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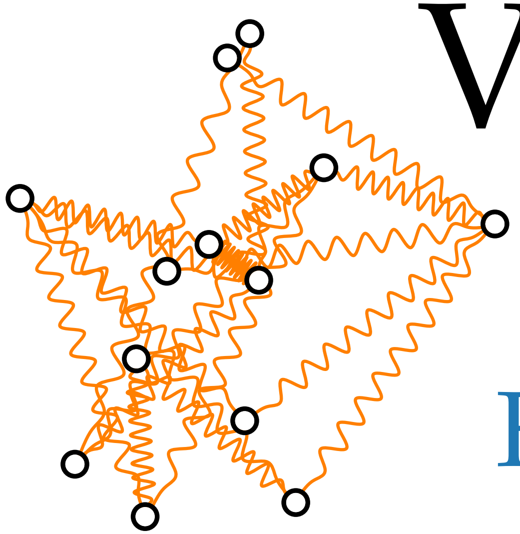
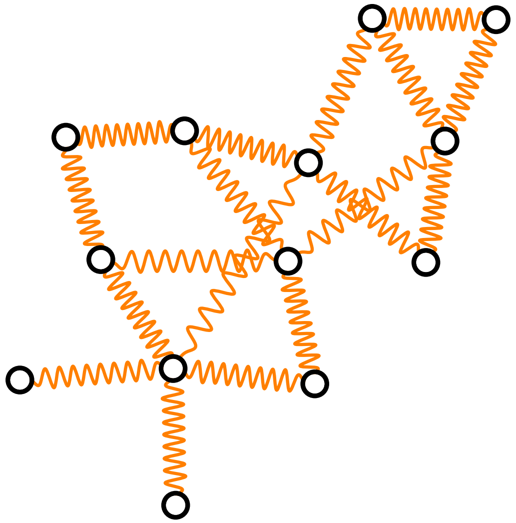


Visualization of Graphs

Lecture 3:
Force-Directed Drawing Algorithms



Part I:
Algorithm Framework

Philipp Kindermann

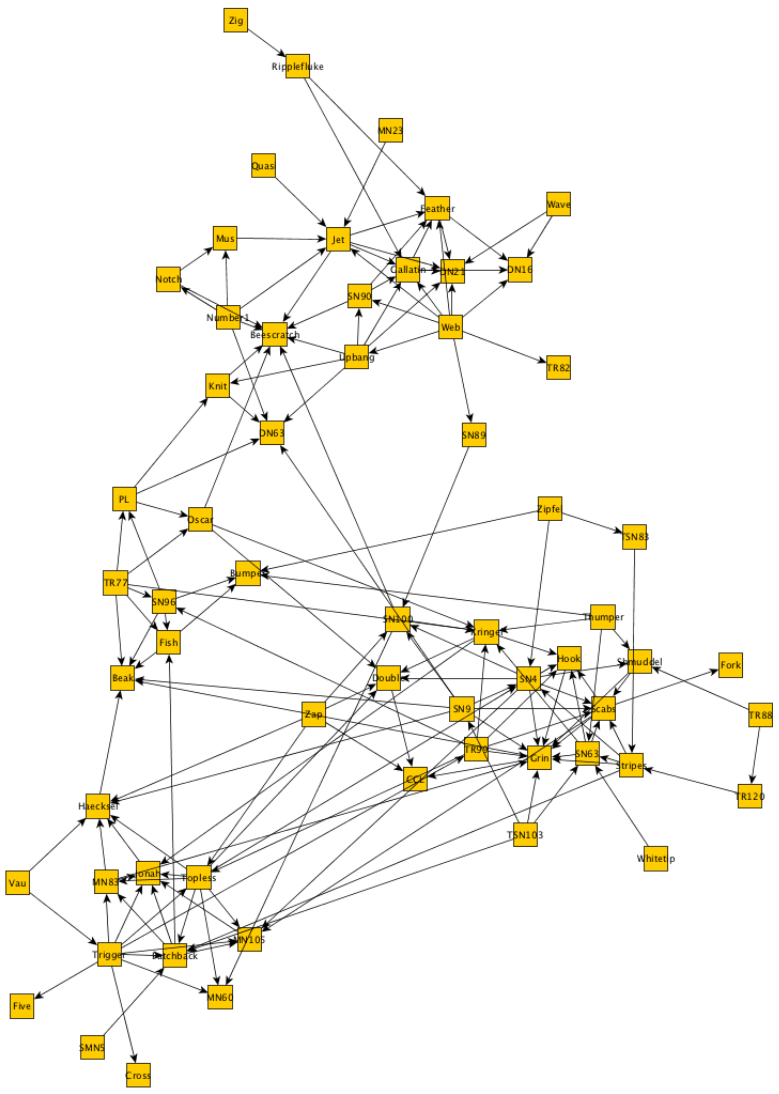
General Layout Problem

Input: Graph $G = (V, E)$

Output: Clear and readable straight-line drawing of G

Drawing aesthetics:

- adjacent vertices are close
- non-adjacent vertices are far apart
- edges short, straight-line, **similar length**
- densely connected parts (clusters) form communities
- as few crossings as possible
- nodes distributed evenly

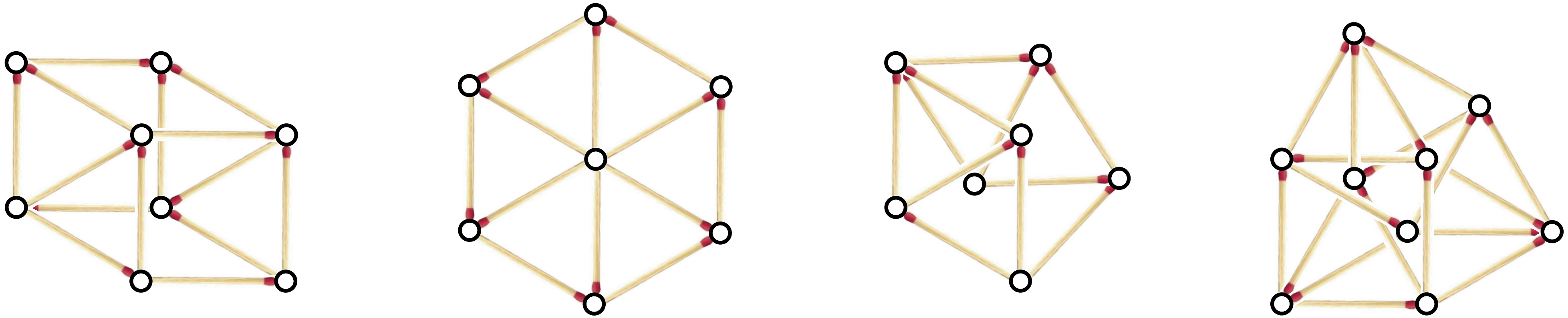


Optimization criteria partially contradict each other

Fixed Edge Lengths?

Input: Graph $G = (V, E)$, required edge length $\ell(e)$, $\forall e \in E$

Output: Drawing of G which realizes all the edge lengths



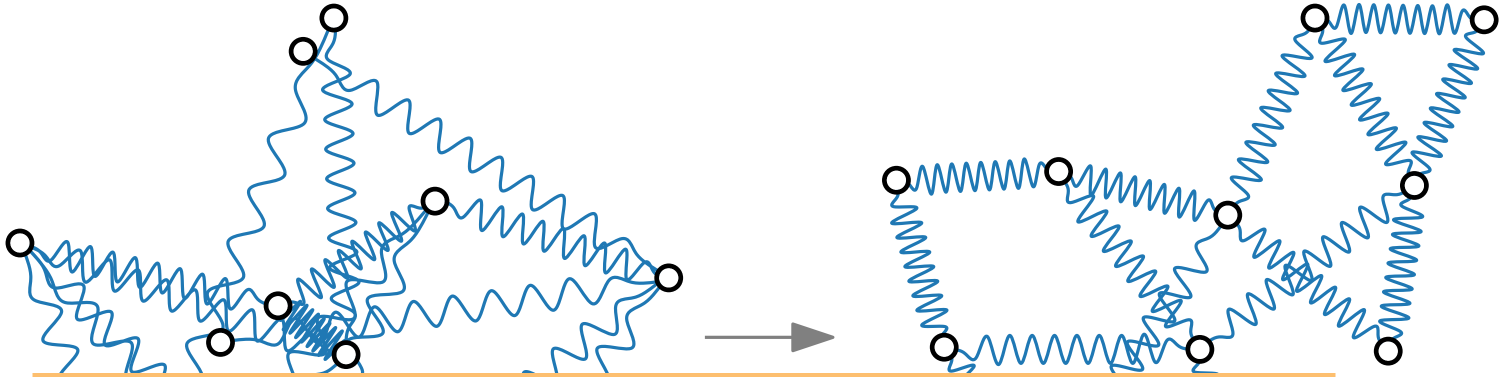
NP-hard for

- uniform edge lengths in any dimension [Johnson '82]
- uniform edge lengths in planar drawings [Eades, Wormald '90]
- edge lengths $\{1, 2\}$ [Saxe '80]

Physical Analogy

Idea. [Eades '84]

"To embed a graph we replace the vertices by steel rings and replace each edge with a **spring** to form a mechanical system ... The vertices are placed in some initial layout and let go so that the spring forces on the rings move the system to a minimal energy state."



Attractive forces.

adjacent vertices u and v:

Repulsive forces.

all vertices x and y:

So-called **spring embedders** or **force-directed** algorithms that work according to this or similar principles are among the most frequently used graph-drawing methods in practice.

Force-Directed Algorithms

initial layout — threshold — max # iterations

ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

$t \leftarrow 1$

while $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**

  foreach $u \in V$ **do**

    $F_u(t) \leftarrow \sum_{v \in V} f_{\text{rep}}(u, v) + \sum_{uv \in E} f_{\text{attr}}(u, v)$

  foreach $u \in V$ **do**

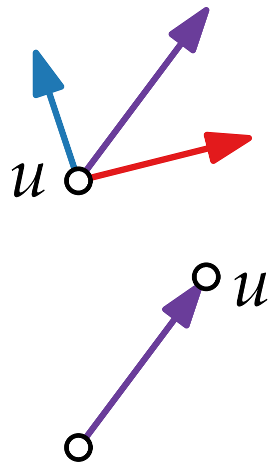
    $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$

  $t \leftarrow t + 1$

return p

cooling factor

end layout



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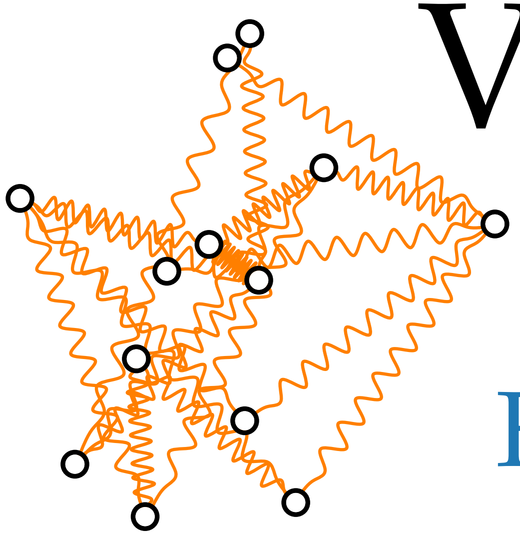
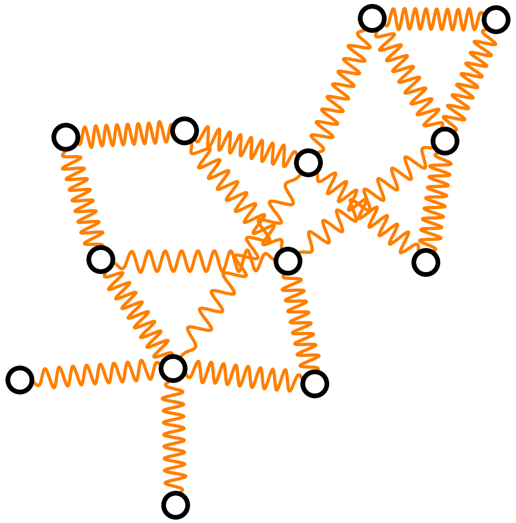


Visualization of Graphs

Lecture 3:
Force-Directed Drawing Algorithms



Part II:
Spring Embedder by Eades

Philipp Kindermann

Spring Embedder by Eades – Model

```
ForceDirected(G = (V, E), p = (p_v)_{v∈V}, ε > 0, K ∈ ℕ)
  t ← 1
  while t < K and max_{v∈V} ||F_v(t)|| > ε do
    foreach u ∈ V do
      F_u(t) ← Σ_{v∈V} f_rep(u, v) + Σ_{uv∈E} f_attr(u, v)
    foreach u ∈ V do
      p_u ← p_u + δ(t) · F_u(t)
    t ← t + 1
  return p
```

- **Repulsive forces**

repulsion constant (e.g. 2.0)

$$f_{\text{rep}}(u,v) = \frac{c_{\text{rep}}}{||p_v - p_u||^2} \cdot \overrightarrow{p_v p_u}$$

- **Attractive forces**

spring constant (e.g. 1.0)

$$f_{\text{spring}}(u,v) = c_{\text{spring}} \cdot \log \frac{||p_v - p_u||}{\ell} \cdot \overrightarrow{p_u p_v}$$

$$f_{\text{attr}}(u,v) = f_{\text{spring}}(u,v) - f_{\text{rep}}(u,v)$$

- **Resulting displacement vector**

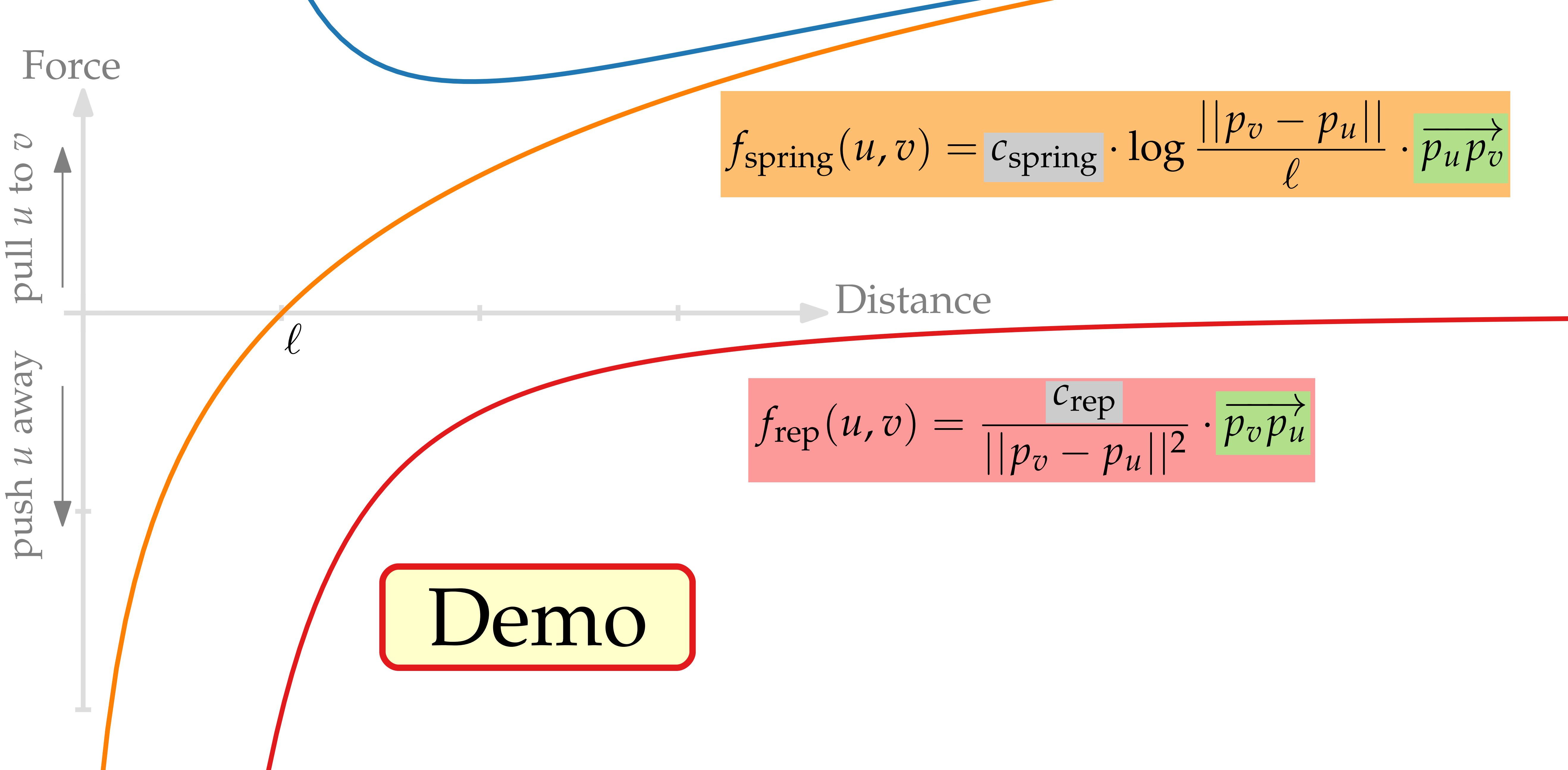
$$F_u = \sum_{v \in V} f_{\text{rep}}(u,v) + \sum_{uv \in E} f_{\text{attr}}(u,v)$$

Notation.

- $||p_u - p_v||$ = Euclidean distance between u and v
- $\overrightarrow{p_u p_v}$ = unit vector pointing from u to v
- ℓ = ideal spring length for edges

Spring Embedder by Eades – Force Diagram

$$f_{\text{attr}}(u,v) = f_{\text{spring}}(u,v) - f_{\text{rep}}(u,v)$$



$$f_{\text{spring}}(u,v) = c_{\text{spring}} \cdot \log \frac{||p_v - p_u||}{\ell} \cdot \overrightarrow{p_u p_v}$$

$$f_{\text{rep}}(u,v) = \frac{c_{\text{rep}}}{||p_v - p_u||^2} \cdot \overrightarrow{p_v p_u}$$

Demo

Spring Embedder by Eades – Discussion

Advantages.

- very simple algorithm
- good results for small and medium-sized graphs
- empirically good representation of symmetry and structure

Disadvantages.

- system is not stable at the end
- converging to local minima
- timewise f_{spring} in $\mathcal{O}(|E|)$ and f_{rep} in $\mathcal{O}(|V|^2)$

Influence.

- original paper by Peter Eades [Eades '84] got $\sim$ 2000 citations
- basis for many further ideas

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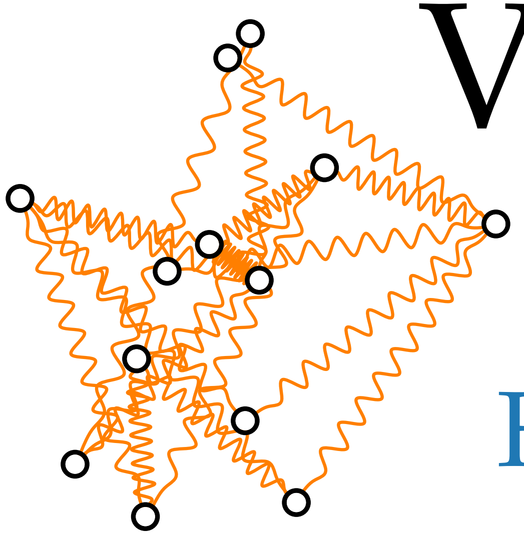
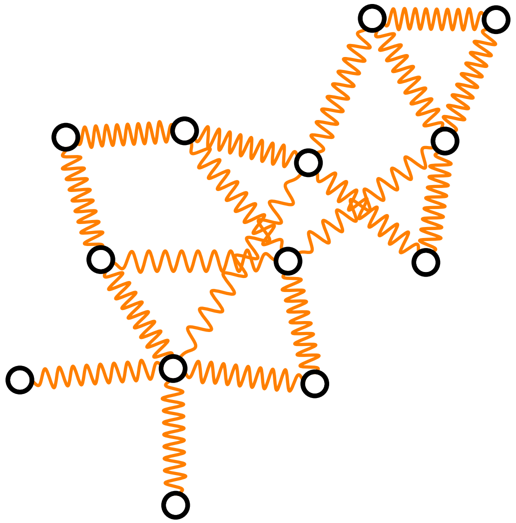


Visualization of Graphs

Lecture 3: Force-Directed Drawing Algorithms

Part III: Variant by Fruchterman & Reingold



Philipp Kindermann

Variant by Fruchterman & Reingold

ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

```
t ← 1
while t < K and max_{v∈V} ||F_v(t)|| > ε do
    foreach u ∈ V do
        F_u(t) ← Σ_{v∈V} f_rep(u,v) + Σ_{uv∈E} f_attr(u,v)
    foreach u ∈ V do
        p_u ← p_u + δ(t) · F_u(t)
    t ← t + 1
return p
```

- **Repulsive forces**

$$f_{\text{rep}}(u,v) = \frac{\ell^2}{||p_v - p_u||} \cdot \overrightarrow{p_v p_u}$$

- **Attractive forces**

$$f_{\text{attr}}(u,v) = \frac{||p_v - p_u||^2}{\ell} \cdot \overrightarrow{p_u p_v}$$

- **Resulting displacement vector**

$$F_u = \sum_{v \in V} f_{\text{rep}}(u,v) + \sum_{uv \in E} f_{\text{attr}}(u,v)$$

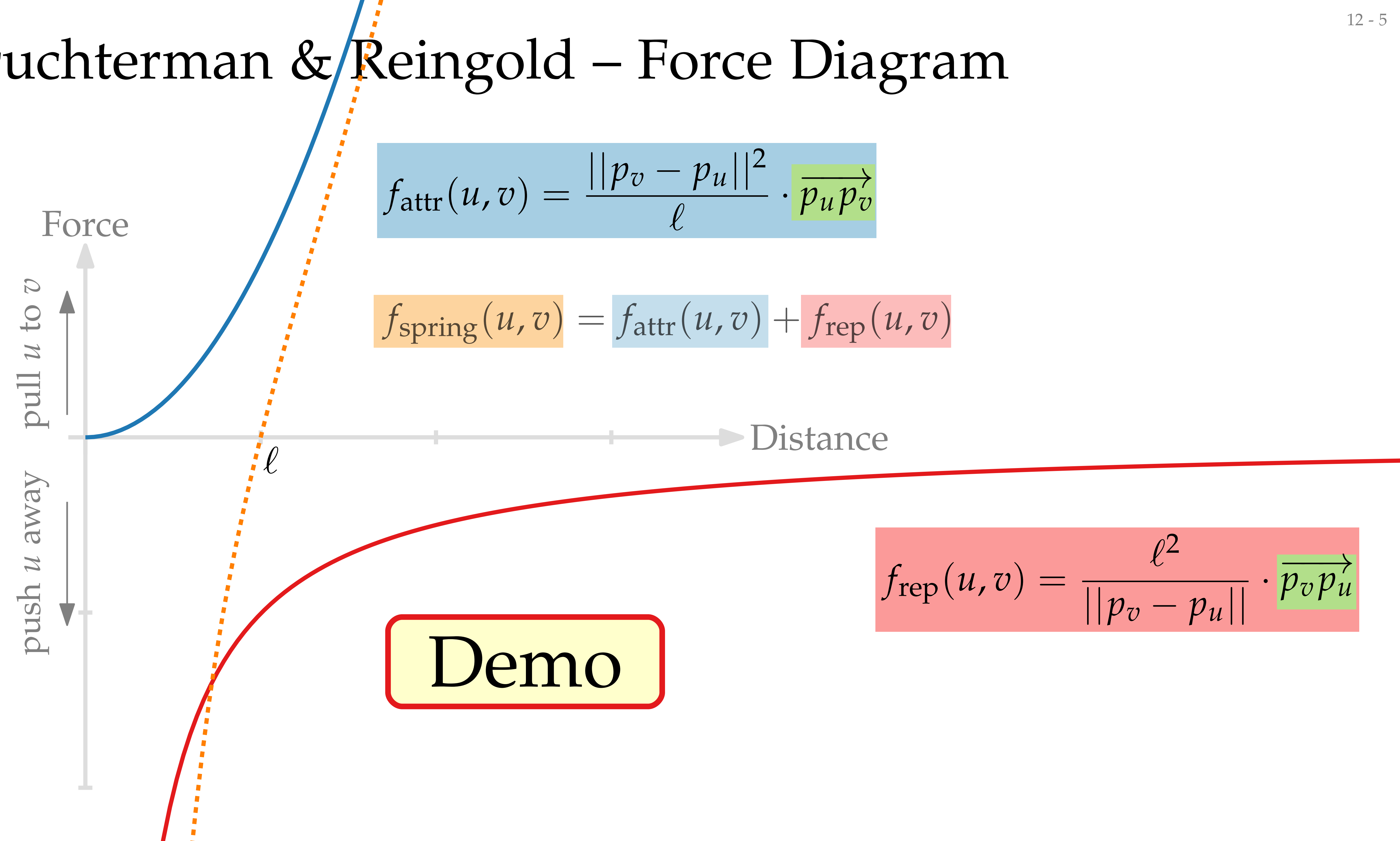
Notation.

- $||p_u - p_v||$ = Euclidean distance between u and v
- $\overrightarrow{p_u p_v}$ = unit vector pointing from u to v
- ℓ = ideal spring length for edges

Fruchterman & Reingold – Force Diagram

$$f_{\text{attr}}(u,v) = \frac{||p_v - p_u||^2}{\ell} \cdot \overrightarrow{p_u p_v}$$

$$f_{\text{spring}}(u,v) = f_{\text{attr}}(u,v) + f_{\text{rep}}(u,v)$$



$$f_{\text{rep}}(u,v) = \frac{\ell^2}{||p_v - p_u||} \cdot \overrightarrow{p_v p_u}$$

Demo

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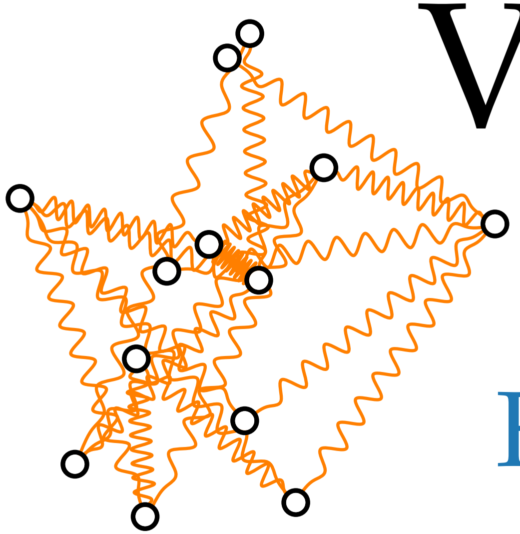
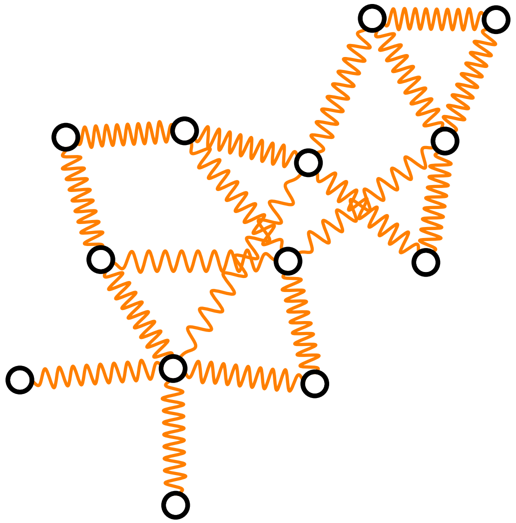


Visualization of Graphs

Lecture 3:
Force-Directed Drawing Algorithms

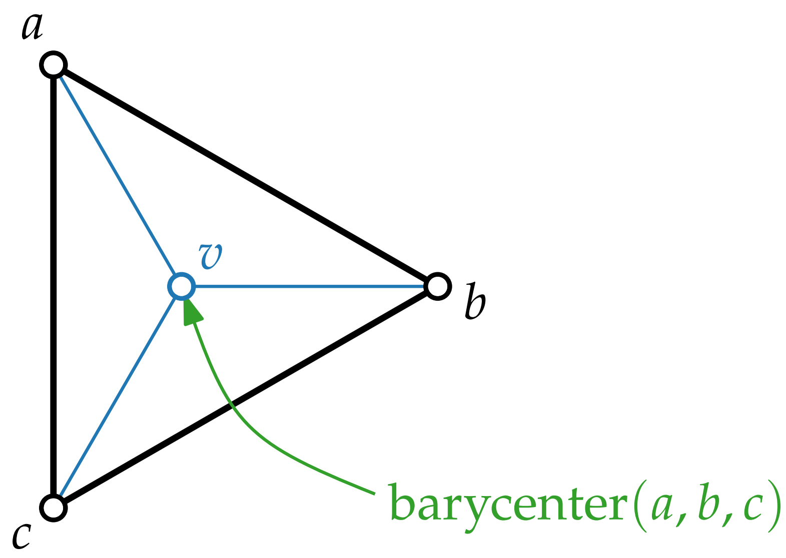


Part IV:
Tutte Drawing

Philipp Kindermann

Idea

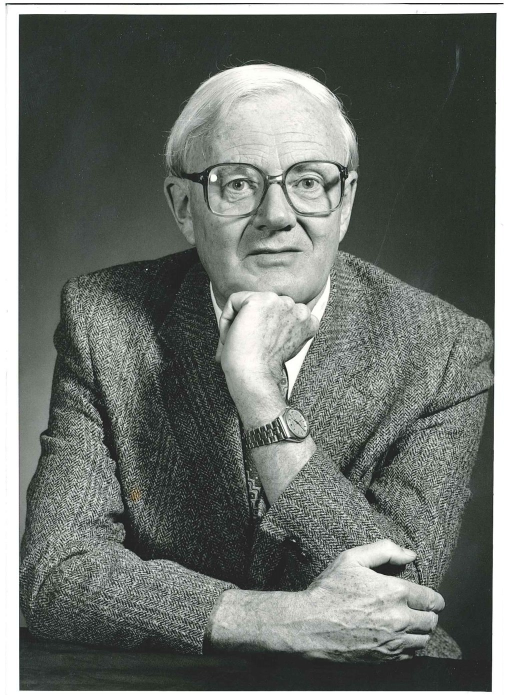
Consider a fixed triangle (a, b, c) with one common neighbor v



Where would you place v?

$$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$$

Idea.
Repeatedly place every vertex at barycenter of neighbors.



William T. Tutte
1917 – 2002

HOW TO DRAW A GRAPH

By W. T. TUTTE

[Received 22 May 1962]

1. Introduction

We use the definitions of (11). However, in deference to some recent attempts to unify the terminology of graph theory we replace the term 'circuit' by 'polygon', and 'degree' by 'valency'.

A graph G is *3-connected* (*nodally 3-connected*) if it is simple and non-separable and satisfies the following condition; if G is the union of two proper subgraphs H and K such that $H \cap K$ consists solely of two vertices u and v, then one of H and K is a link-graph (arc-graph) with ends u and v.

It should be noted that the union of two proper subgraphs H and K of G can be the whole of G only if each of H and K includes at least one edge or vertex not belonging to the other. In this paper we are concerned mainly with nodally 3-connected graphs, but a specialization to 3-connected graphs is made in § 12.

Tutte's Forces

Goal.

$p_u = \text{barycenter}(\bigcup_{uv \in E} v)$
$= \sum_{uv \in E} p_v / \deg(u)$

$F_u(t) = \sum_{uv \in E} p_v / \deg(u) - p_u$
$= \sum_{uv \in E} (p_v - p_u) / \deg(u)$
$= \sum_{uv \in E} ||p_u - p_v|| / \deg(u)$

ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

- $t \leftarrow 1$
- **while** $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**
  - **foreach** $u \in V$ **do**
    - $F_u(t) \leftarrow \sum_{v \in V} f_{\text{rep}}(u, v) + \sum_{uv \in E} f_{\text{attr}}(u, v)$
  - **foreach** $u \in V$ **do**
    - $p_u \leftarrow p_u + \delta(t)$ ~~$\delta(t)$~~ $1 \cdot F_u(t)$
  - $t \leftarrow t + 1$
- **return** p

$\text{barycenter}(x_1, \dots, x_k) = \sum_{i=1}^{k} x_i / k$

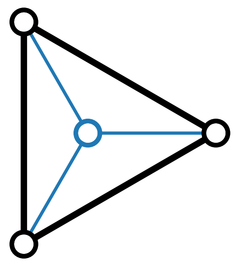
- **Repulsive forces**

$$f_{\text{rep}}(u, v) = 0$$

- **Attractive forces**

$$f_{\text{attr}}(u, v) = \begin{cases} 0 & u \text{ fixed} \\ \frac{1}{\deg(u)} \cdot ||p_u - p_v|| & \text{else} \end{cases}$$

Solution: $p_u = (0, 0)\ \forall u \in V$



Fix coordinates of outer face!

Demo

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

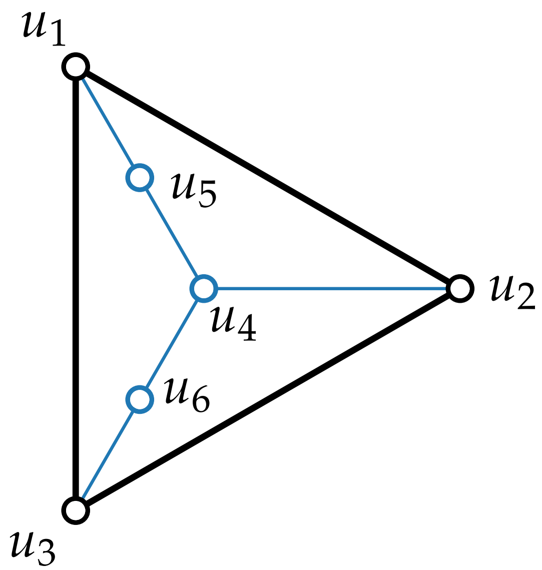
$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

$Ax = b \quad Ay = b \quad b = (0)_n$

2 Systems of linear equations

$x_u = \sum_{uv \in E} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$

$y_u = \sum_{uv \in E} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{uv \in E} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{uv \in E} y_v = 0$



A

	u_1	u_2	u_3	u_4	u_5	u_6
u_1	3	−1	−1	0	−1	0
u_2	−1	3	−1	−1	0	0
u_3	−1	−1	3	0	0	−1
u_4	0	−1	0	3	−1	−1
u_5	−1	0	0	−1	2	0
u_6	0	0	−1	−1	0	2

Laplacian matrix of G

$A_{ii} = \deg(u_i)$

$$A_{ij, i \neq j} = \begin{cases} -1 & u_i u_j \in E \\ 0 & u_i u_j \notin E \end{cases}$$

k variables, k constraints, $\det(A) > 0$

$k = \#\text{free vertices}$

$\Rightarrow$ unique solution

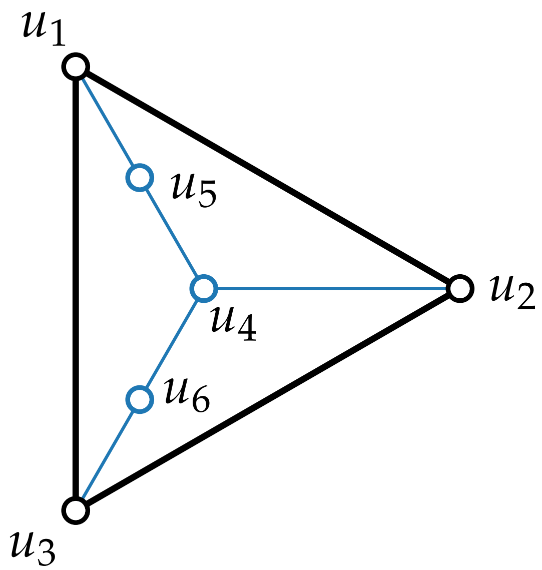
Linear System of Equations

Theorem. **Tutte drawing**
Tutte's barycentric algorithm admits a unique solution.
It can be computed in polynomial time.

Goal. $p_u = (x_u, y_u)$
$p_u = \text{barycenter}(\bigcup_{uv \in E} v)$

$x_u = \sum_{uv \in E} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$

$y_u = \sum_{uv \in E} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{uv \in E} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{uv \in E} y_v = 0$



A

	u_1	u_2	u_3	u_4	u_5	u_6
u_1	3	−1	−1	0	−1	0
u_2	−1	3	−1	−1	0	0
u_3	−1	−1	3	0	0	−1
u_4	0	−1	0	3	−1	−1
u_5	−1	0	0	−1	2	0
u_6	0	0	−1	−1	0	2

Laplacian matrix of G

$A_{ii} = \deg(u_i)$

$$A_{ij, i \neq j} = \begin{cases} -1 & u_i u_j \in E \\ 0 & u_i u_j \notin E \end{cases}$$

k variables, k constraints, $\det(A) > 0$

$k = \#\text{free vertices}$ $\Rightarrow$ unique solution

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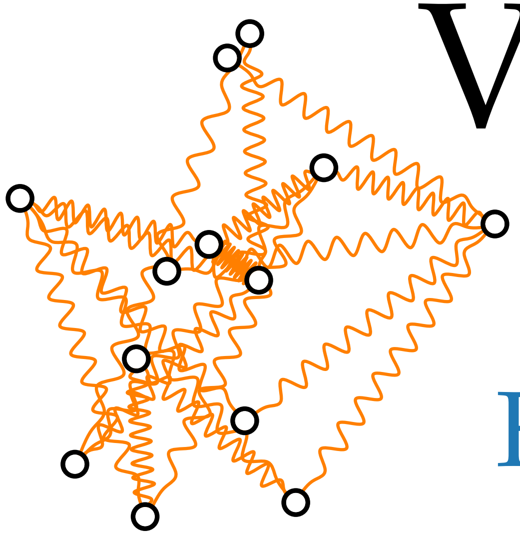
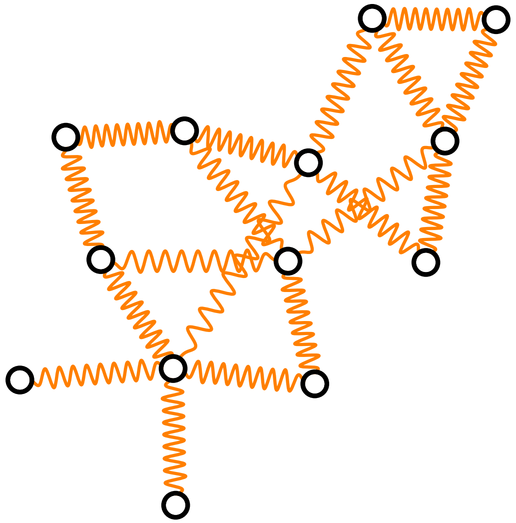


Visualization of Graphs

Lecture 3:
Force-Directed Drawing Algorithms



Part V:
Tutte's Theorem

Philipp Kindermann

3-Connected Planar Graphs

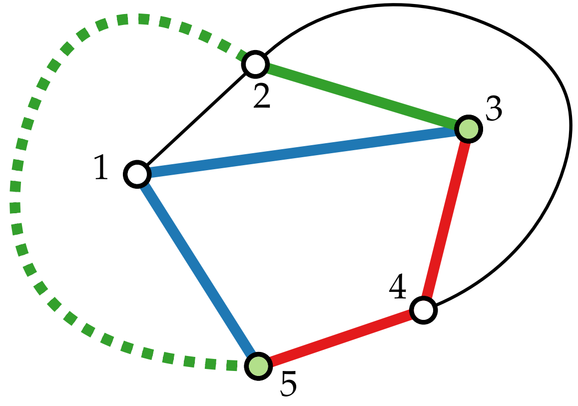
planar: G can be drawn in such a way that no edges cross each other

connected: There is a u-v-path for every $u, v \in V$

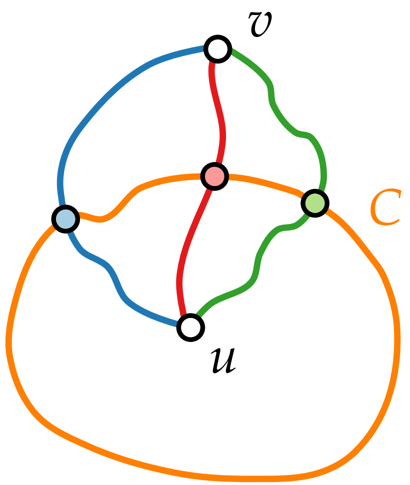
***k*-connected:** $G - \{v_1, \ldots, v_{k-1}\}$ is connected for **any** $v_1 \ldots, v_{k-1} \in V$
or (equivalently)
There are at least k vertex-disjoint u-v-paths for every $u, v \in V$

Theorem. [Whitney 1933]
Every 3-connected planar graph has a unique planar embedding.

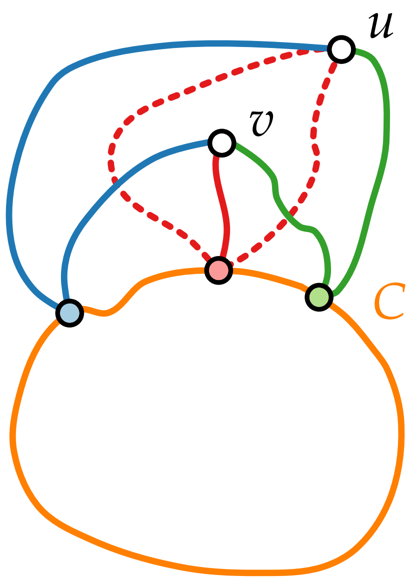
Proof sketch.
Γ_1, Γ_2 embeddings of G
C face of Γ_2, but not Γ_1
u inside C in Γ_1, v outside C in Γ_1
both on same side in Γ_2



Γ_1

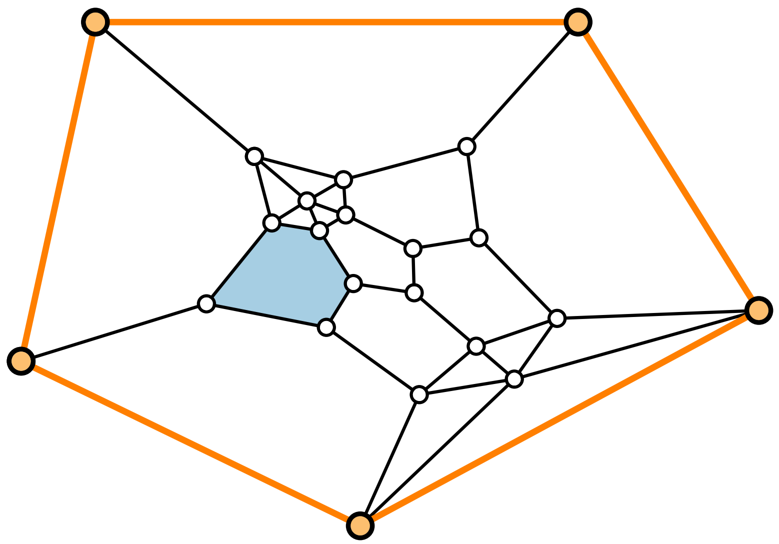


Γ_2



Tutte's Theorem

Theorem. [Tutte 1963]
Let G be a 3-connected planar graph, and let C be a face of its unique embedding.
If we fix C on a strictly convex polygon, then the Tutte drawing of G is planar and all its faces are strictly convex.



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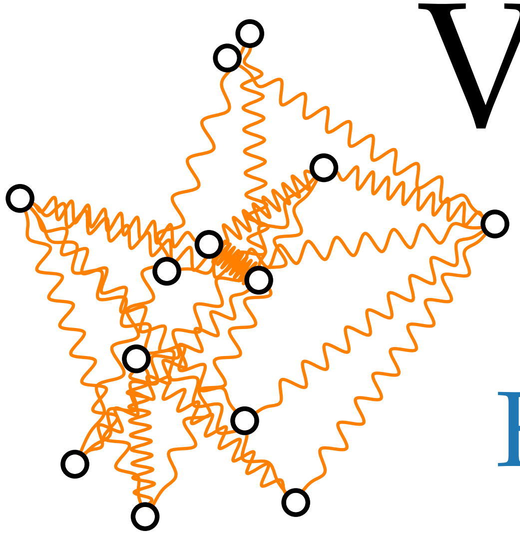
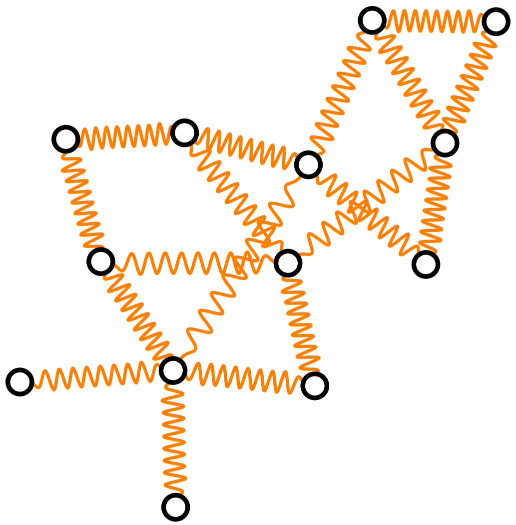


Visualization of Graphs

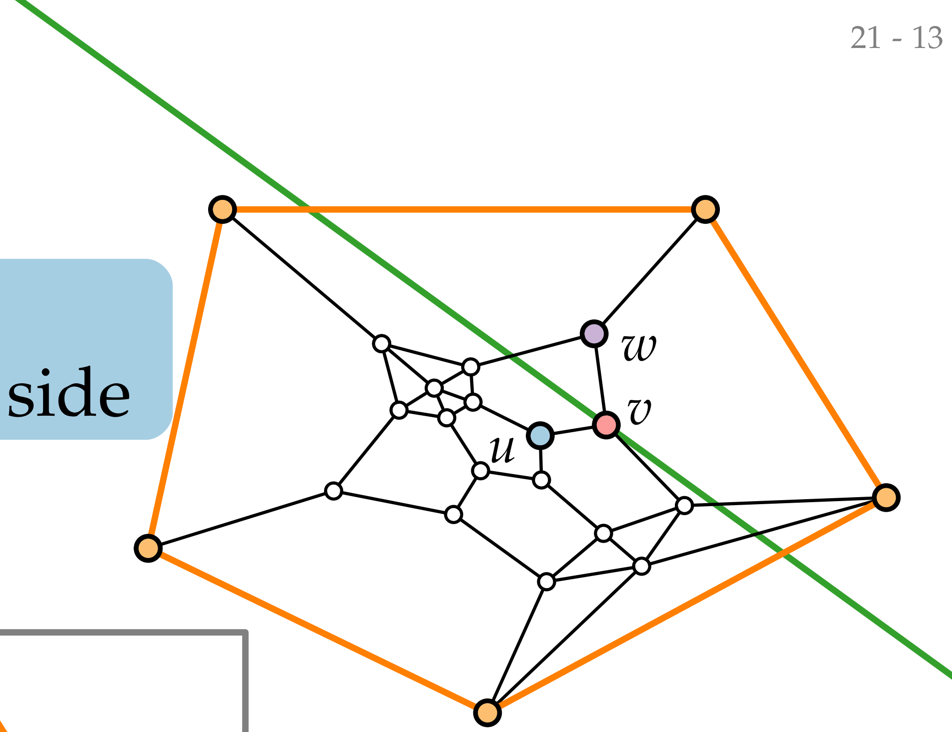
Lecture 3:
Force-Directed Drawing Algorithms



Part VI:
Proof of Tutte's Theorem

Philipp Kindermann

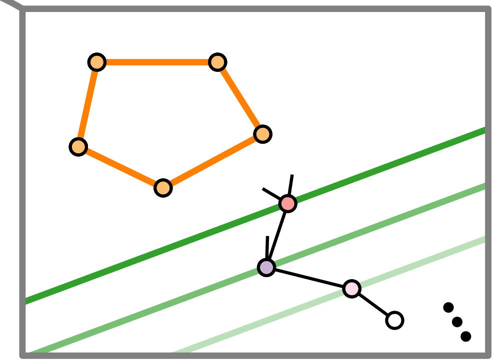
Properties of Tutte Drawings



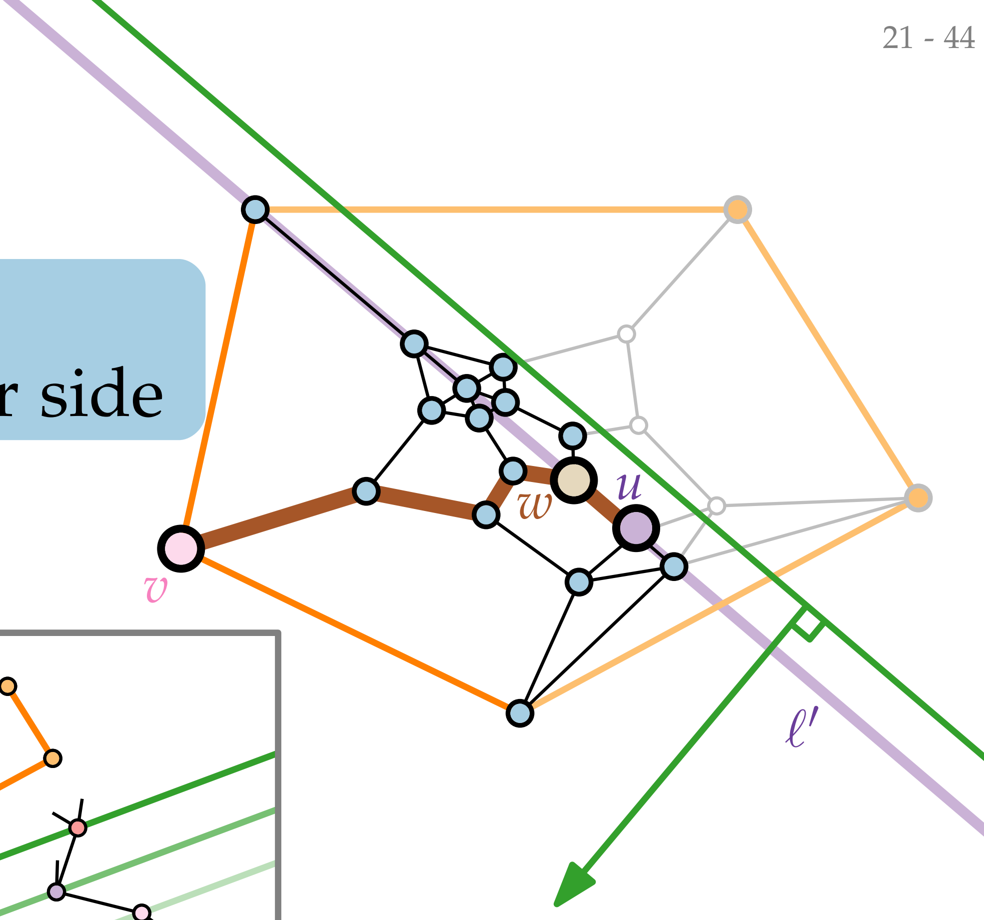
Property 1. Let $v \in V$ free, ℓ line through v.
$\exists uv \in E$ on one side of $\ell \Rightarrow \exists vw \in E$ on other side

Otherwise, all forces to same side . . .

Property 2. All free vertices lie inside C.



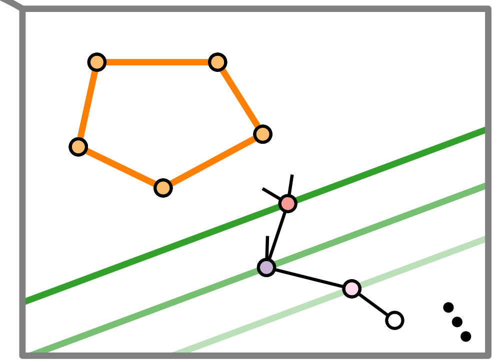
Properties of Tutte Drawings



Property 1. Let $v \in V$ free, ℓ line through v.
$\exists uv \in E$ on one side of $\ell \Rightarrow \exists vw \in E$ on other side

Otherwise, all forces to same side …

Property 2. All free vertices lie inside C.

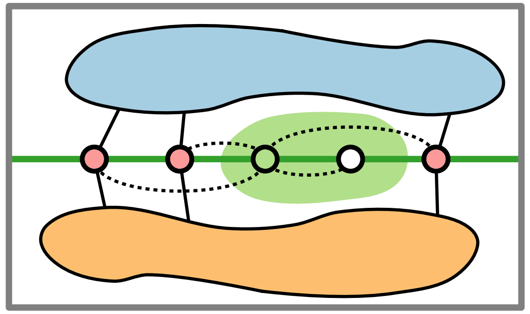


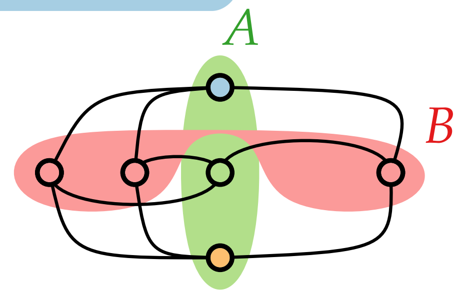
Property 3. Let ℓ be any line.
Let V_ℓ be all vertices on one side of ℓ.
Then $G[V_\ell]$ is connected.

v furthest away from ℓ
Pick any vertex u, ℓ' parallel to ℓ throught u
G connected, v not on $\ell' \Rightarrow \exists w$ on ℓ' with neighbor further away from ℓ
$\Rightarrow \exists$ path from u to v

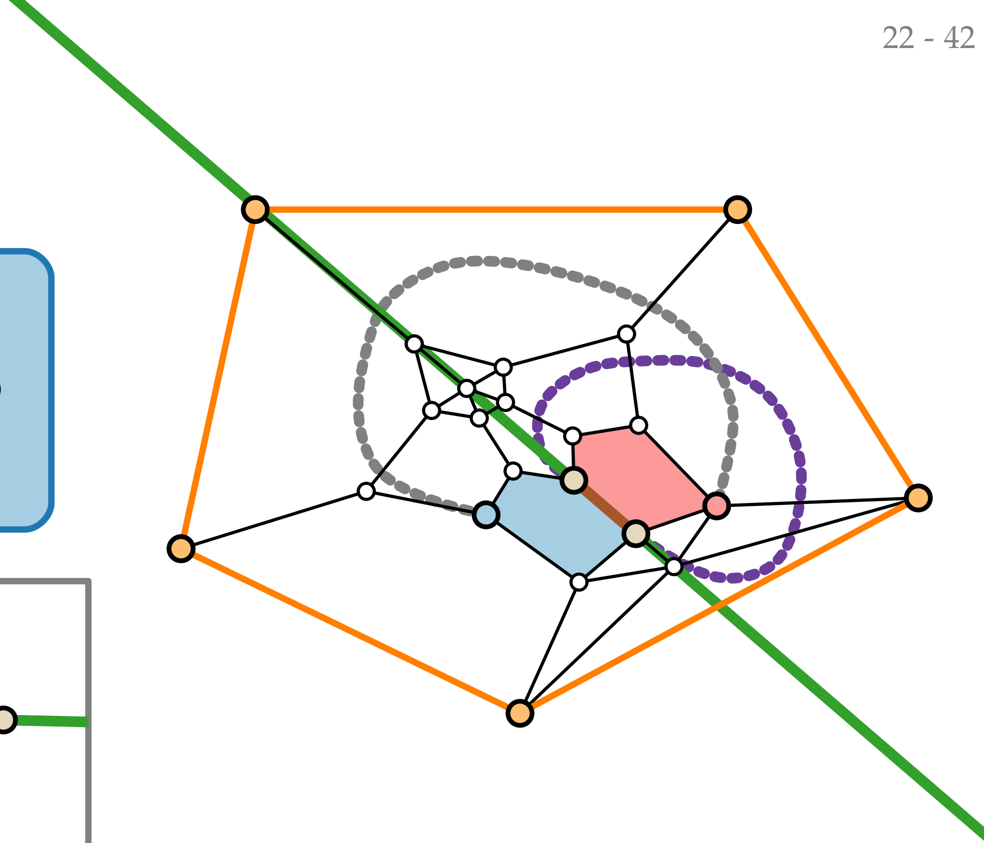
Property 4. No vertex is collinear with all of its neighbors.

Not all vertices collinear
G 3-connected
$\Rightarrow K_{3,3}$ minor





Proof of Tutte's Theorem

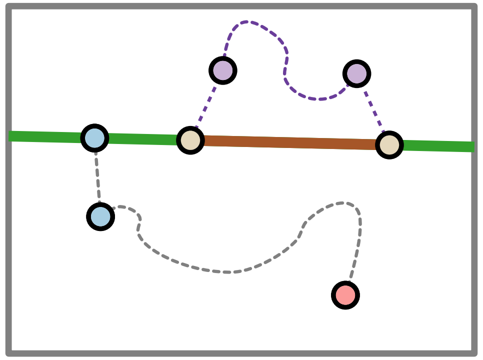


Lemma. Let $uv \in E$ be a non-boundary edge, ℓ line through uv. Then the two faces f_1, f_2 incident to uv lie completely on opposite sides of ℓ.

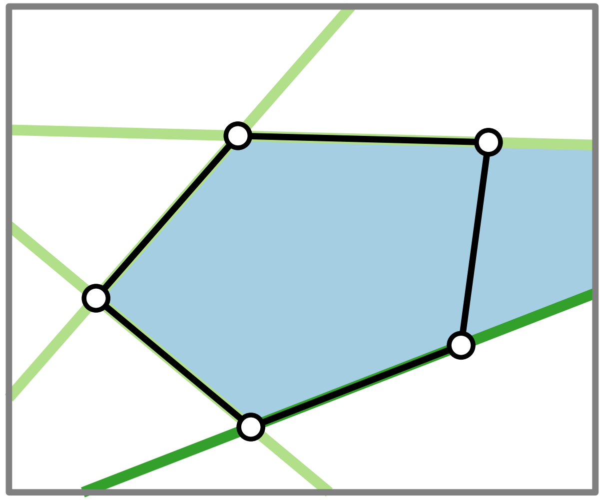
Property 1. Let $v \in V$ free, ℓ line through v.
$\exists uv \in E$ on one side of $\ell \Rightarrow \exists vw \in E$ on other side

Property 3. Let ℓ be any line.
Let V_ℓ be all vertices on one side of ℓ.
Then $G[V_\ell]$ is connected.

Property 4. No vertex is collinear with all of its neighbors.



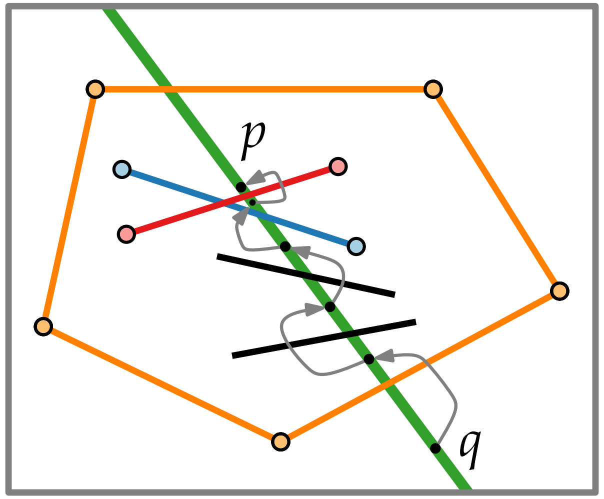
Lemma. All faces are strictly convex.



Lemma. The drawing is planar.

Property 2. All free vertices lie inside C.

p inside two faces
$\Rightarrow q$ in one face
jumping over edge
$\rightarrow$ #faces the same
$\Rightarrow p$ inside one face ↯



Literature

Main sources:

- [GD Ch. 10] Force-Directed Methods
- [DG Ch. 4] Drawing on Physical Analogies

Referenced papers:

- [Johnson 1982] The NP-completeness column: An ongoing guide
- [Eades, Wormald 1990] Fixed edge-length graph drawing is NP-hard
- [Saxe 1980] Two papers on graph embedding problems
- [Eades 1984] A heuristic for graph drawing
- [Fruchterman, Reingold 1991] Graph drawing by force-directed placement
- [Frick, Ludwig, Mehldau 1994] A fast adaptive layout algorithm for undirected graphs
- [Tutte 1963] How to draw a graph