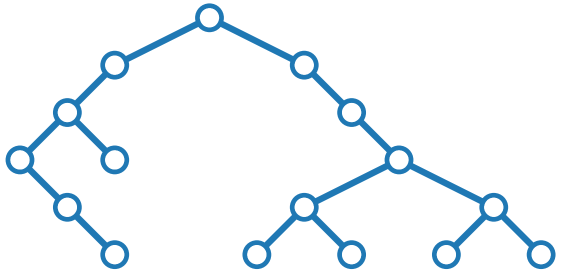
Visualization of Graphs



Lecture 2:
Drawing Trees and Series-Parallel Graphs



Part III:
HV-Drawings



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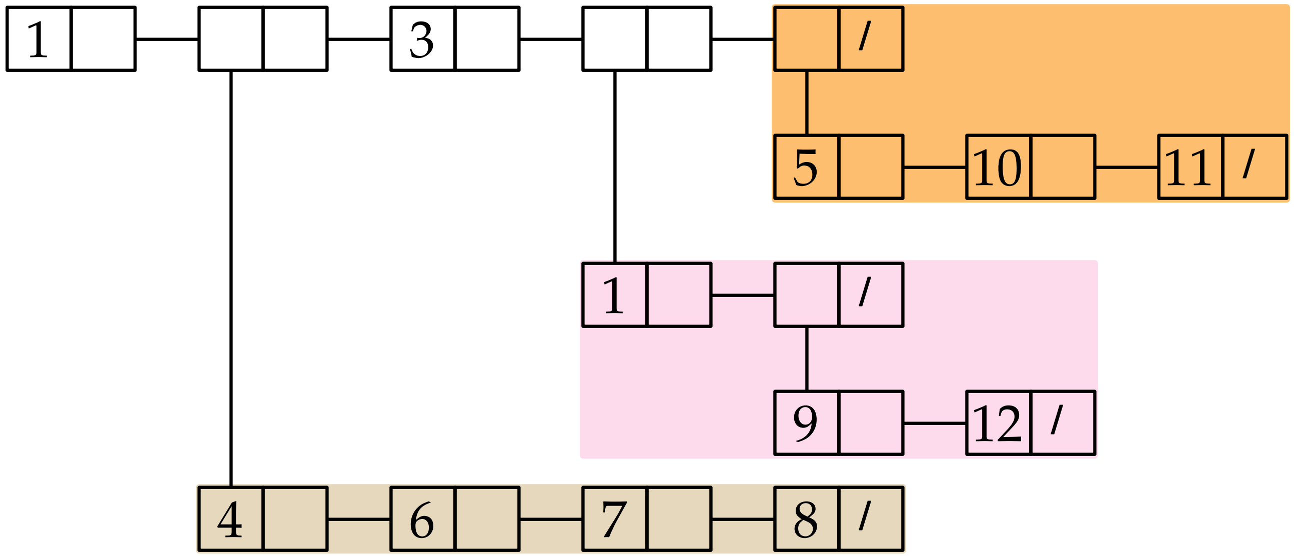
HV-Drawings – Drawing Style

Applications

- Cons cell diagram in LISP
- *Cons*(constructs) are memory objects which hold two values or pointers to values



Source: after gajon.org/trees-linked-lists-common-lisp/

Drawing conventions

- Children are vertically or horizontally aligned with their parent
- The bounding boxes of the subtrees of the children are disjoint
- Edges are strictly down- or rightwards

Drawing aesthetics

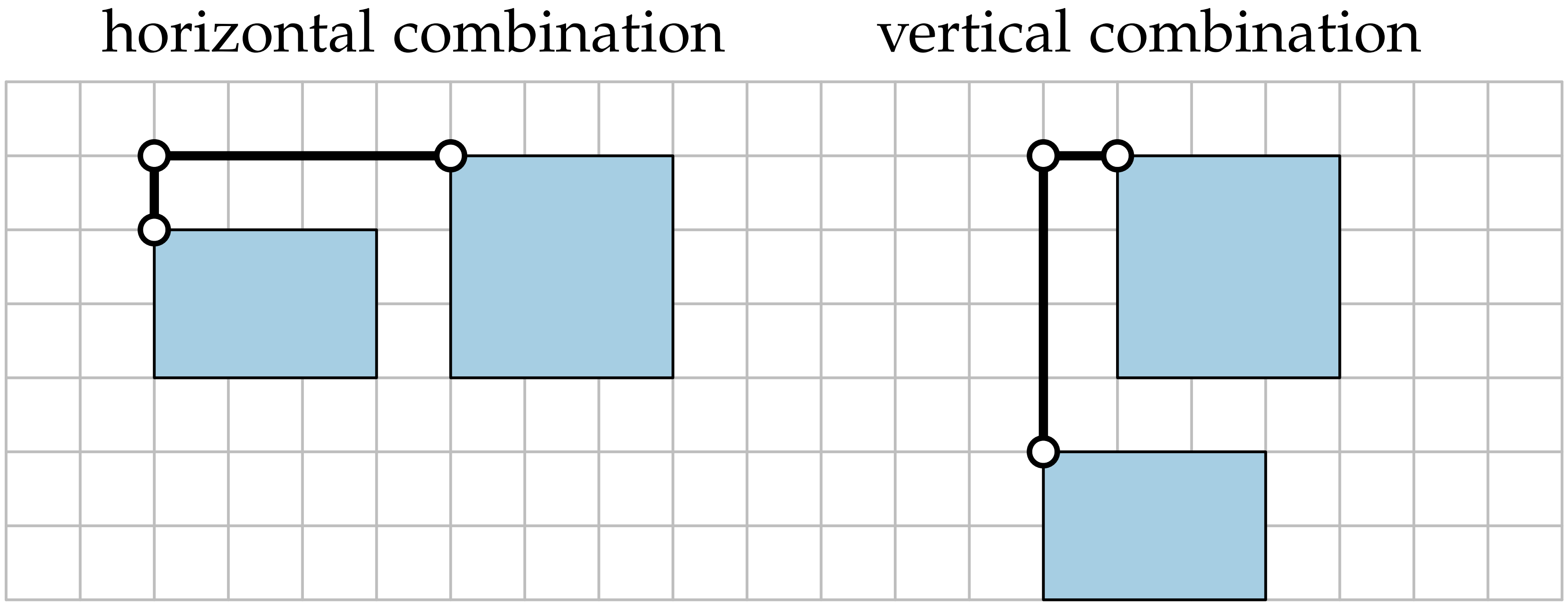
- Height, width, area

HV-Drawings – Algorithm

Input: A binary tree T
Output: An HV-drawing of T

Base case: 
Divide: Recursively apply the algorithm to draw the left and right subtrees

Conquer:



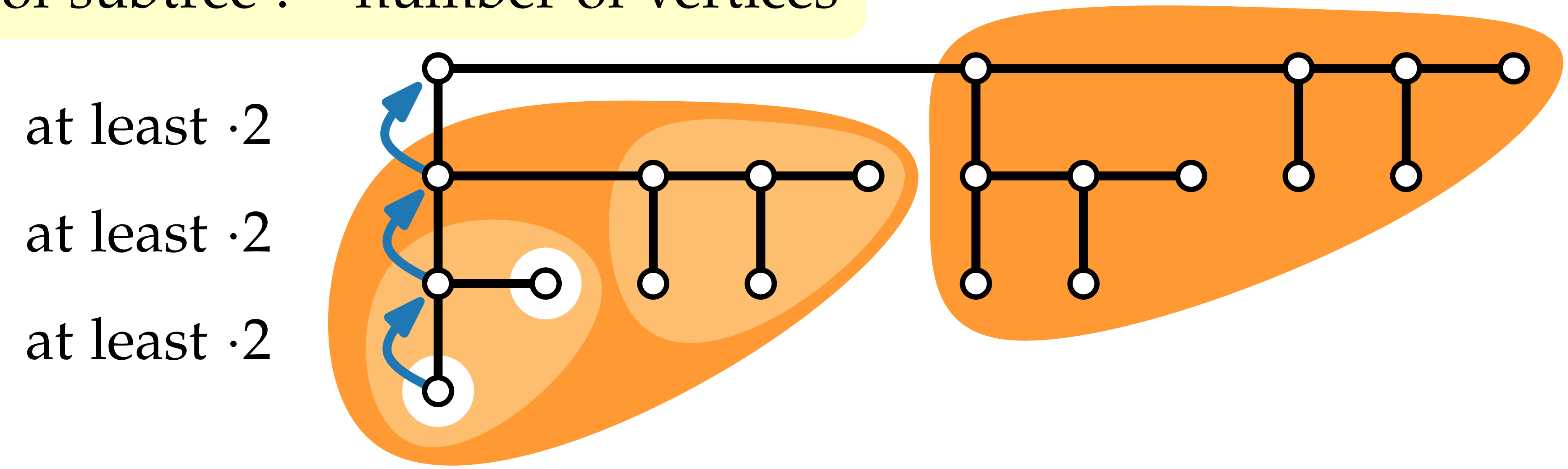
HV-Drawings – Right-Heavy HV-Layout

Right-heavy approach

- Always apply horizontal combination
- Place the larger subtree to the right

Size of subtree := number of vertices

How to implement this in linear time?



Lemma. Let T be a binary tree. The drawing constructed by the right-heavy approach has

- width at most $n - 1$ and
- height at most $\log n$.

HV-Drawings – Result

Theorem.
Let T be a binary tree with n vertices. The right-heavy algorithm constructs in $O(n)$ time a drawing Γ of T s.t.:

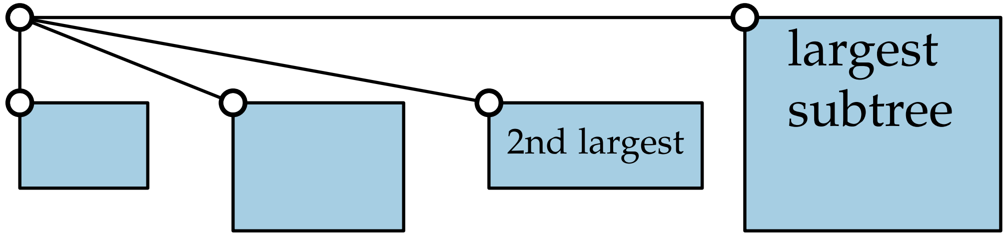
- Γ is an HV-drawing
  (planar, orthogonal, strictly right-/downward)
- Width is at most $n - 1$
- Height is at most $\log n$
- Area is in $\mathcal{O}(n \log n)$
- Simply and axially isomorphic subtrees have congruent drawings up to translation

HV-Drawings – Result

Theorem.
Let T be a ~~binary~~ rooted tree with n vertices. The right-heavy algorithm constructs in $O(n)$ time a drawing Γ of T s.t.:

- Γ is an HV-drawing (planar, ~~orthogonal~~, strictly right-/downward)
- Width is at most $n - 1$
- Height is at most $\log n$
- Area is in $\mathcal{O}(n \log n)$
- Simply and axially isomorphic subtrees have congruent drawings up to translation

General rooted tree



Optimal area?

Not with divide & conquer approach, but can be computed with Dynamic Programming.

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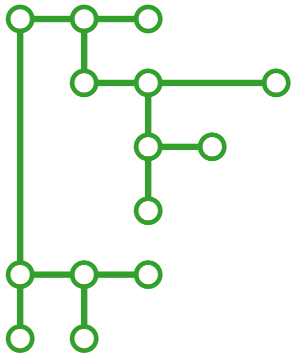
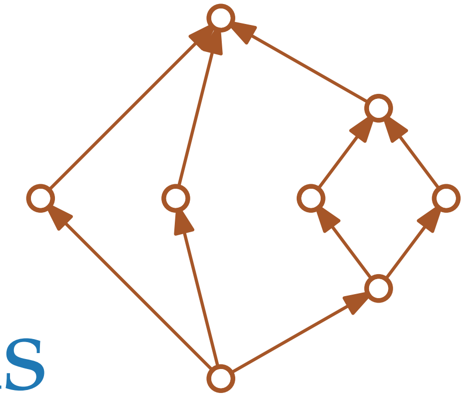
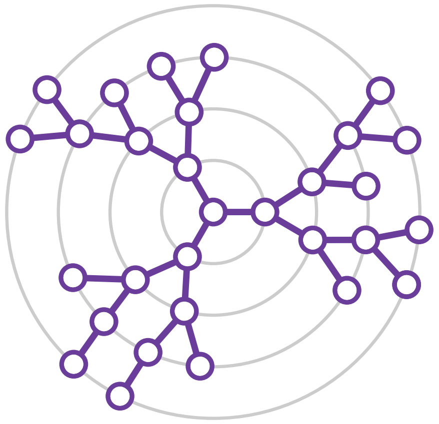


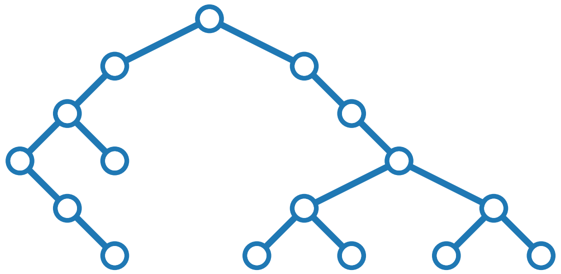
Visualization of Graphs



Lecture 2:
Drawing Trees and Series-Parallel Graphs

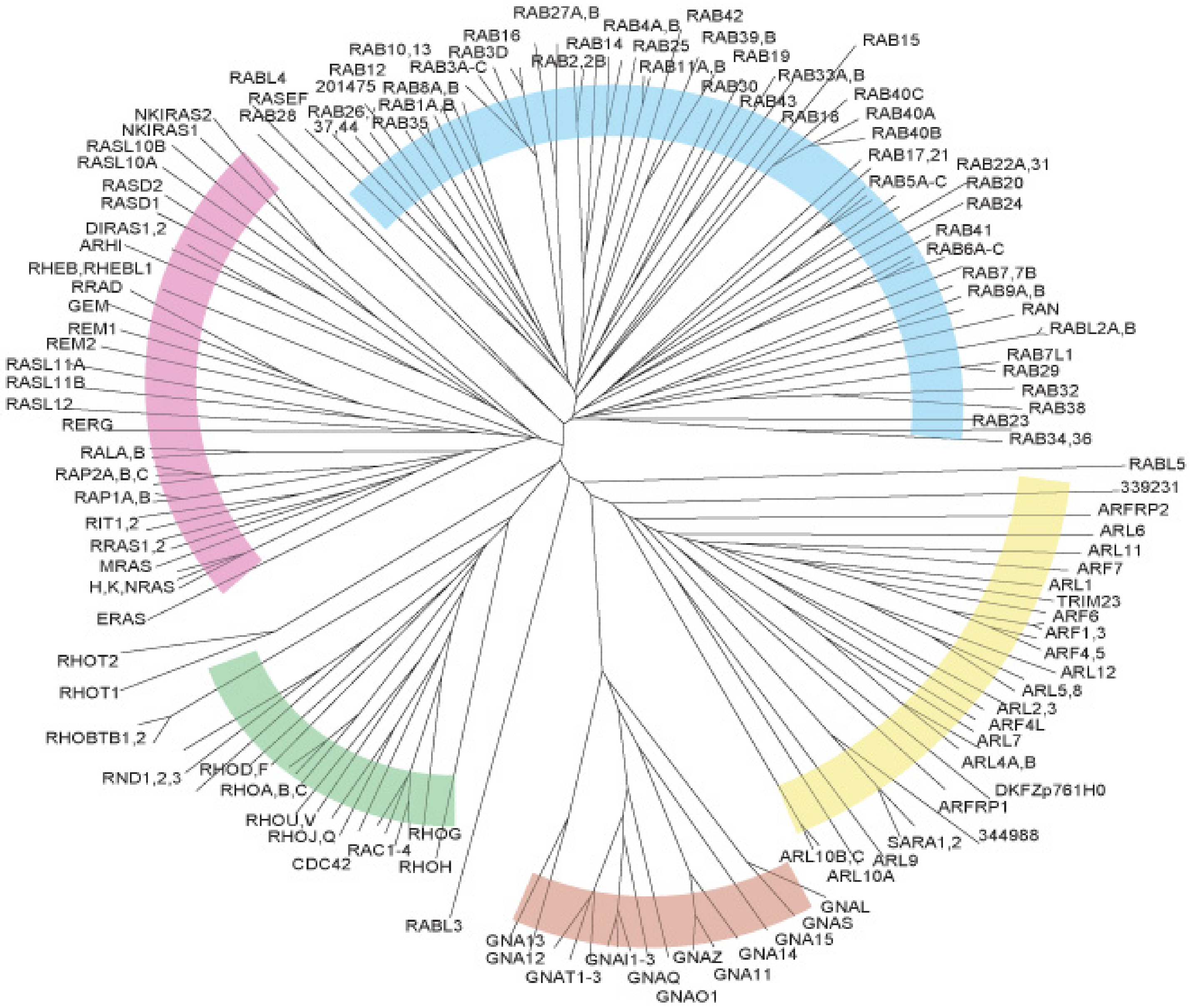


Part IV:
Radial Layouts



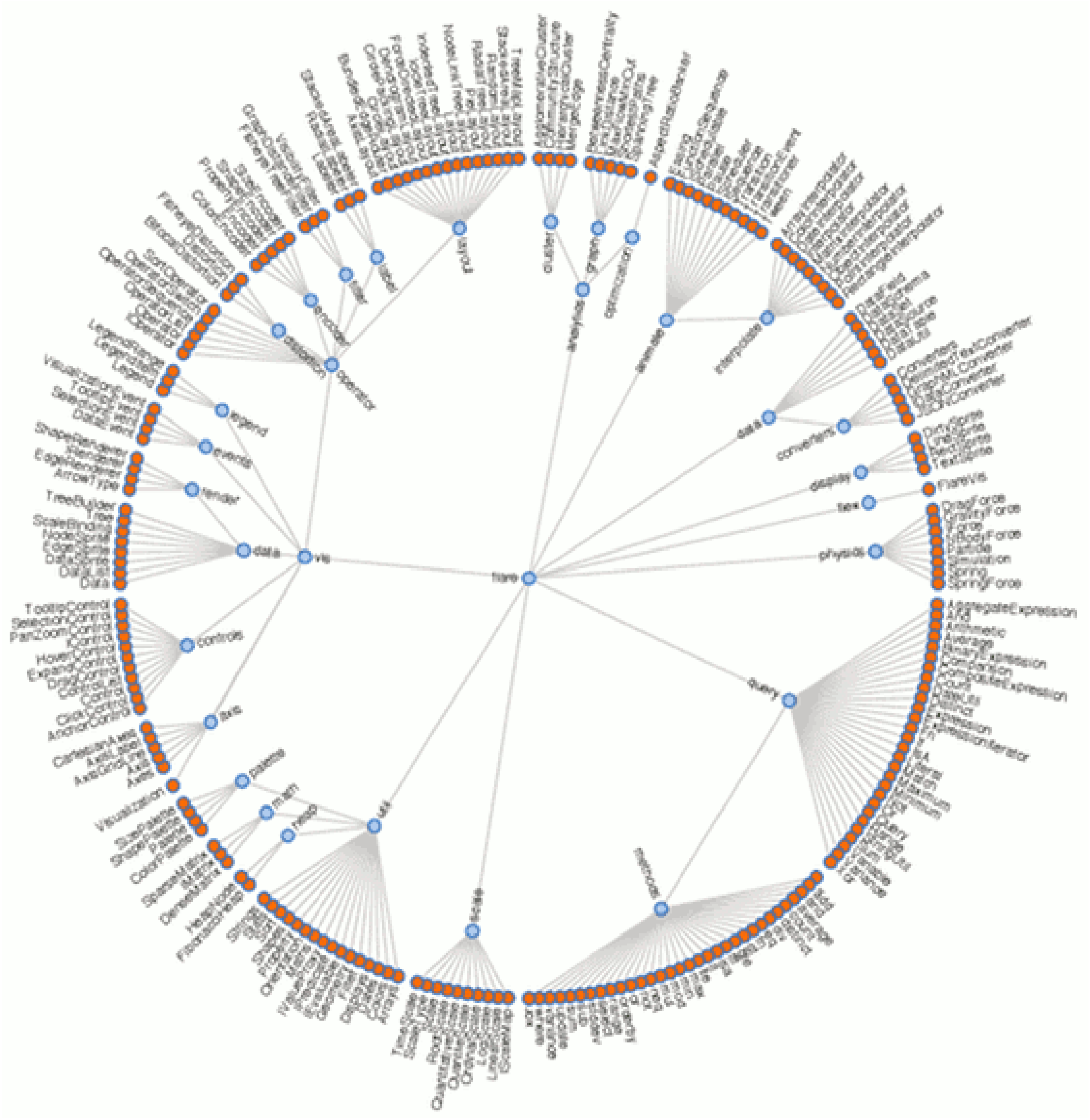
Philipp Kindermann

Radial Layouts – Applications

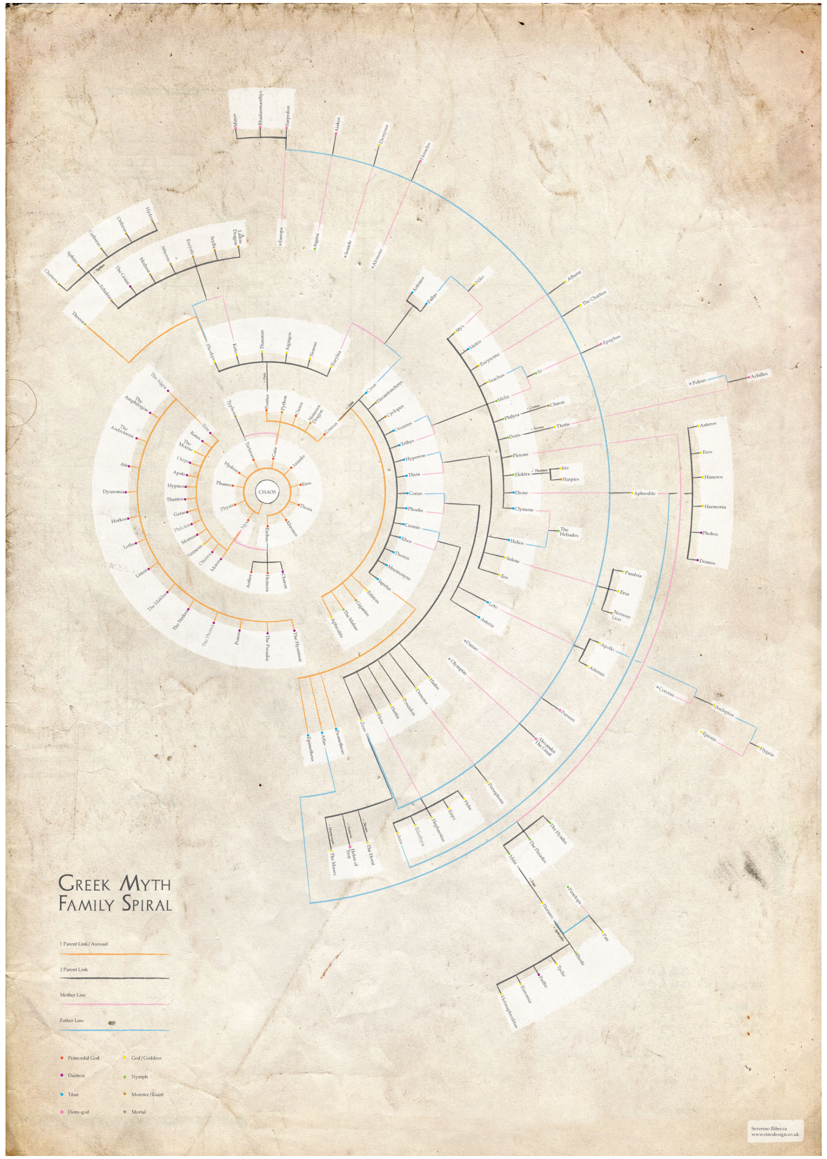


Phylogenetic tree
by Colicelli, ScienceSignaling, 2004

Radial Layouts – Applications

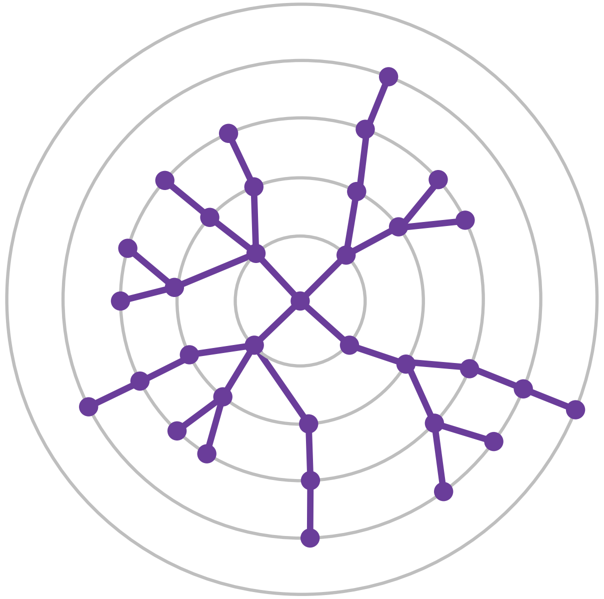


Flare Visualization Toolkit code structure
by Heer, Bostock and Ogievetsky, 2010



Greek Myth Family
by Ribecca, 2011

Radial Layouts – Drawing Style



Drawing conventions

- Vertices lie on circular layers according to their depth
- Drawing is planar

Drawing aesthetics

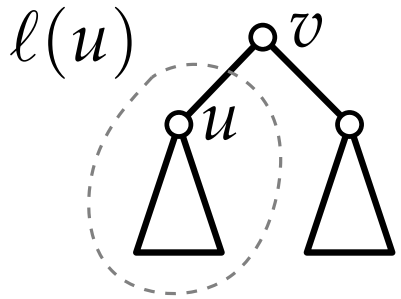
- Distribution of the vertices

How can an algorithm optimize
the distribution of the vertices?

Radial Layouts – Algorithm Attempt

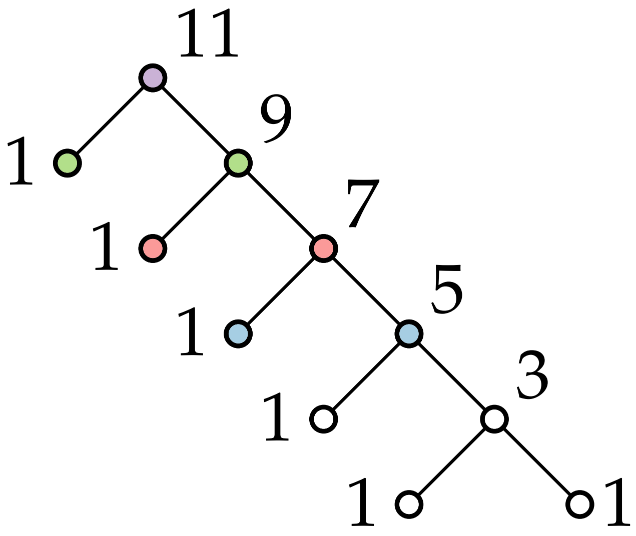
Idea

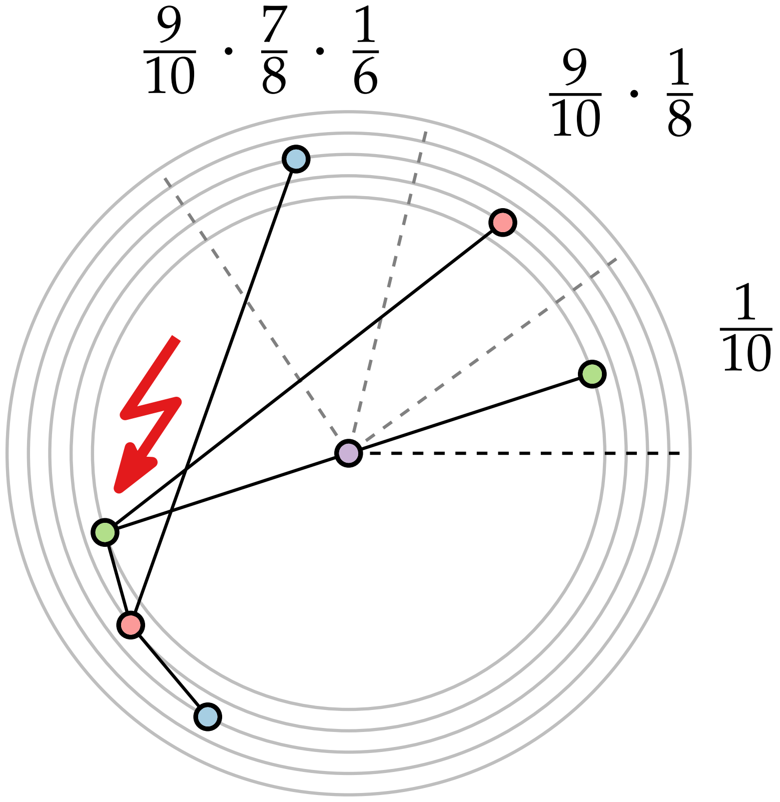
- Reserve area corresponding to size $\ell(u)$ of $T(u)$:



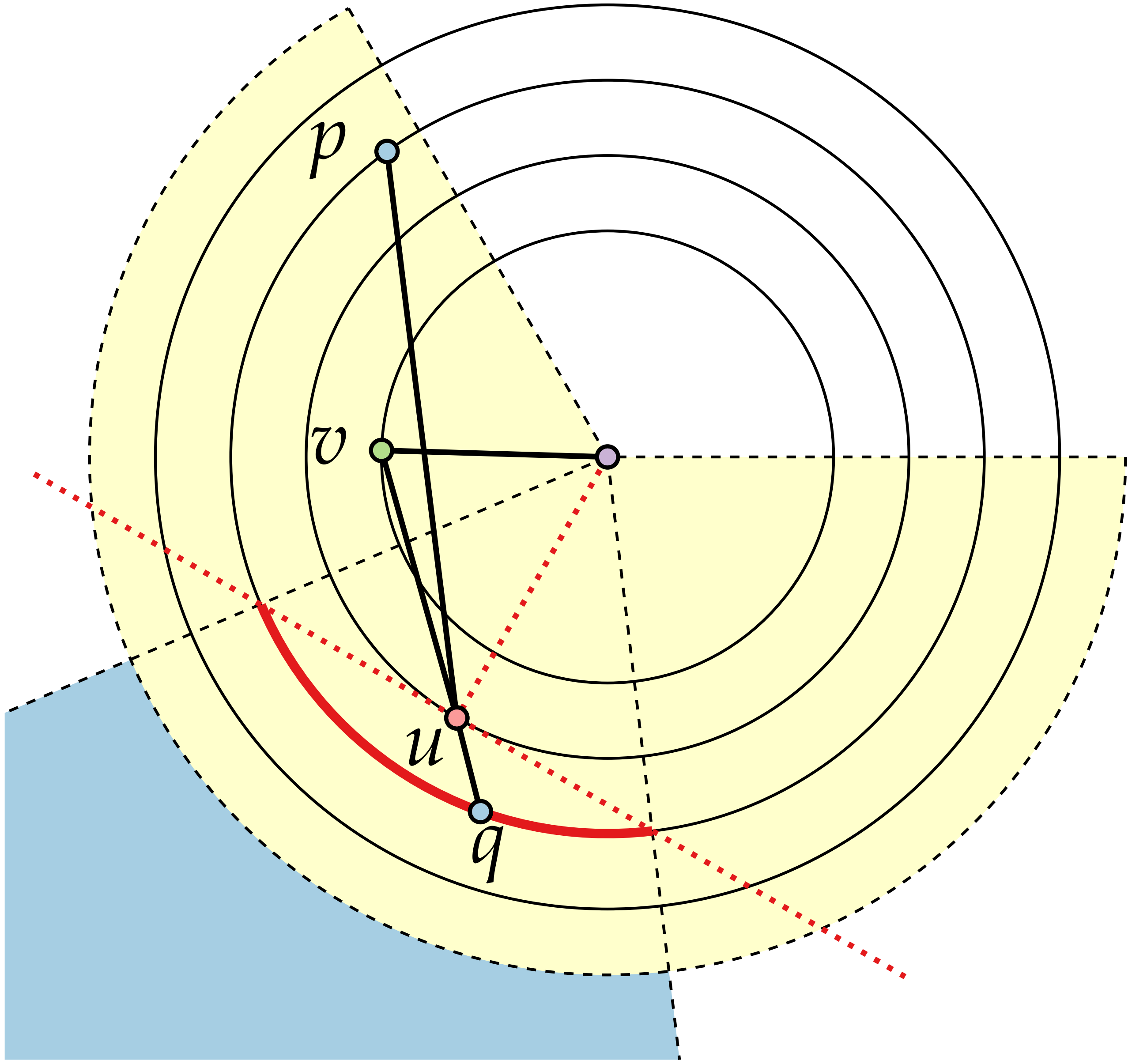
$$\tau_u = \frac{\ell(u)}{\ell(v) - 1}$$

- Place u in middle of area

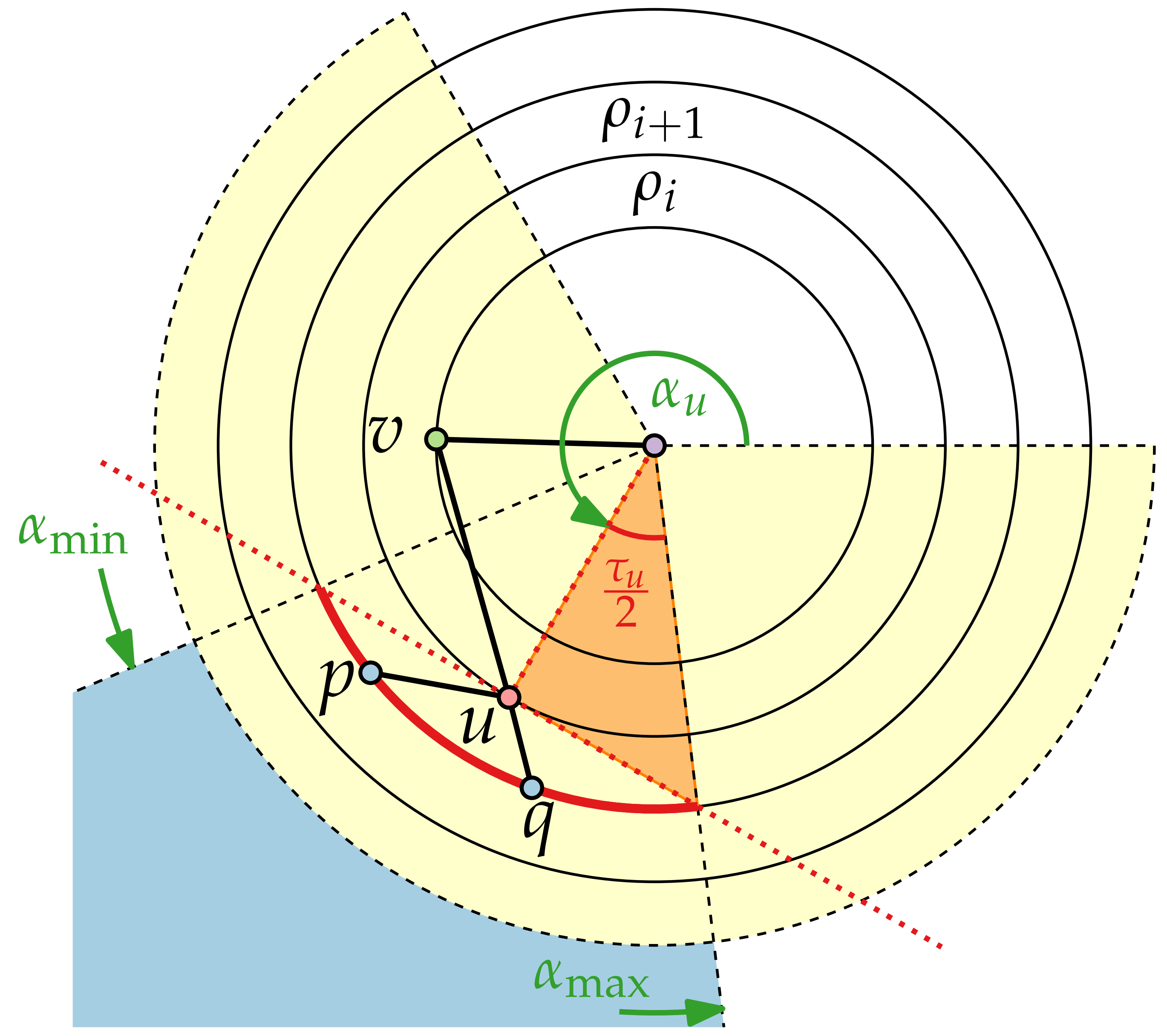




Radial Layouts – How To Avoid Crossings



Radial Layouts – How To Avoid Crossings



- τ_u – angle of the wedge corresponding to vertex u
- $\ell(u)$ – number of nodes in the subtree rooted at u
- ρ_i – radius of layer i
- $\cos \frac{\tau_u}{2} = \frac{\rho_i}{\rho_{i+1}}$
- $\tau_u = \min\{\frac{\ell(u)}{\ell(v)-1}, 2 \arccos \frac{\rho_i}{\rho_{i+1}}\}$
- Alternative:

  $\alpha_{\min} = \alpha_u - \arccos \frac{\rho_i}{\rho_{i+1}}$

  $\alpha_{\max} = \alpha_u + \arccos \frac{\rho_i}{\rho_{i+1}}$

Radial Layouts – Pseudocode

RadialTreeLayout(tree T, root $r \in T$, radii $\rho_1 < \cdots < \rho_k$)

begin

- *postorder*(r)
- *preorder*$(r, 0, 0, 2\pi)$
- **return** $(d_v, \alpha_v)_{v \in V(T)}$
- // vertex pos./polar coord.

postorder(vertex v)

- $\ell(v) \leftarrow 1$
- **foreach** child w of v **do**
  - *postorder*(w)
  - $\ell(v) \leftarrow \ell(v) + \ell(w)$

preorder(vertex v, $t, \alpha_{\min}, \alpha_{\max}$)

- $d_v \leftarrow \rho_t$
- $\alpha_v \leftarrow (\alpha_{\min} + \alpha_{\max})/2$ *//output*
- **if** $t > 0$ **then**
  - $\alpha_{\min} \leftarrow \max\{\alpha_{\min}, \alpha_v - \arccos \frac{\rho_t}{\rho_{t+1}}\}$
  - $\alpha_{\max} \leftarrow \min\{\alpha_{\max}, \alpha_v + \arccos \frac{\rho_t}{\rho_{t+1}}\}$
- *left* $\leftarrow \alpha_{\min}$
- **foreach** child w of v **do**
  - *right* $\leftarrow$ *left* $+ \frac{\ell(w)}{\ell(v)-1} \cdot (\alpha_{\max} - \alpha_{\min})$
  - *preorder*$(w, t+1, \mathit{left}, \mathit{right})$
  - *left* $\leftarrow$ *right*

Runtime? $\mathcal{O}(n)$

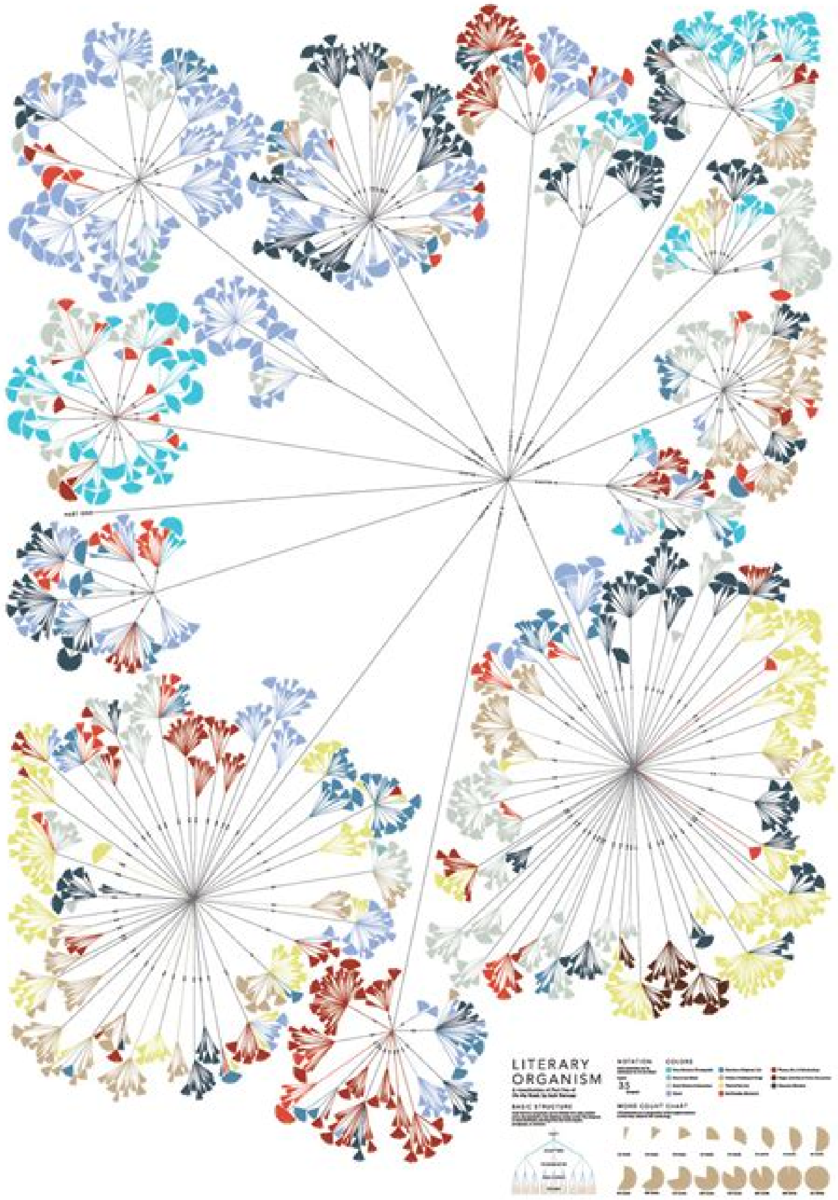
Correctness? ✓

Radial Layouts – Result

Theorem.
Let T be a tree with n vertices. The RadialTreeLayout algorithm constructs in $O(n)$ time a drawing Γ of T s.t.:

- Γ is radial drawing
- Vertices lie on circle according to their depth
- Area quadratic in max degree times height of T
  (see [GD Ch. 3.1.3] if interested)

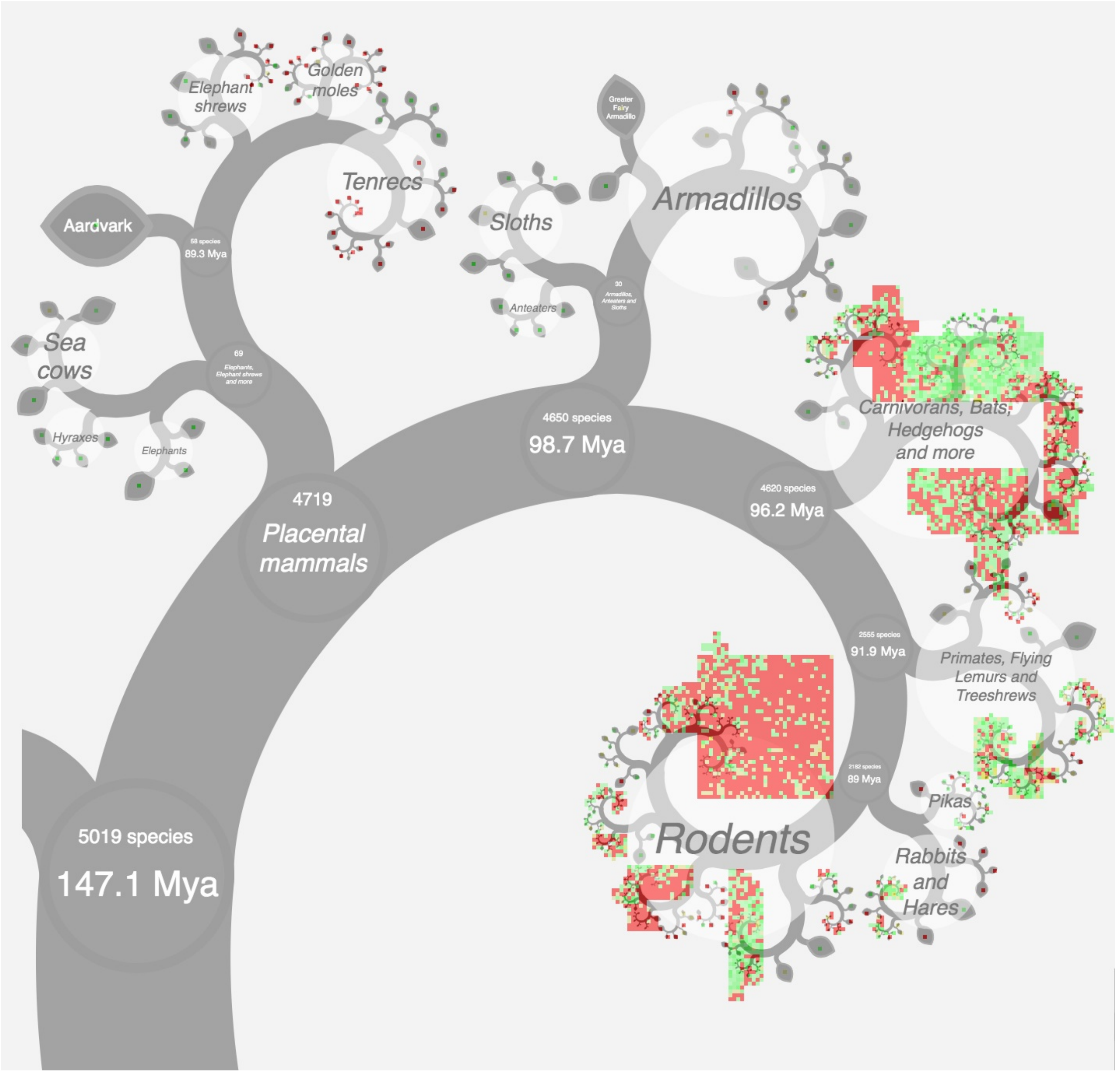
Other Tree Visualization Styles



Writing Without Words:
The project explores methods to visualizes the differences in writing styles of different authors.

Similar to ballon layout

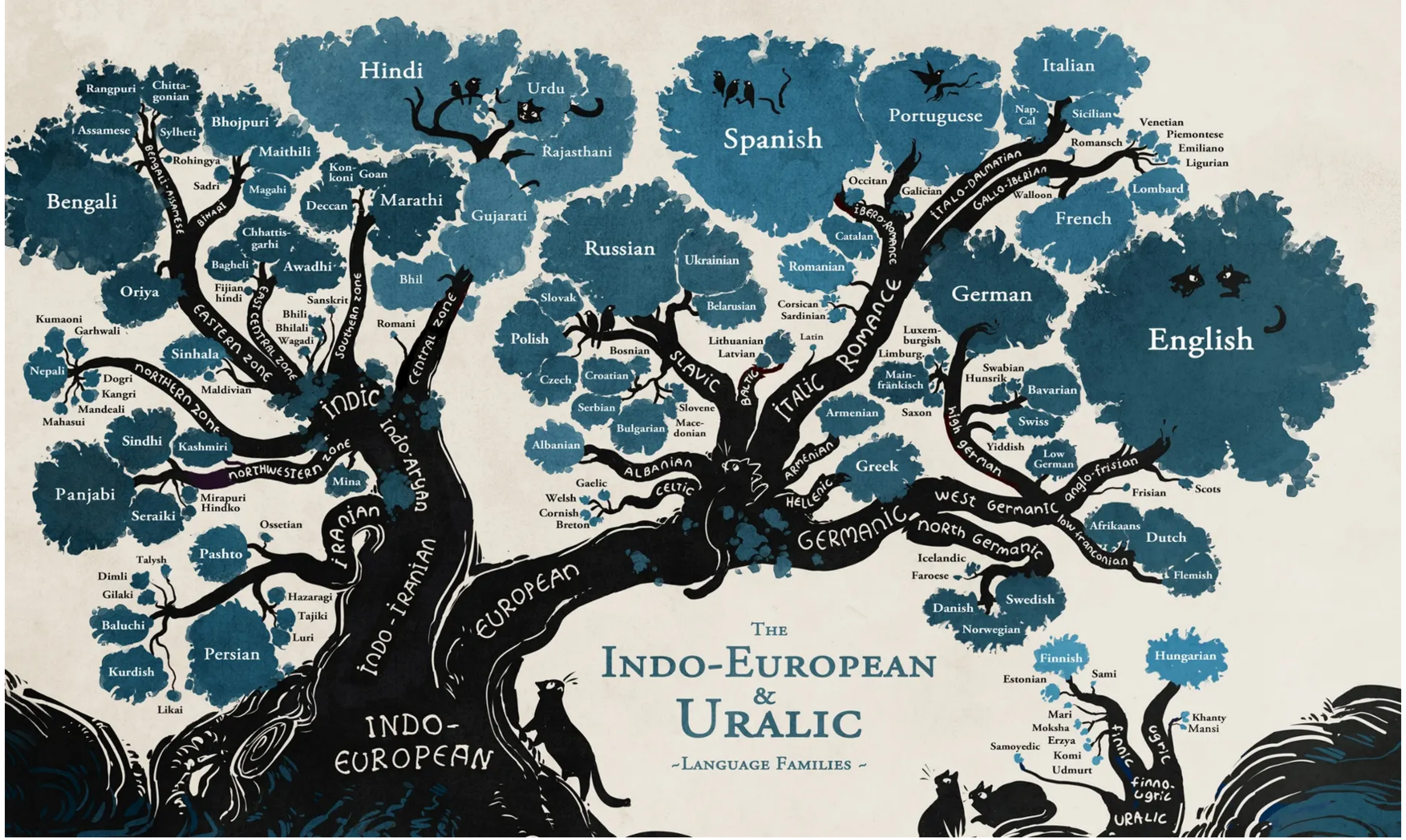
Other Tree Visualization Styles



A phylogenetically organised display of data for all placental mammal species.

Fractal layout

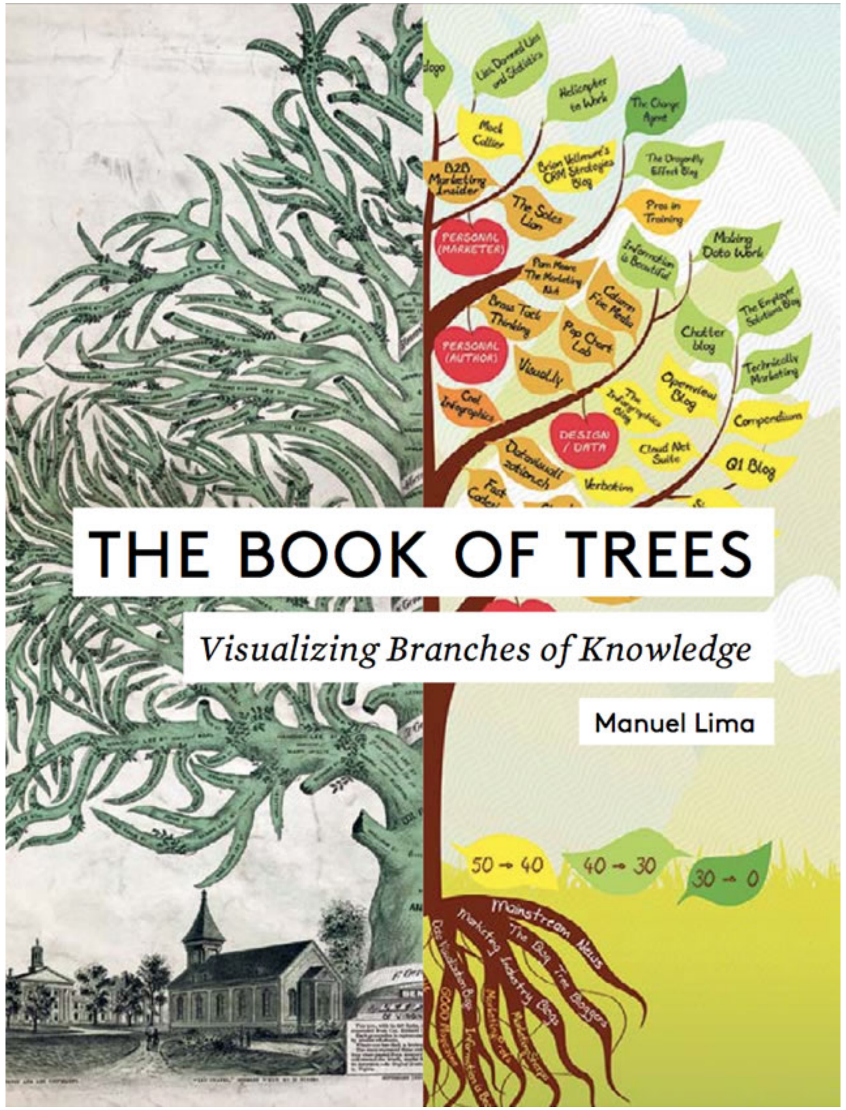
Other Tree Visualization Styles



A language family tree – in pictures

Fractal layout

Other Tree Visualization Styles



treevis.net

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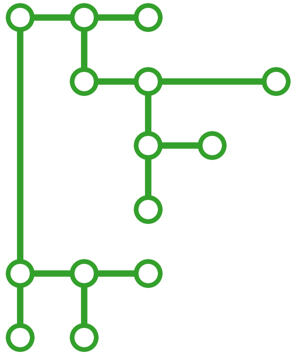
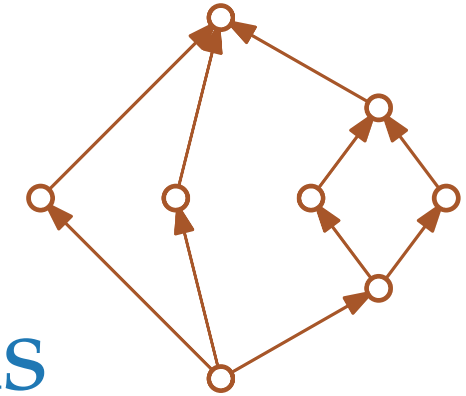
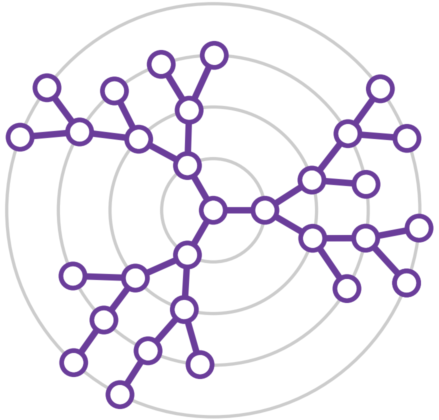


Visualization of Graphs

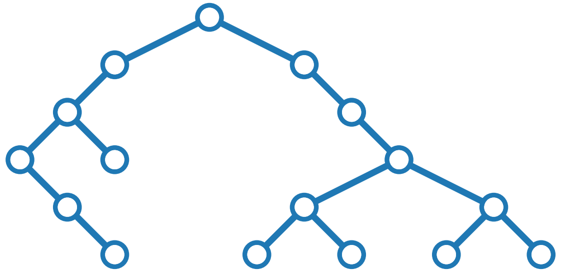


Lecture 2:
Drawing Trees and Series-Parallel Graphs



Part V:
Series-Parallel Graphs

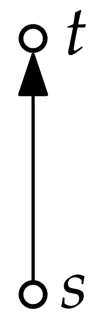
Philipp Kindermann



Series-Parallel Graphs

A graph G is **series-parallel**, if

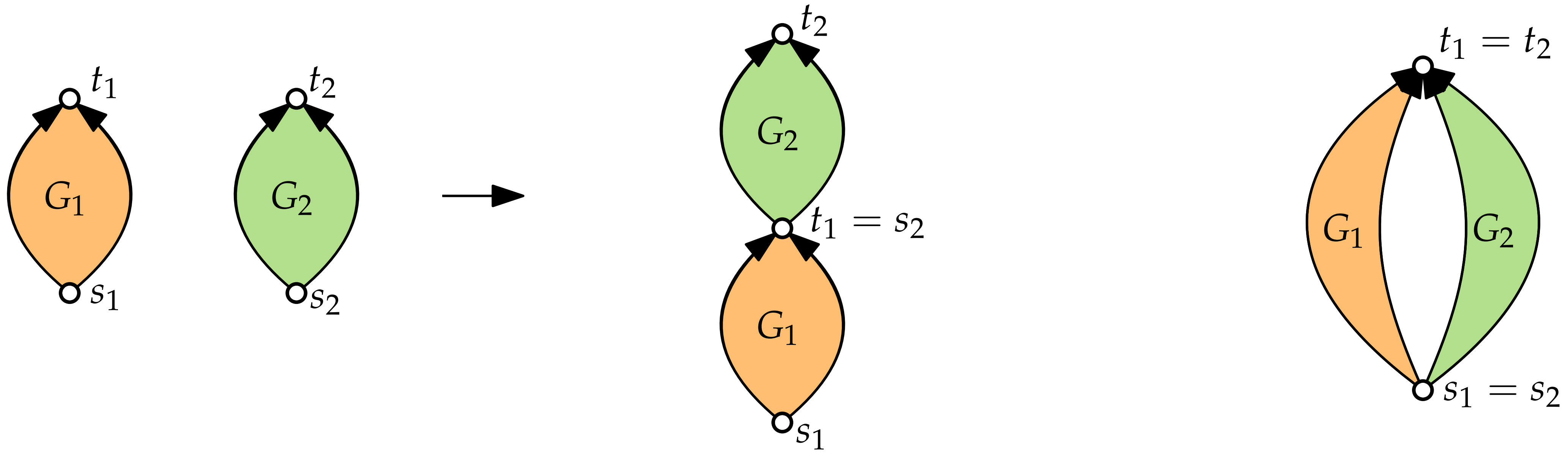
- it contains a single (directed) edge (s, t), or
- it consists of two series-parallel graphs G_1, G_2 with sources s_1, s_2 and sinks t_1, t_2 that are combined using one of the following rules:



convince yourself that series-parallel graphs are planar

Series composition

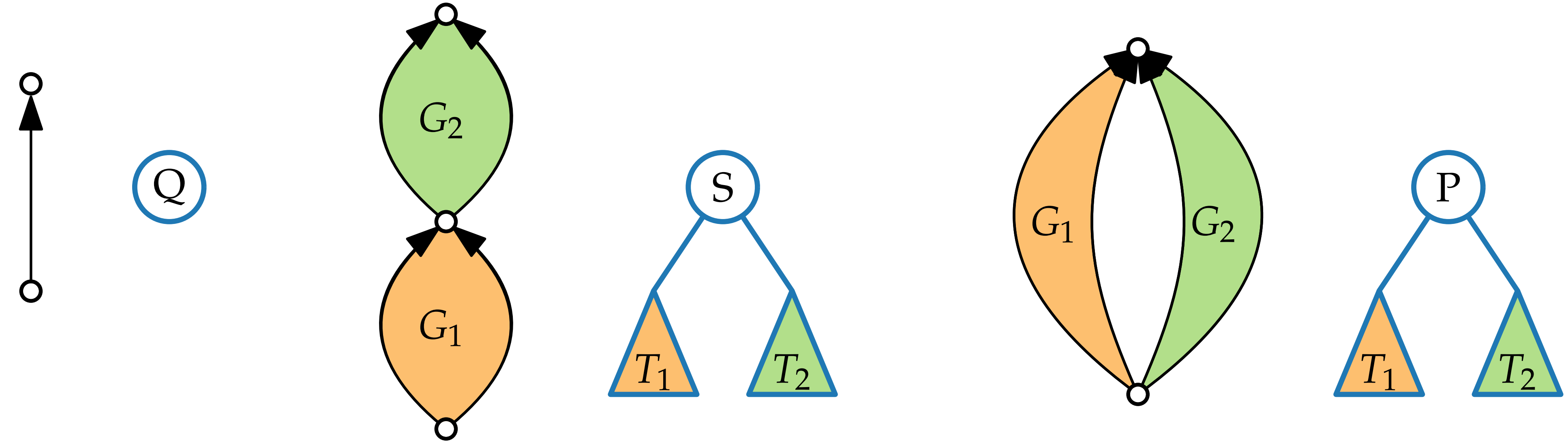
Parallel composition



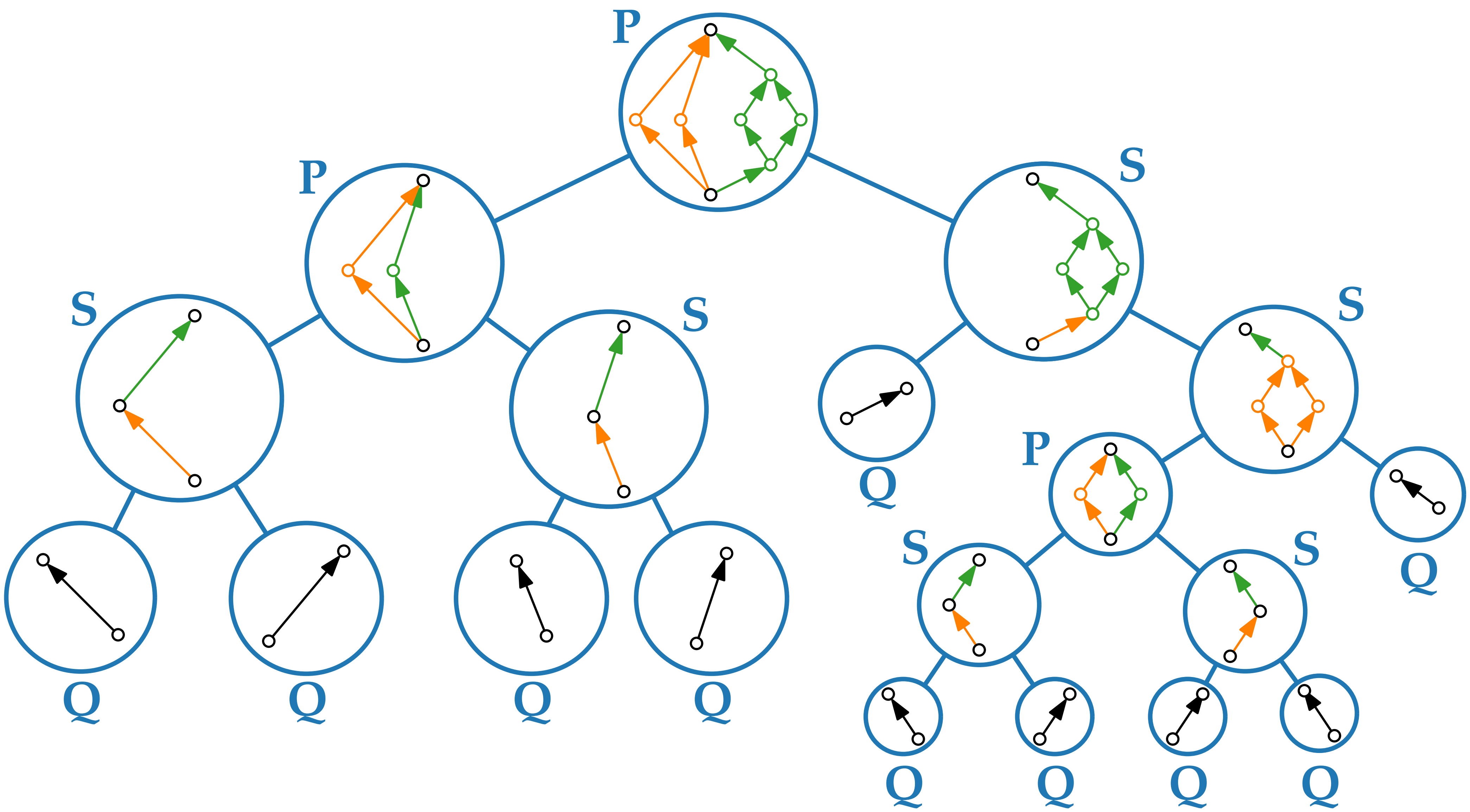
Series-Parallel Graphs – Decomposition Tree

A **decomposition tree** of G is a binary tree T with nodes of three types: **S**, **P** and **Q**-type

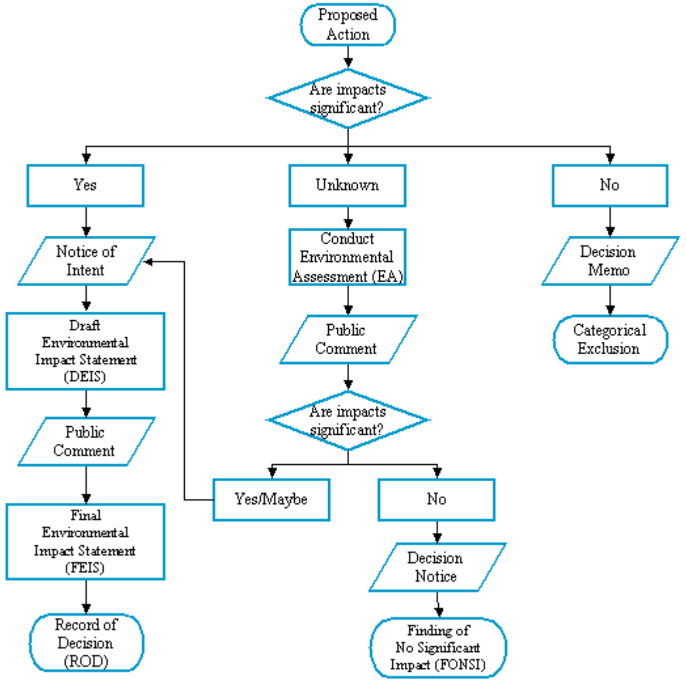
- A **Q**-node represents a single edge
- An **S**-node represents a series composition; its children T_1 and T_2 represent G_1 and G_2
- A **P**-node represents a parallel composition; its children T_1 and T_2 represent G_1 and G_2



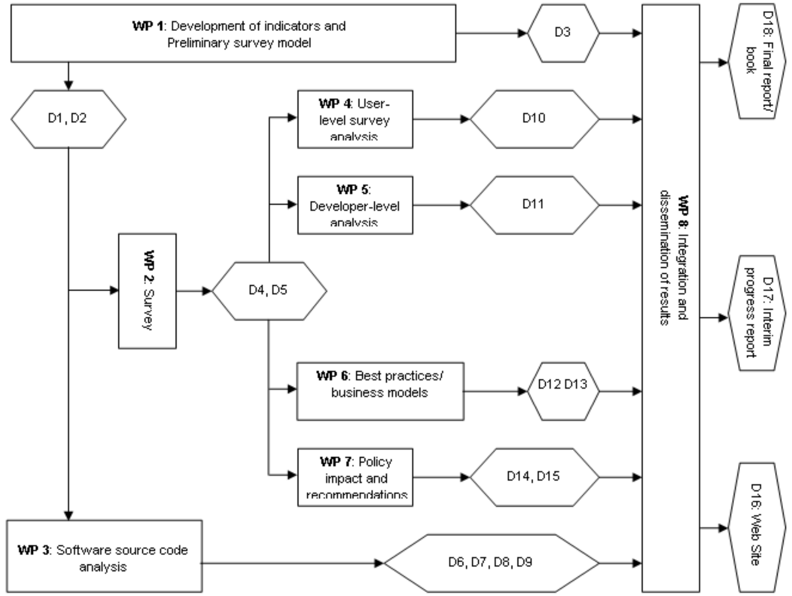
Series-Parallel Graphs – Decomposition Example



Series-Parallel Graphs – Applications



Flowcharts



PERT-Diagrams
(Program Evaluation and Review Technique)

Computational complexity:
Linear time algorithms for $\mathcal{NP}$-hard problems
(e.g. Maximum Matching, MIS, Hamiltonian Completion)

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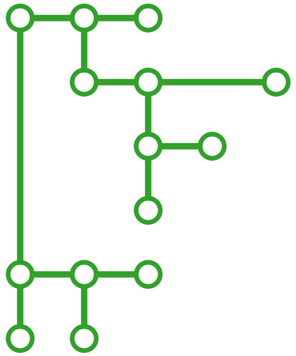
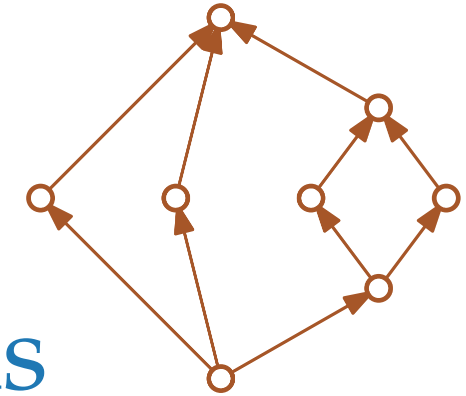
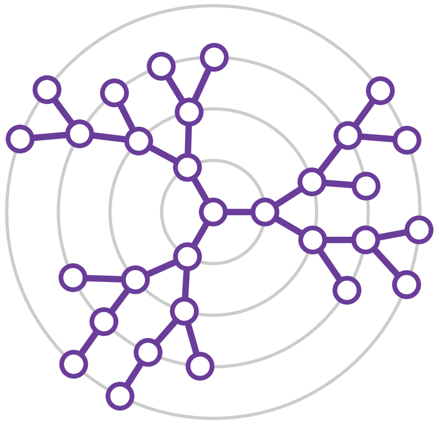


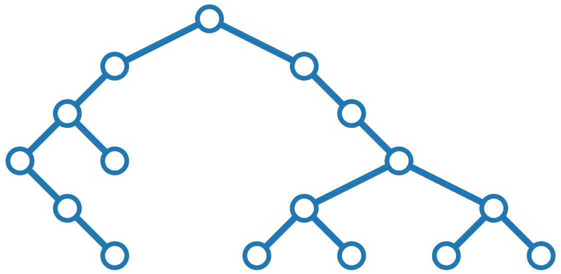
Visualization of Graphs



Lecture 2:
Drawing Trees and Series-Parallel Graphs



Part VI:
Drawings of Series-Parallel Graphs



Philipp Kindermann

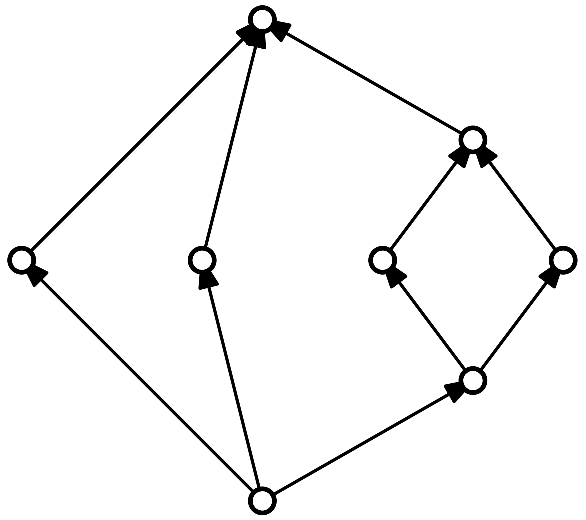
Series-Parallel Graphs – Drawing Style

Drawing conventions

- Planarity
- Straight-line edges
- Upward

Drawing aesthetics

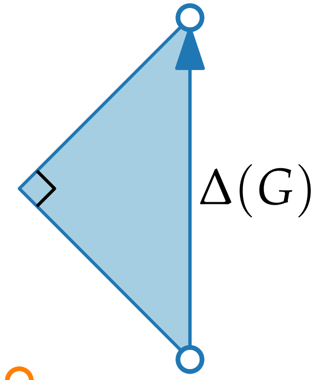
- Area



Series-Parallel Graphs – Straight-Line Drawings

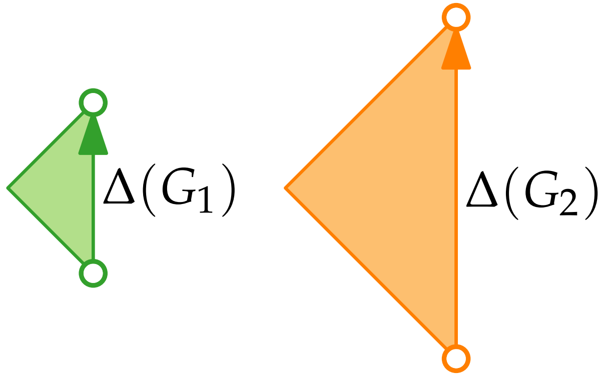
Divide & conquer algorithm using the decomposition tree

- Draw G inside a right-angled isosceles bounding triangle $\Delta(G)$



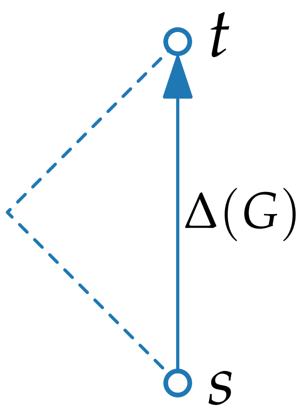
Base case: Q-nodes

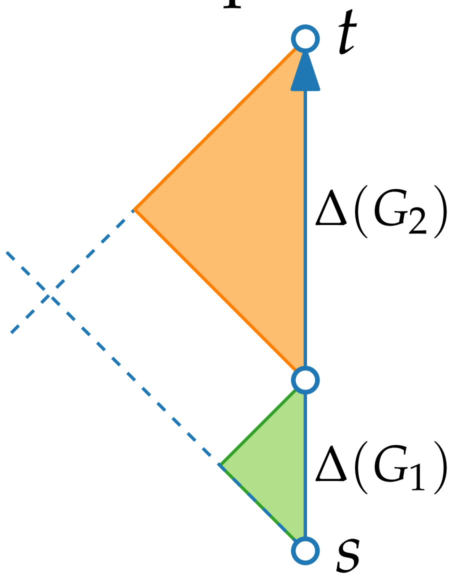
Divide: Draw G_1 and G_2 first

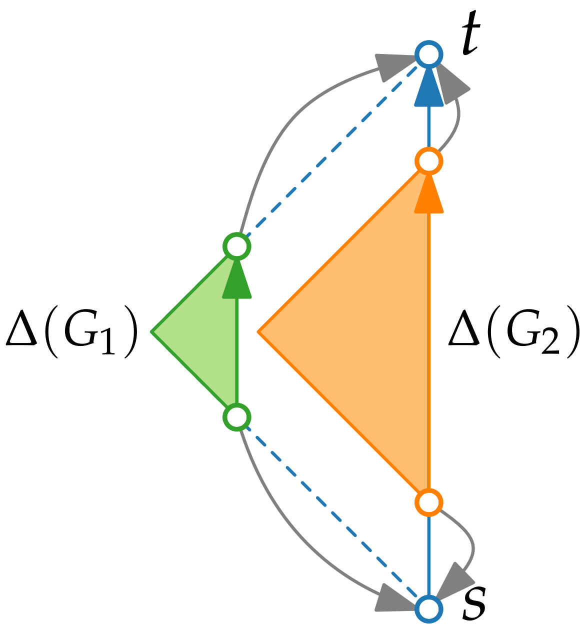


Conquer:

- S-nodes / series composition
- P-nodes / parallel composition







Do you see any problem?

Series-Parallel Graphs – Straight-Line Drawings

Divide & conquer algorithm using the decomposition tree

- Draw G inside a right-angled isosceles bounding triangle $\Delta(G)$

Base case: Q-nodes

Divide: Draw G_1 and G_2 first

Conquer:

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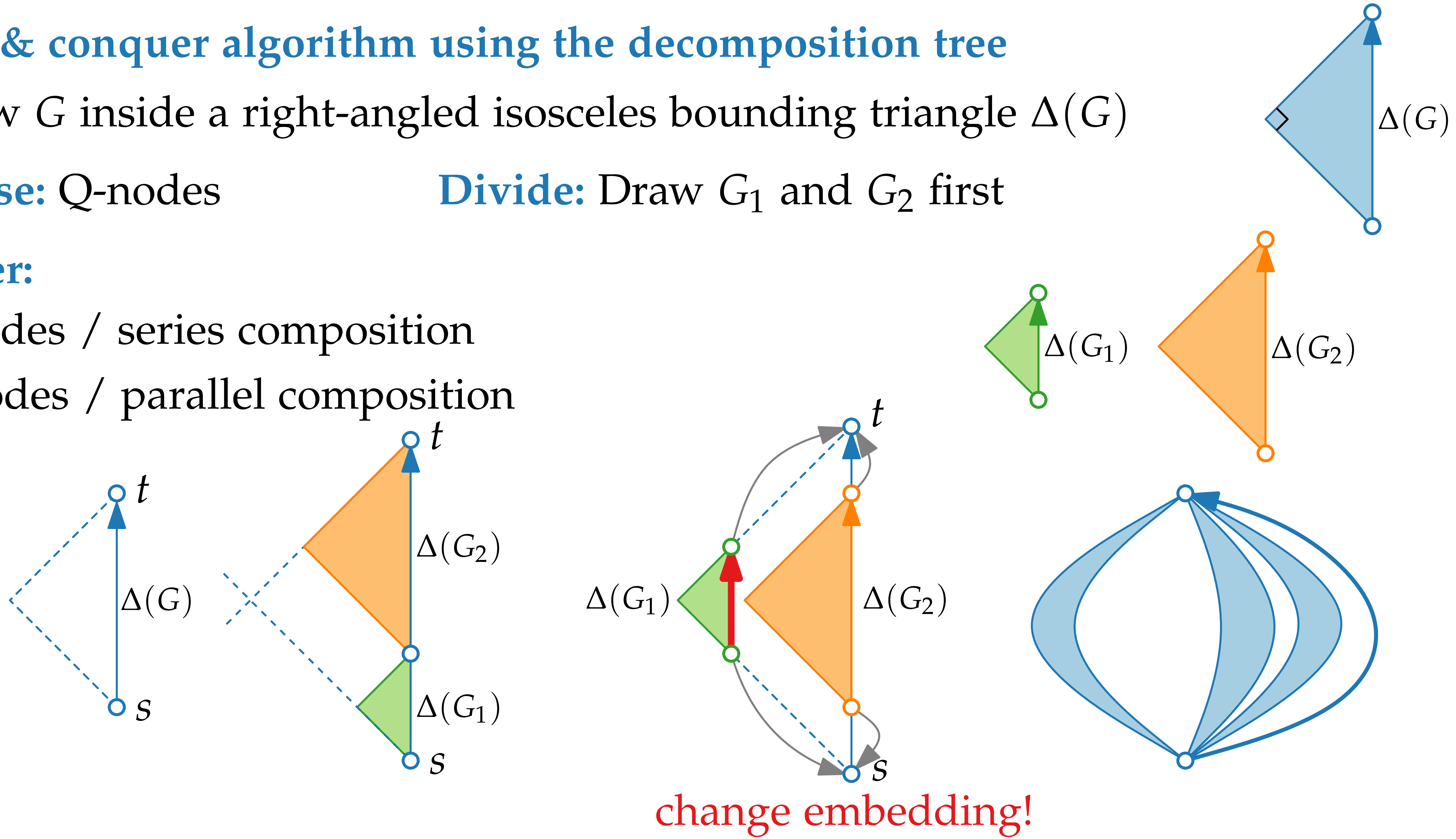
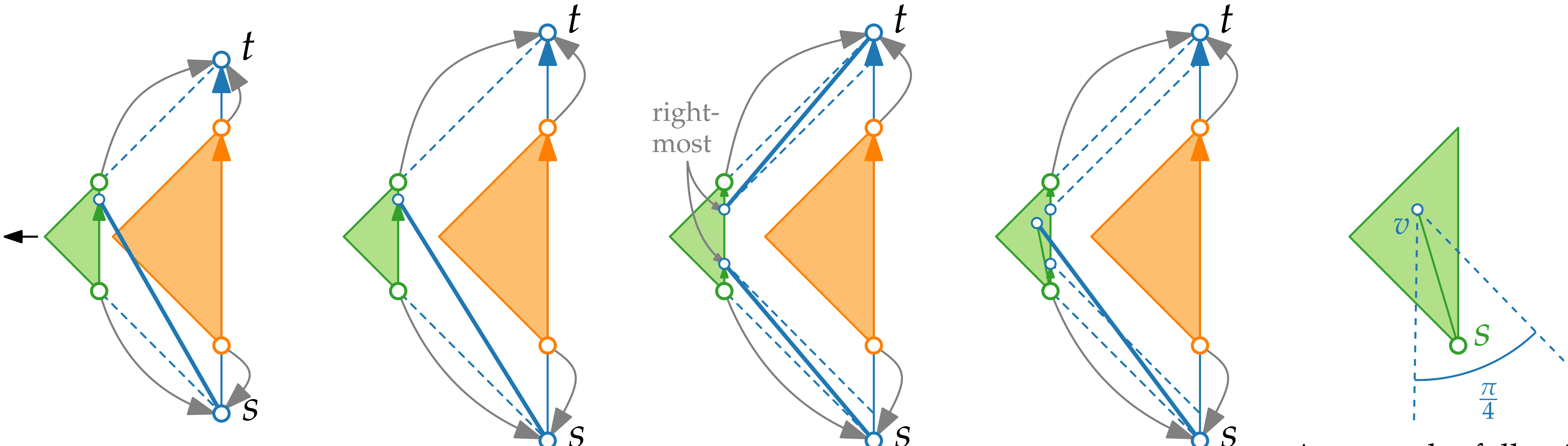


Series-Parallel Graphs – Straight-Line Drawings

- What makes parallel composition possible without creating crossings?



Assume the following holds:
the only vertex in angle(v) is s

- This condition **is** preserved during the induction step.

Lemma.
The drawing produced by the algorithm is planar.

Series-Parallel Graphs – Result

Theorem.
Let G be a series-parallel graph. Then G (with **variable embedding**) admits a drawing Γ that

- is upward planar and
- a straight-line drawing
- with area in $\mathcal{O}(n^2)$.
- Isomorphic components of G have congruent drawings up to translation.

Γ can be computed in $\mathcal{O}(n)$ time.

Series-Parallel Graphs – Fixed Embedding

Theorem. [Bertolazzi et al. 94]

There exists a $2n$-vertex series-parallel graph G_n such that any upward planar drawing of G_n that respects the embedding requires $\Omega(4^n)$ area.

- $2 \cdot Area(G_n) < Area(\Pi)$
- $2 \cdot Area(\Pi) \leq Area(G_{n+1})$
- $4 \cdot Area(G_n) \leq Area(G_{n+1})$