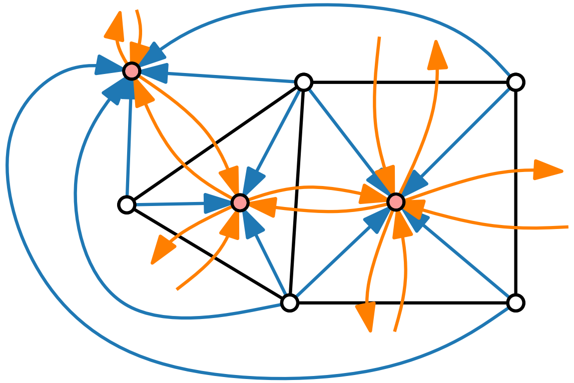
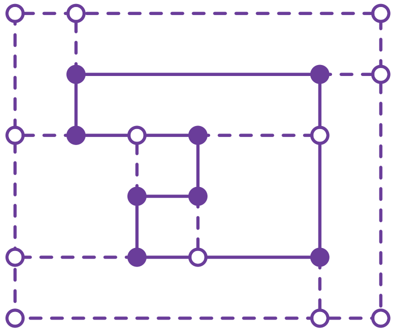
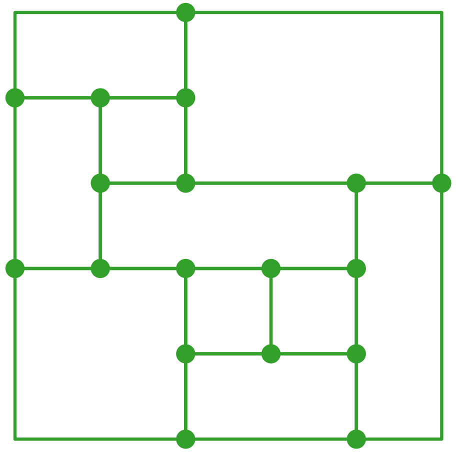
Universität Trier

Visualization of Graphs





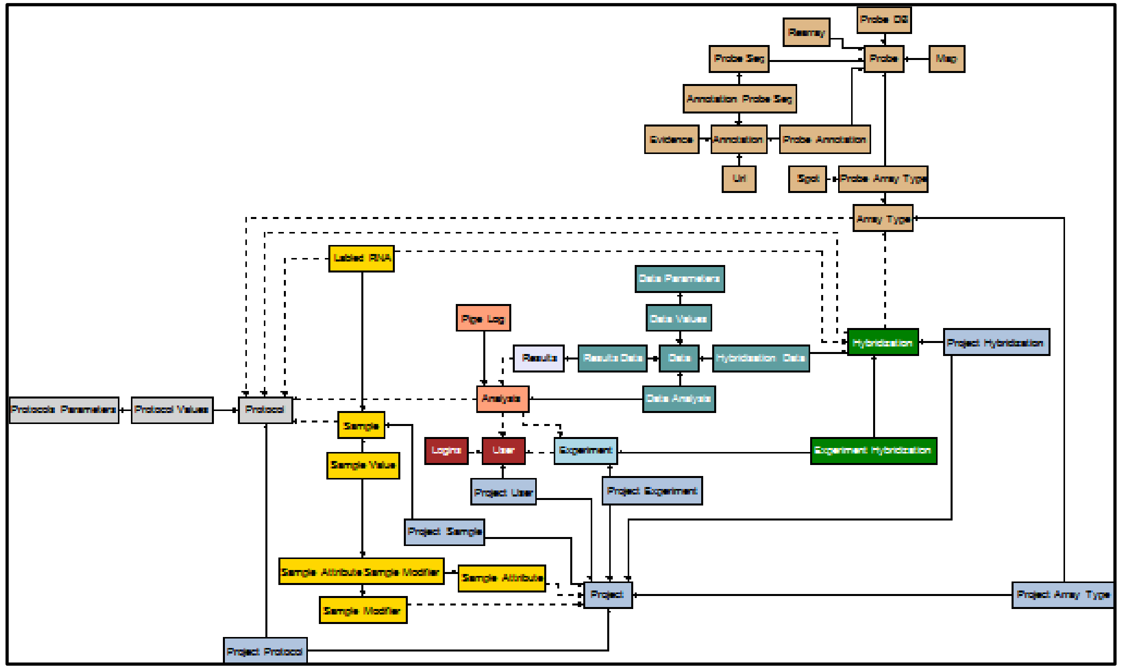
Lecture 6:
Orthogonal Layouts



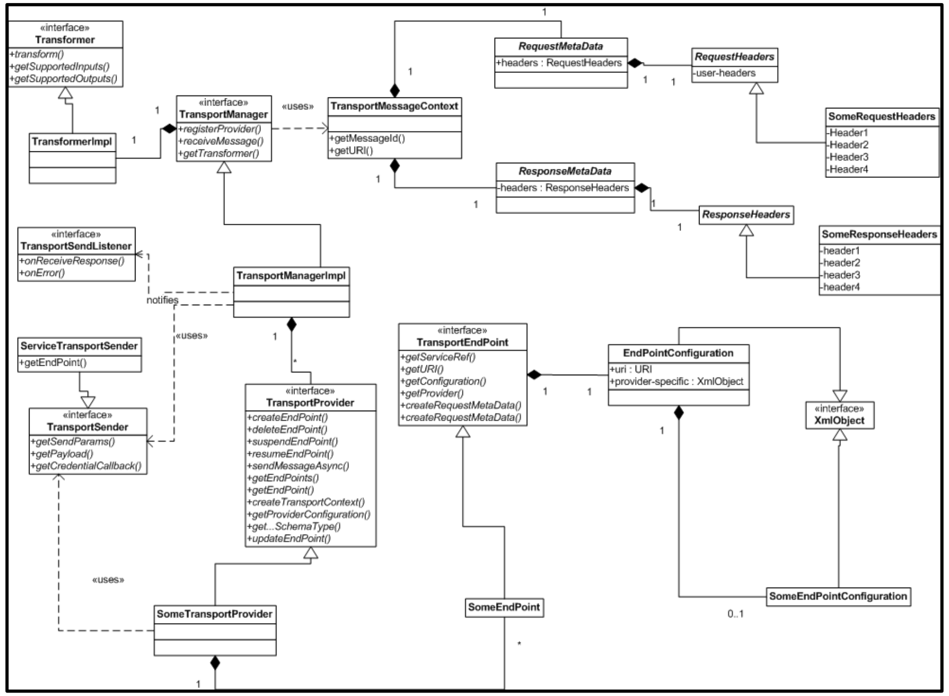
Part I:
Topology – Shape – Metrics

Philipp Kindermann

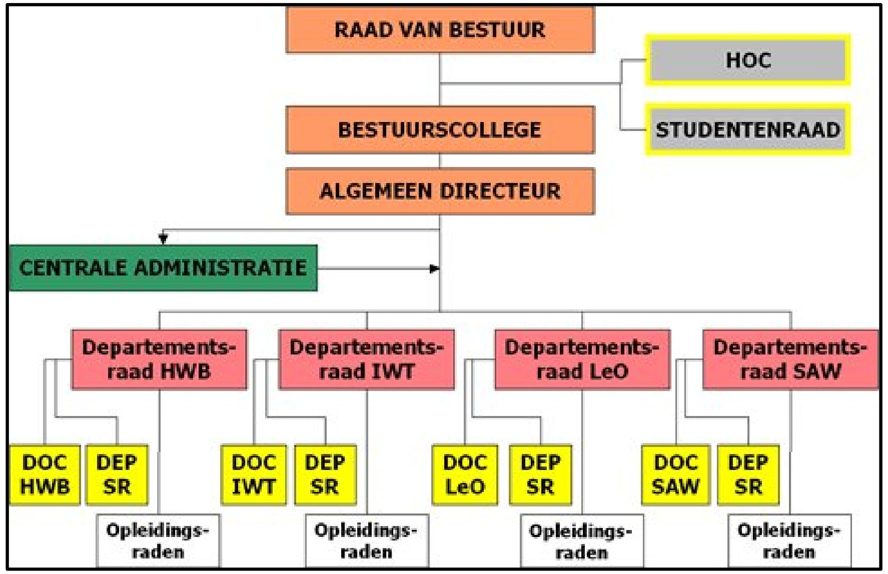
Orthogonal Layout – Applications



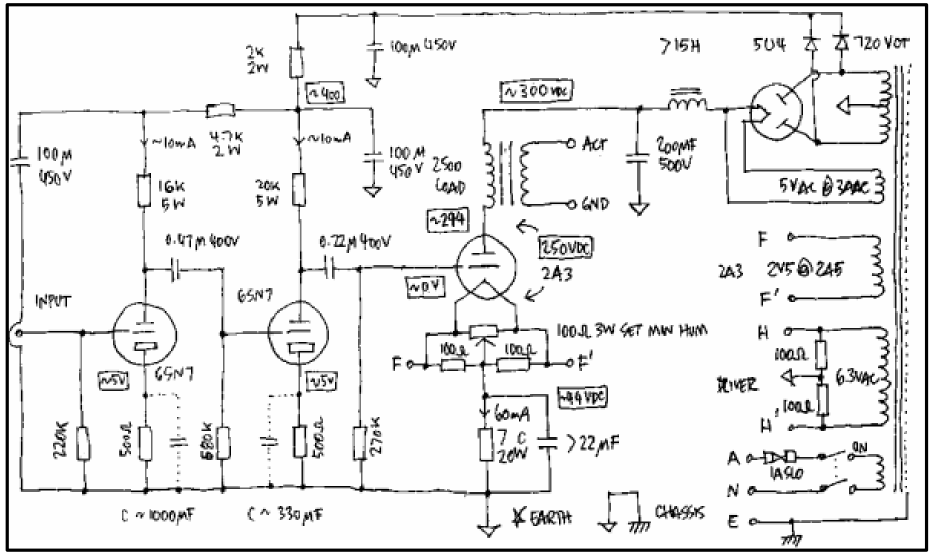
ER diagram in OGDF



UML diagram by Oracle

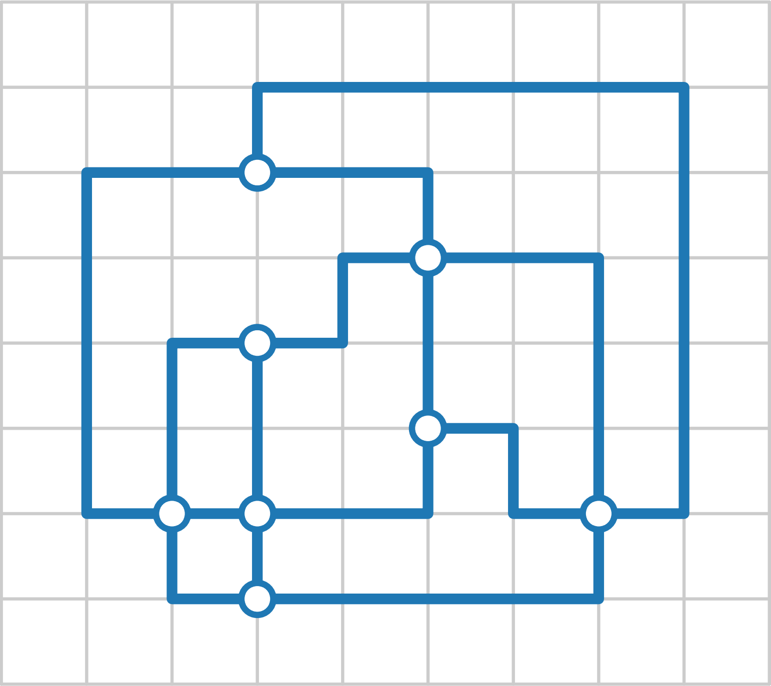


Organigram of HS Limburg



Circuit diagram by Jeff Atwood

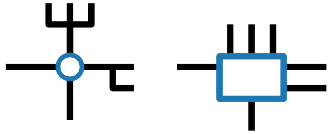
Orthogonal Layout – Definition



Definition.
A drawing Γ of a graph $G = (V, E)$ is called **orthogonal** if
- vertices are drawn as points on a grid,
- each edge is represented as a sequence of alternating horizontal and vertical segments, and
- pairs of edges are disjoint or cross orthogonally.

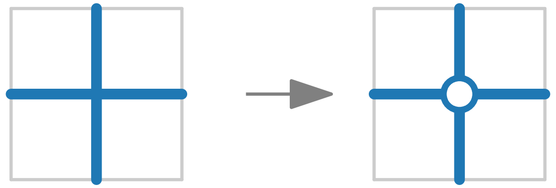
Observations.

- Edges lie on grid $\Rightarrow$ **bends** lie on grid points
- Max degree of each vertex is at most 4
- Otherwise



Planarization.

- Fix embedding
- Crossings become vertices

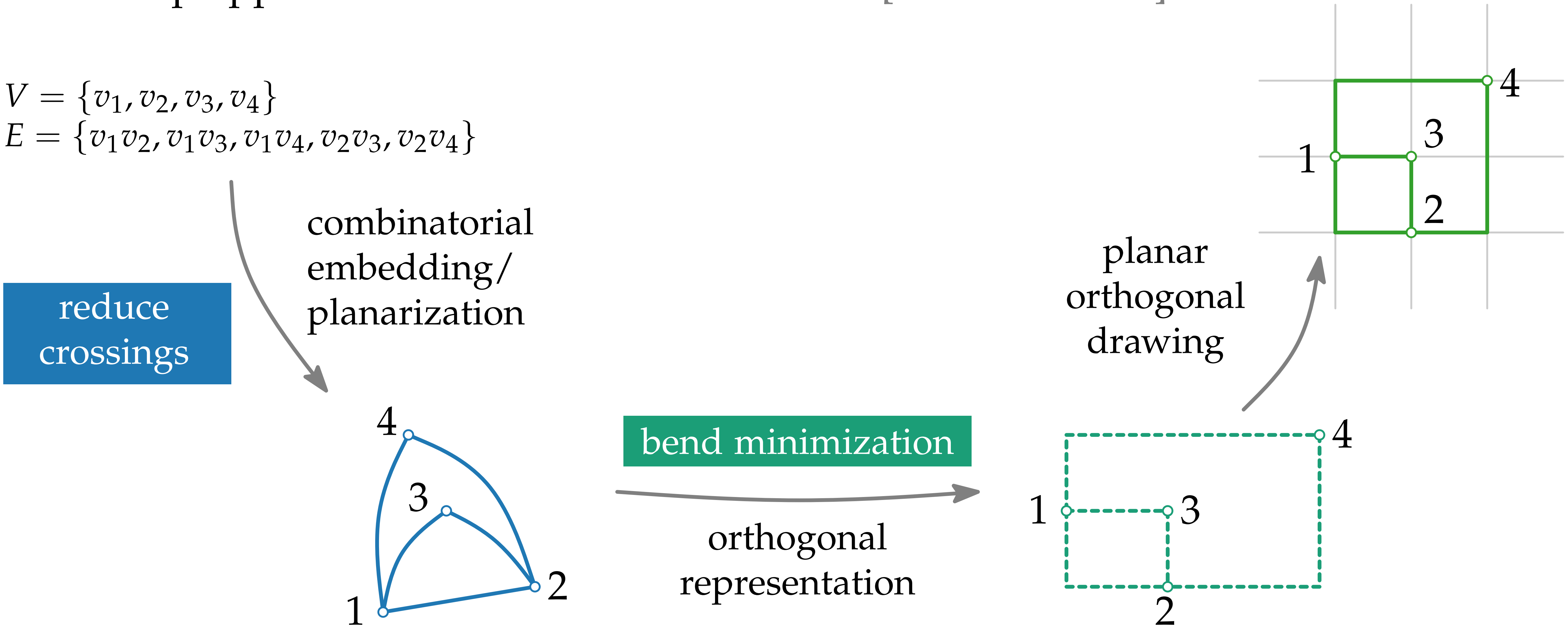


Aesthetic criteria.

- Number of bends
- Length of edges
- Width, height, area
- Monotonicity of edges
- ...

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]



$V = \{v_1, v_2, v_3, v_4\}$
$E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$

combinatorial embedding/ planarization

reduce crossings

bend minimization

orthogonal representation

planar orthogonal drawing

Topology — Shape — Metrics

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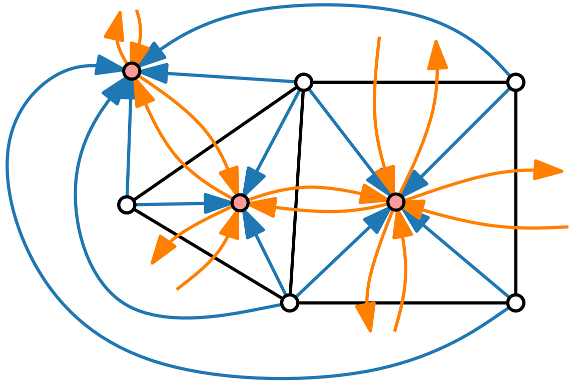
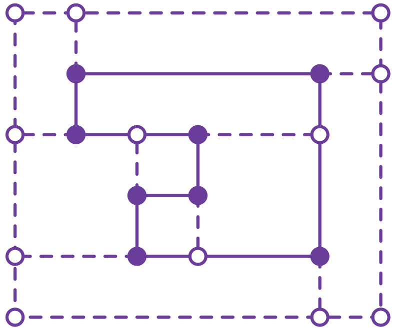
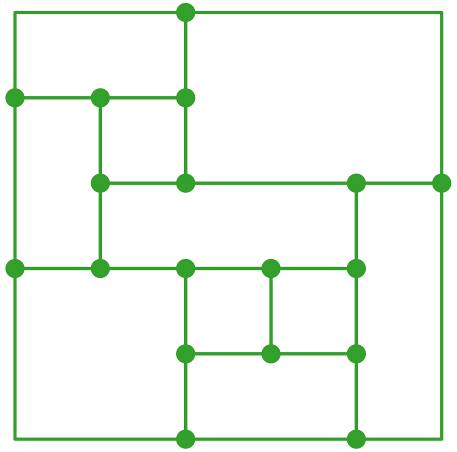
Visualization of Graphs



Lecture 6:
Orthogonal Layouts





Part II:
Orthogonal Representation

Philipp Kindermann

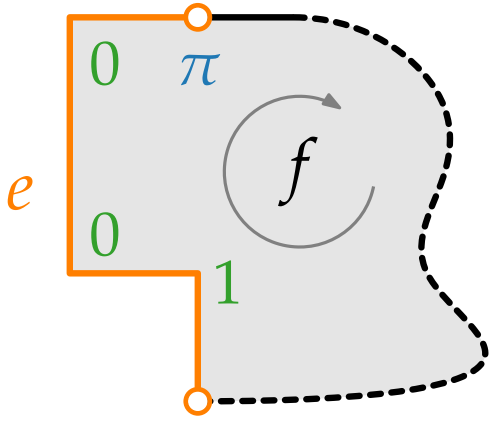
Orthogonal Representation

Idea.
Describe orthogonal drawing combinatorically.

Definitions.
Let $G = (V, E)$ be a plane graph with faces F and outer face f_0.

- Let e be an edge with the face f to the right.
  An **edge description** of e wrt f is a triple (e, δ, α) where
  - δ is a sequence of $\{0,1\}^*$ (0 = right bend, 1 = left bend)
  - α is angle $\in \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ between e and next edge e'
- A **face representation** $H(f)$ of f is a clockwise ordered sequence of edge descriptions (e, δ, α).
- An **orthogonal representation** $H(G)$ of G is defined as

$$H(G) = \{H(f) \mid f \in F\}.$$



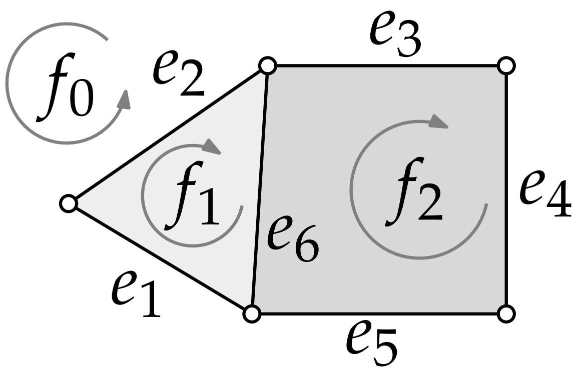
$(e, 100, \pi)$

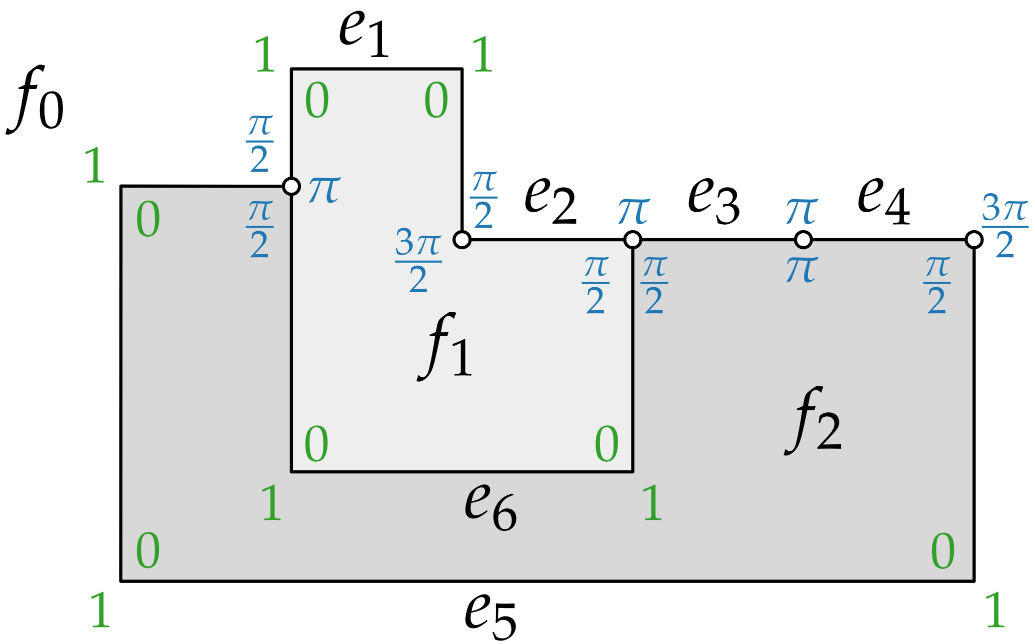
Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$

$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$

$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Concrete coordinates are not fixed yet!

Correctness of an Orthogonal Representation

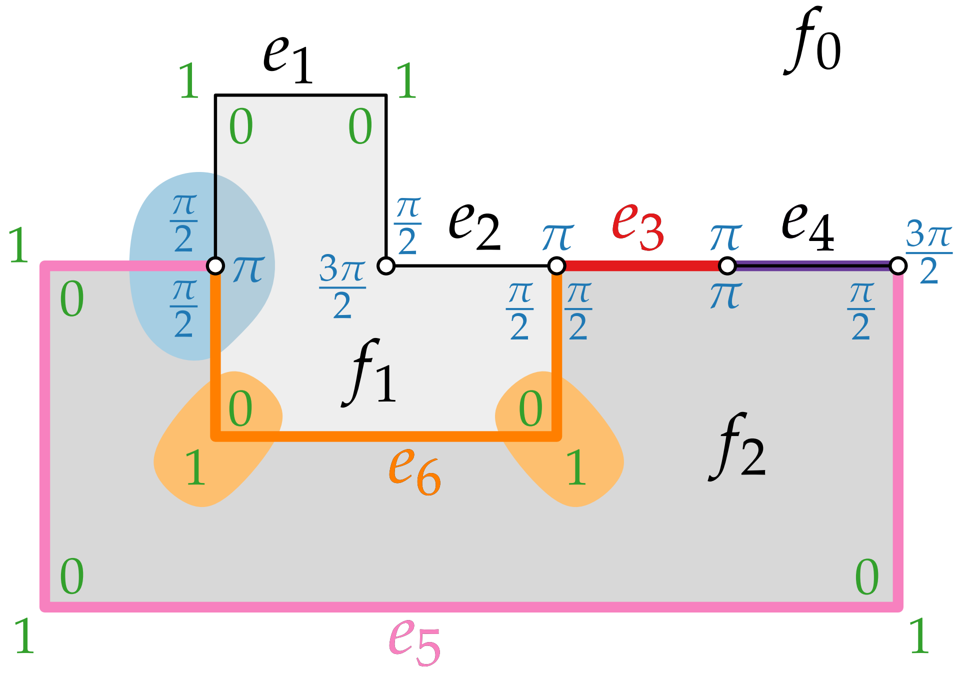
(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g with $((u,v), \delta_1, \alpha_1) \in H(f)$ and $((v,u), \delta_2, \alpha_2) \in H(g)$ sequence δ_1 is reversed and inverted δ_2.

(H3) Let $|\delta|_0$ (resp. $|\delta|_1$) be the number of zeros (resp. ones) in δ and $r = (e, \delta, \alpha)$.
Let $C(r) := |\delta|_0 - |\delta|_1 + 2 - \alpha \cdot 2/\pi$.
For each **face** f it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex** v the sum of incident angles is 2π.



$$C(e_3) = 0 - 0 + 2 - \pi \cdot \frac{2}{\pi} = 0$$

$$C(e_4) = 0 - 0 + 2 - \frac{\pi}{2} \cdot \frac{2}{\pi} = 1$$

$$C(e_5) = 3 - 0 + 2 - \frac{\pi}{2} \cdot \frac{2}{\pi} = 4$$

$$C(e_6) = 0 - 2 + 2 - \frac{\pi}{2} \cdot \frac{2}{\pi} = -1$$

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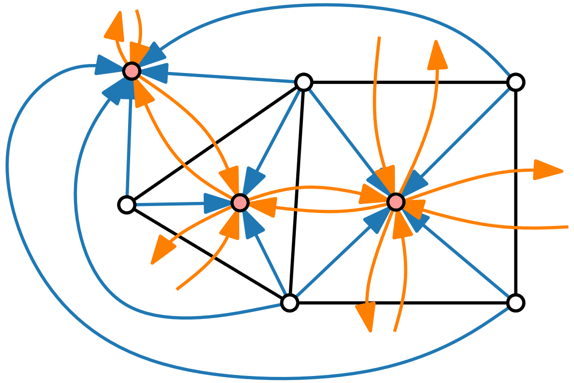
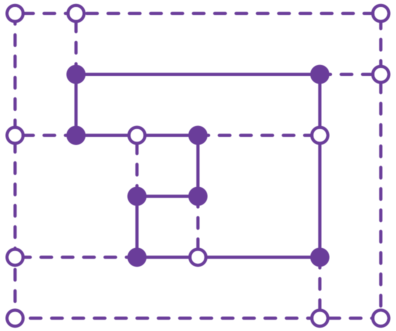
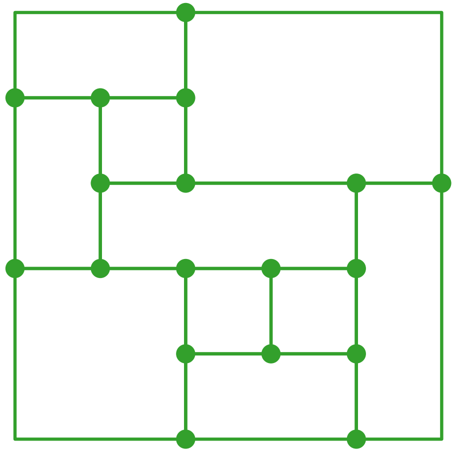
Visualization of Graphs





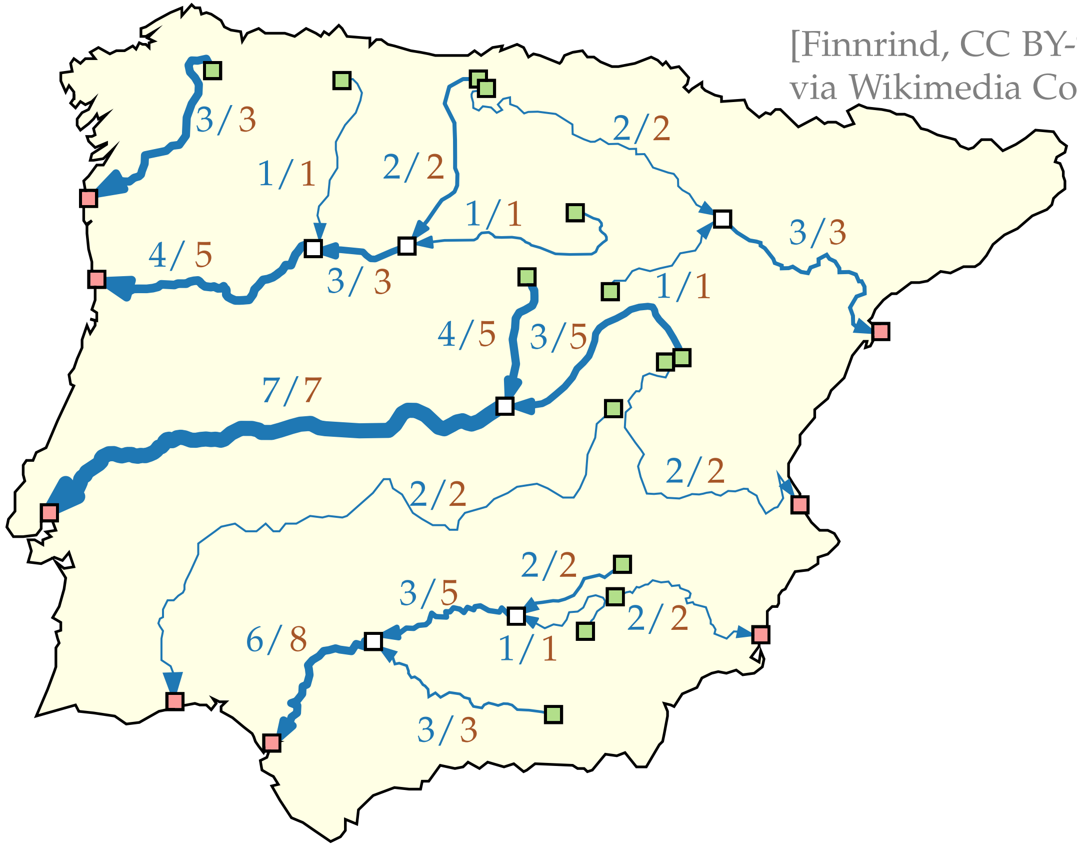
Lecture 6:
Orthogonal Layouts



Part III:
Flow Networks

Philipp Kindermann

Flow Networks



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Flow network $(G = (V, E); S, T; u)$ with

- directed graph $G = (V, E)$
- *sources* $S \subseteq V$, *sinks* $T \subseteq V$
- edge *capacity* $u \colon E \to \mathbb{R}_0^+$

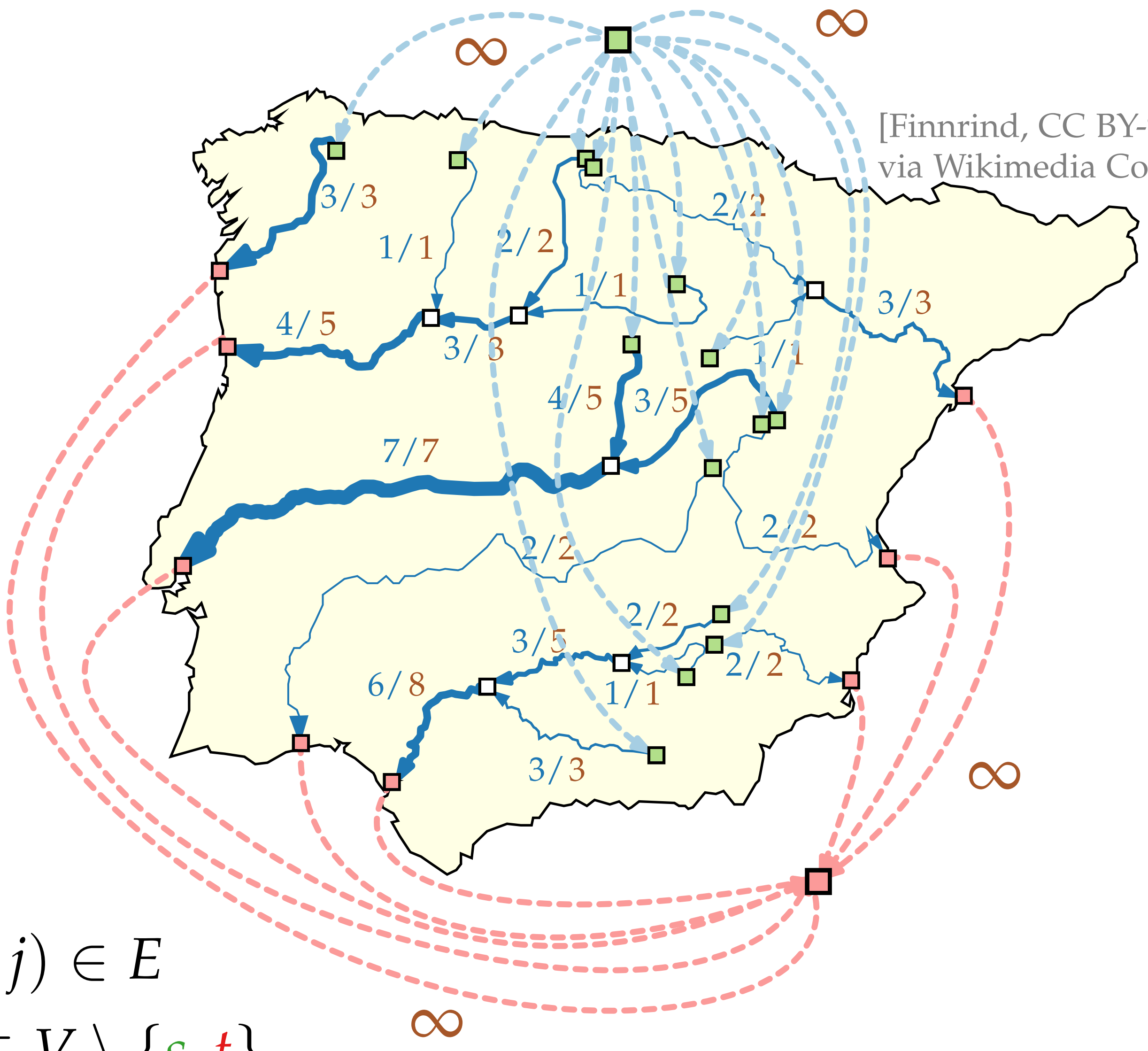
A function $X \colon E \to \mathbb{R}_0^+$ is called ***S*-*T*-flow**, if:

$$0 \le X(i, j) \le u(i, j) \qquad \forall (i, j) \in E$$

$$\sum_{(i,j) \in E} X(i, j) - \sum_{(j,i) \in E} X(j, i) = 0 \qquad \forall i \in V \setminus (S \cup T)$$

A **maximum** S-T-flow is an S-T-flow where $\sum_{(i,j) \in E, i \in S} X(i, j)$ is maximized.

s-*t*-Flow Networks



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Flow network $(G = (V, E); s, t; u)$ with

- directed graph $G = (V, E)$
- *source* $s \in V$, *sink* $t \in V$
- edge *capacity* $u \colon E \to \mathbb{R}_0^+$

A function $X \colon E \to \mathbb{R}_0^+$ is called ***s*-*t*-flow**, if:

$$0 \le X(i,j) \le u(i,j) \qquad \forall (i,j) \in E$$

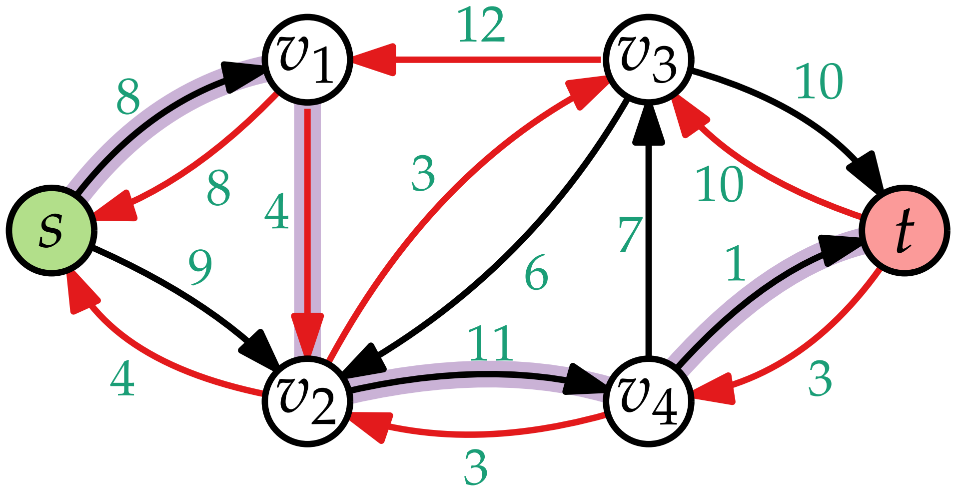
$$\sum_{(i,j)\in E} X(i,j) - \sum_{(j,i)\in E} X(j,i) = 0 \qquad \forall i \in V \setminus \{s,t\}$$

A **maximum** *s*-*t*-flow is an *s*-*t*-flow where $\sum_{(s,j)\in E} X(s,j)$ is maximized.

Residual Network

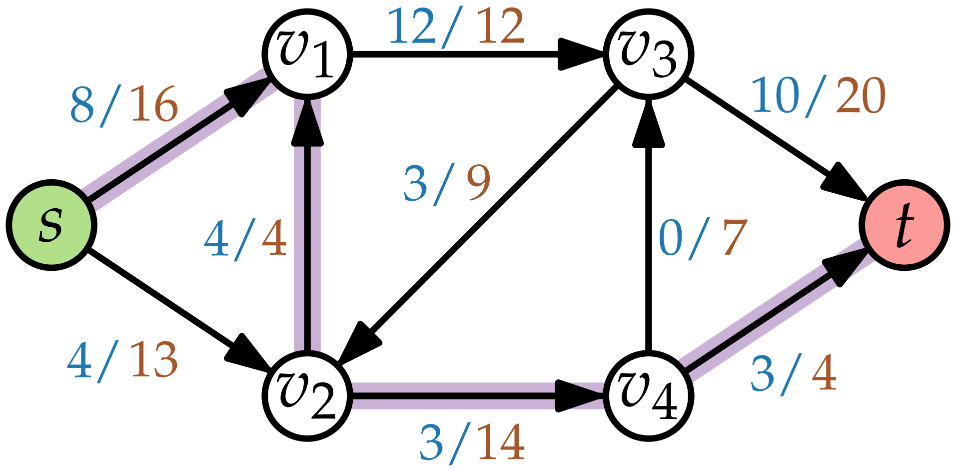
Residual network $G_X = (V, E')$:

- $X(v, v') < u(v, v') \Rightarrow (v, v') \in E'$
  $c(v, v') = u(v, v') - (v, v')$
- $X(v, v') > 0 \Rightarrow (v', v) \in E'$
  $c(v, v') = u(v, v')$



Flow-increasing path W

Flow network $(G = (V, E); s, t; u)$

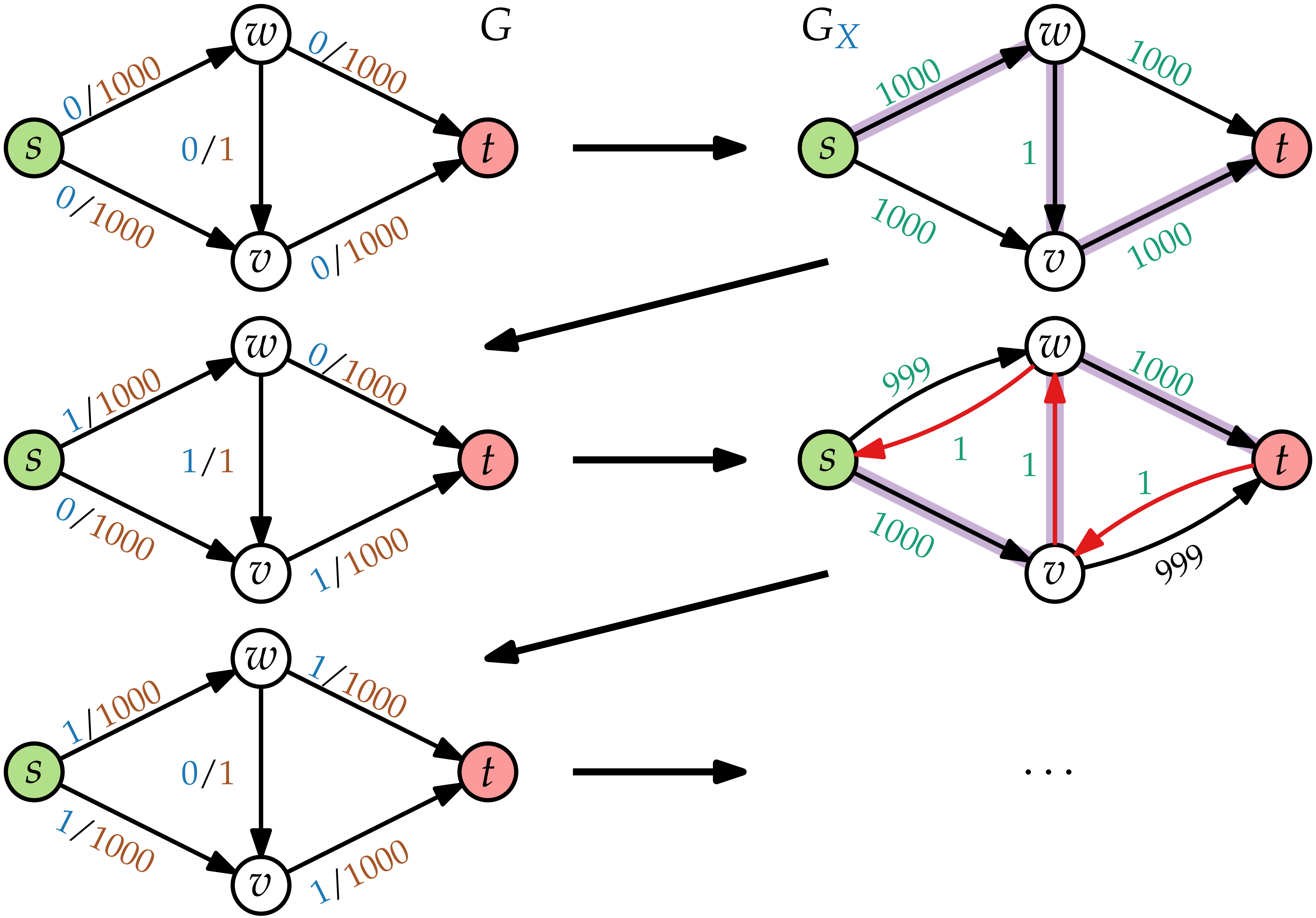


FordFulkerson

```
FordFulkerson($G = (V, E); s, t; u$)
    foreach $(v, v') \in E$ do                                  } Initialization with Zero-flow
        $X(v, v') = 0$
    while $G_X$ contains $s$-$t$-path $W$ do
        $\Delta_W = \min_{(v,v') \in W} c(v, v')$               } Capacity of $W$
        foreach $(v, v') \in W$ do                              } Increasing flow along $W$
            if $(v, v') \in E$ then
                $X(v, v') = X(v, v') + \Delta_W$
            else
                $X(v, v') = X(v, v') - \Delta_W$
    return $X$                                                  } Max Flow
```

FordFulkerson finds a maximum s-t-flow in $O(|X^*| \cdot n)$ time.

FordFulkerson – Example



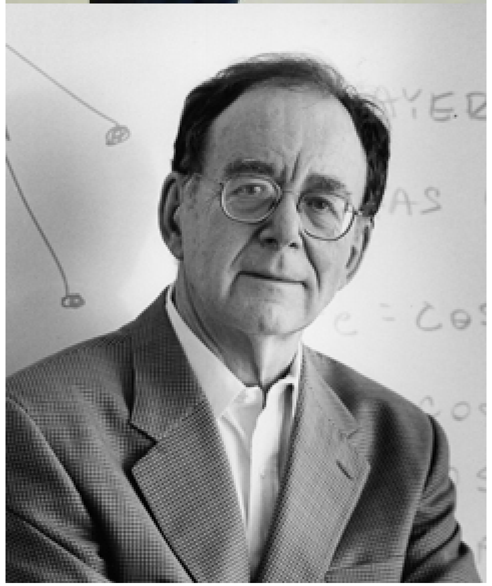
EdmondsKarp

```
EdmondsKarp
~~FordFulkerson~~$(G = (V, E); s, t; u)$
    foreach $(v, v') \in E$ do
        $X(v, v') = 0$
    while $G_X$ contains $s$-$t$-path $W$ do
        $W =$ shortest $s$-$t$-path in $G_X$
        $\Delta_W = \min_{(v,v') \in c(v,v')}$
        foreach $(v, v') \in W$ do
            if $(v, v') \in E$ then
                $X(v, v') = X(v, v') + \Delta_W$
            else
                $X(v, v') = X(v, v') - \Delta_W$
    return $X$
```

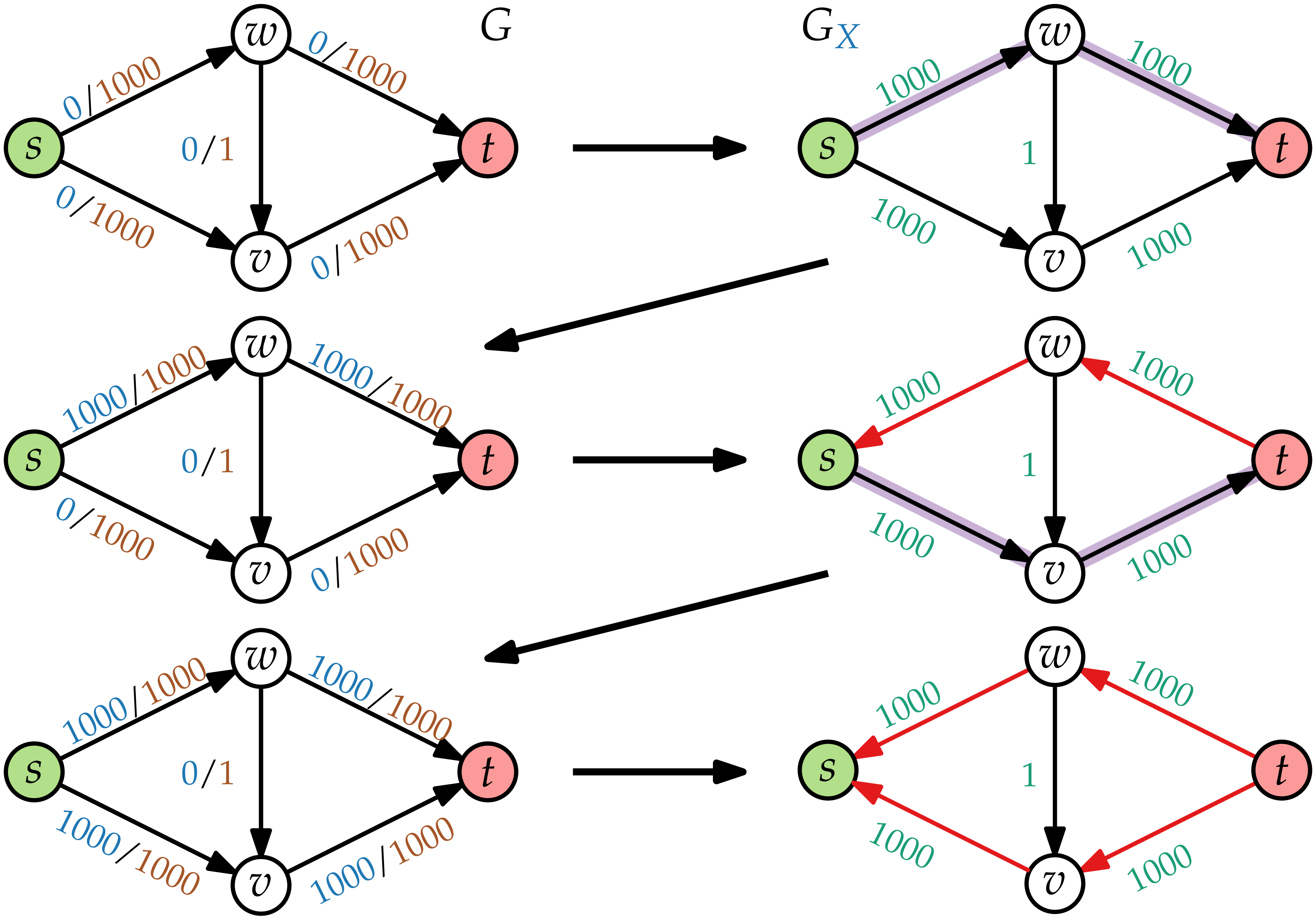
EdmondsKarp finds a maximum s-t-flow in $O(nm^2)$ time.

Jack R. Edmonds
*1934

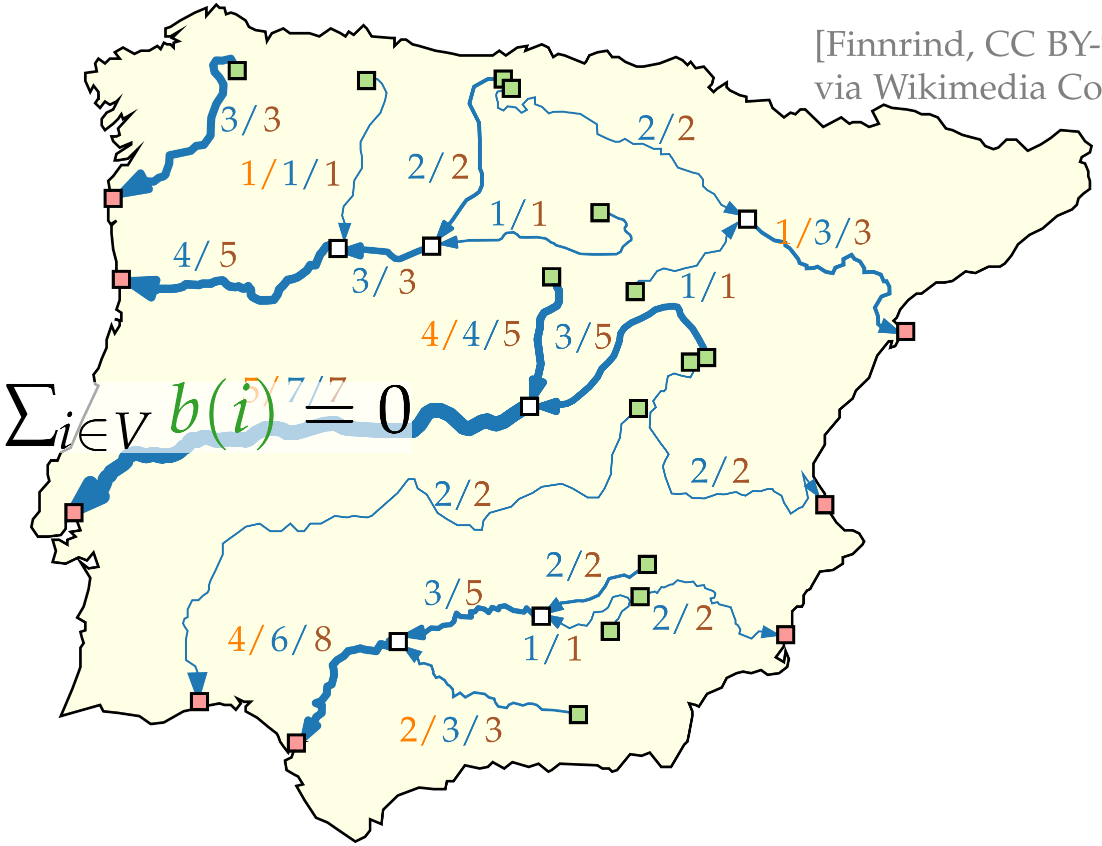
Richard M. Karp
*1935 Boston, MA



EdmondsKarp – Example



General Flow Network



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Flow network $(G = (V, E); b; \ell; u)$ with

- directed graph $G = (V, E)$
- node *production/consumption* $b\colon V \to \mathbb{R}$ with $\sum_{i \in V} b(i) = 0$
- edge *lower bound* $\ell : E \to \mathbb{R}_0^+$
- edge *capacity* $u\colon E \to \mathbb{R}_0^+$

A function $X\colon E \to \mathbb{R}_0^+$ is called **valid flow**, if:

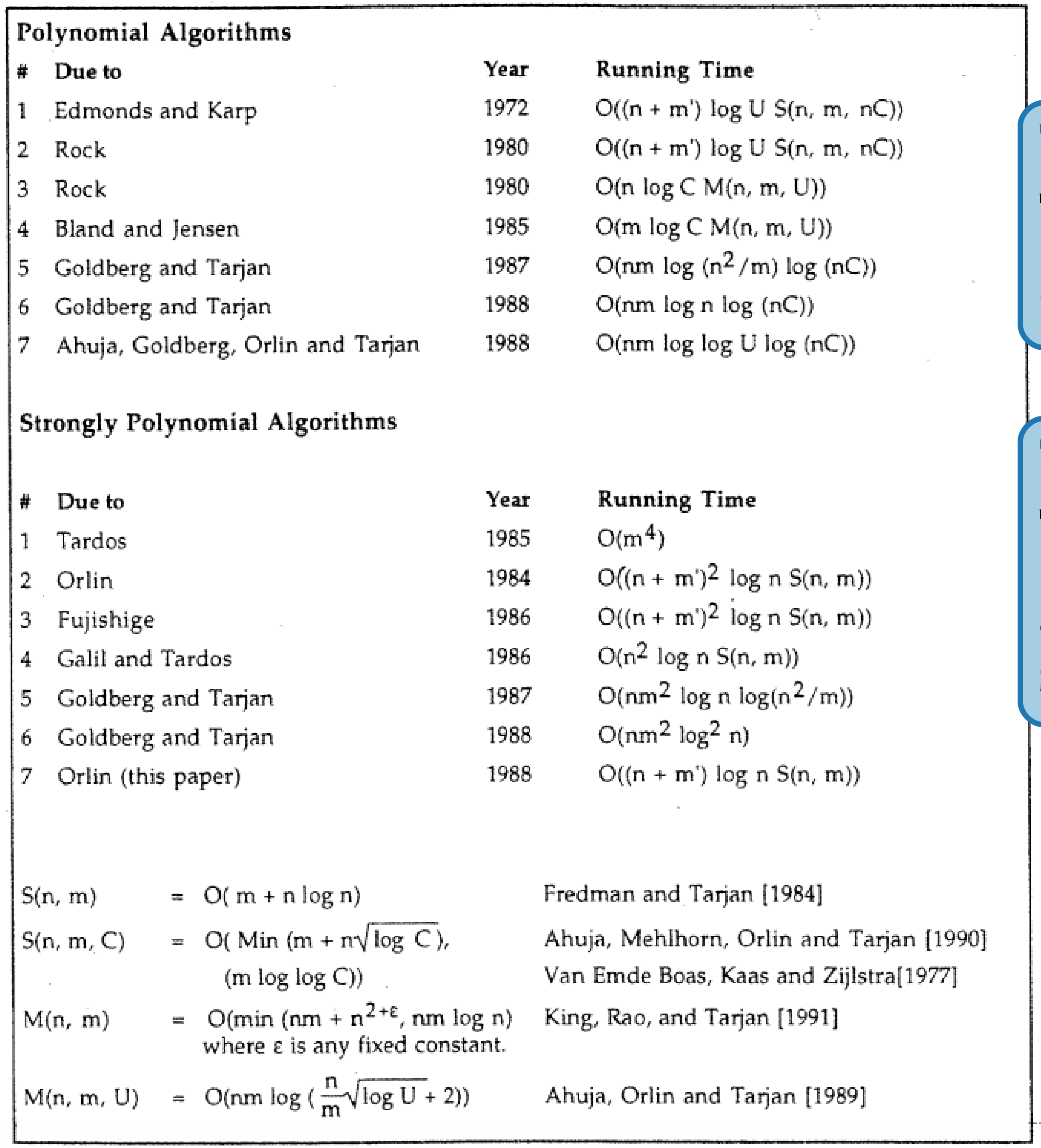
$$\ell(i,j) \leq X(i,j) \leq u(i,j) \qquad \forall (i,j) \in E$$

$$\sum_{(i,j)\in E} X(i,j) - \sum_{(j,i)\in E} X(j,i) = b(i) \qquad \forall i \in V$$

- *Cost function* $\mathrm{cost}\colon E \to \mathbb{R}_0^+$ and $\mathrm{cost}(X) := \sum_{(i,j)\in E} \mathrm{cost}(i,j) \cdot X(i,j)$

A **minimum cost flow** is a valid flow where $\mathrm{cost}(X)$ is minimized.

General Flow Network – Algorithms



Polynomial Algorithms

#	Due to	Year	Running Time
1	Edmonds and Karp	1972	$O((n + m') \log U\ S(n, m, nC))$
2	Rock	1980	$O((n + m') \log U\ S(n, m, nC))$
3	Rock	1980	$O(n \log C\ M(n, m, U))$
4	Bland and Jensen	1985	$O(m \log C\ M(n, m, U))$
5	Goldberg and Tarjan	1987	$O(nm \log (n^2/m) \log (nC))$
6	Goldberg and Tarjan	1988	$O(nm \log n \log (nC))$
7	Ahuja, Goldberg, Orlin and Tarjan	1988	$O(nm \log \log U \log (nC))$

Strongly Polynomial Algorithms

#	Due to	Year	Running Time
1	Tardos	1985	$O(m^4)$
2	Orlin	1984	$O((n + m')^2 \log n\ S(n, m))$
3	Fujishige	1986	$O((n + m')^2 \log n\ S(n, m))$
4	Galil and Tardos	1986	$O(n^2 \log n\ S(n, m))$
5	Goldberg and Tarjan	1987	$O(nm^2 \log n \log(n^2/m))$
6	Goldberg and Tarjan	1988	$O(nm^2 \log^2 n)$
7	Orlin (this paper)	1988	$O((n + m') \log n\ S(n, m))$

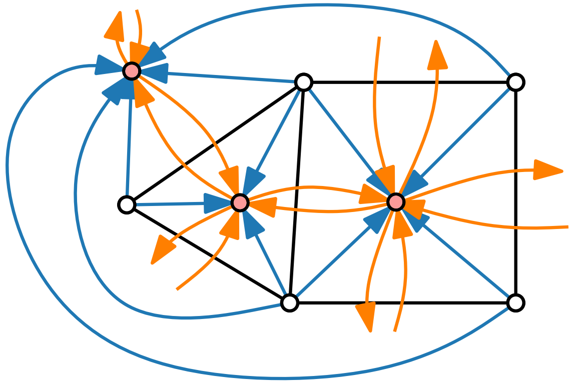
$S(n, m)$	=	$O(m + n \log n)$	Fredman and Tarjan [1984]
$S(n, m, C)$	=	$O(\text{Min } (m + n\sqrt{\log C}),$	Ahuja, Mehlhorn, Orlin and Tarjan [1990]
		$(m \log \log C))$	Van Emde Boas, Kaas and Zijlstra[1977]
$M(n, m)$	=	$O(\min (nm + n^{2+\varepsilon}, nm \log n)$ where ε is any fixed constant.	King, Rao, and Tarjan [1991]
$M(n, m, U)$	=	$O(nm \log (\frac{n}{m}\sqrt{\log U} + 2))$	Ahuja, Orlin and Tarjan [1989]

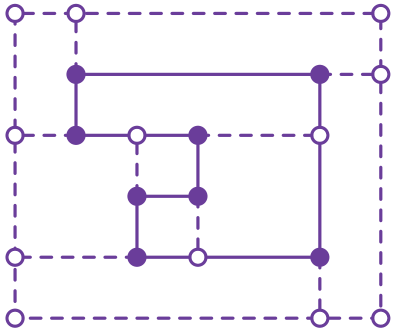
[Orlin 1991]

Theorem. [Orlin 1991]
The minimum cost flow problem can be solved in $O(n^2 \log^2 n + m^2 \log n)$ time.

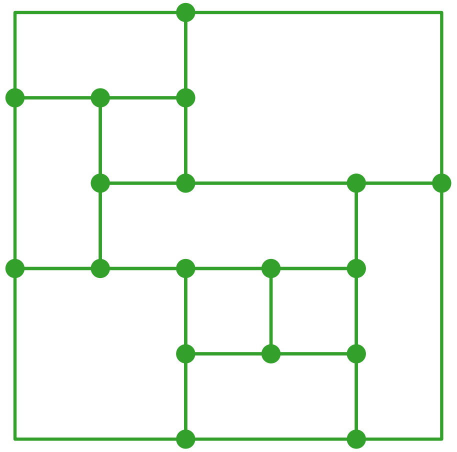
Theorem. [Cornelsen & Karrenbauer 2011]
The minimum cost flow problem for planar graphs with bounded costs and faze sizes can be solved in $O(n^{3/2})$ time.

Visualization of Graphs





Lecture 6:
Orthogonal Layouts



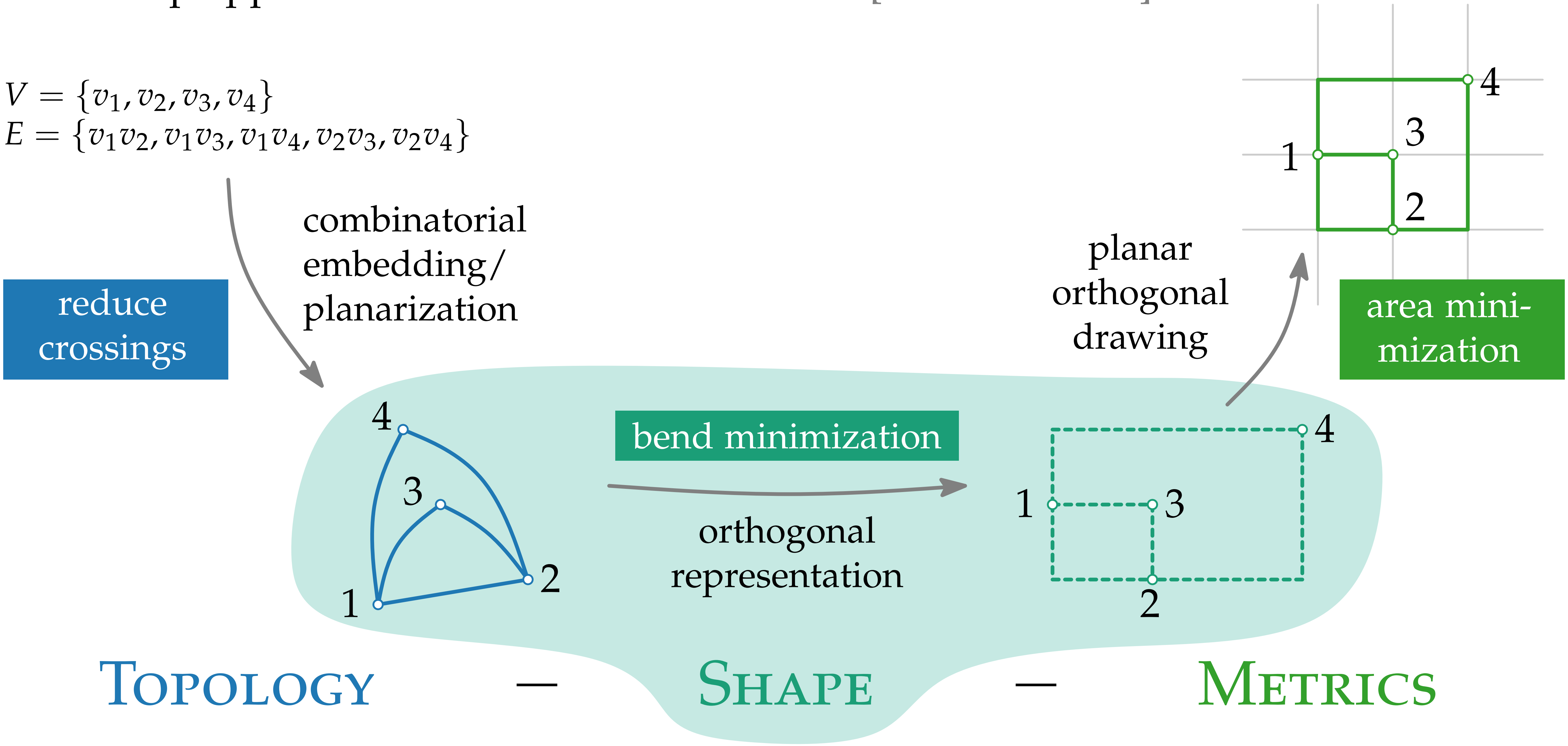
Part IV:
Bend Minimization

Philipp Kindermann

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

$V = \{v_1, v_2, v_3, v_4\}$
$E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$



combinatorial embedding/ planarization

reduce crossings

bend minimization

orthogonal representation

planar orthogonal drawing

area minimization

TOPOLOGY – SHAPE – METRICS

Bend Minimization with Given Embedding

Geometric bend minimization.

Given:
- Plane graph $G = (V, E)$ with maximum degree 4
- Combinatorial embedding F and outer face f_0

Find: Orthogonal drawing with minimum number of bends that preserves the embedding.

Compare with the following variation.

Combinatorial bend minimization.

Given:
- Plane graph $G = (V, E)$ with maximum degree 4
- Combinatorial embedding F and outer face f_0

Find: **Orthogonal representation** $H(G)$ with minimum number of bends that preserves the embedding.

Combinatorial Bend Minimization

Combinatorial bend minimization.

Given:
- Plane graph $G = (V, E)$ with maximum degree 4
- Combinatorial embedding F and outer face f_0

Find: **Orthogonal representation** $H(G)$ with minimum number of bends that preserves the embedding

Idea.

Formulate as a network flow problem:

- a unit of flow $= \measuredangle \frac{\pi}{2}$
- vertices $\xrightarrow{\measuredangle}$ faces (# $\measuredangle \frac{\pi}{2}$ per face)
- faces $\xrightarrow{\measuredangle}$ neighbouring faces (# bends toward the neighbour)

Flow Network for Bend Minimization

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g, sequence δ_1 is reversed and inverted δ_2.

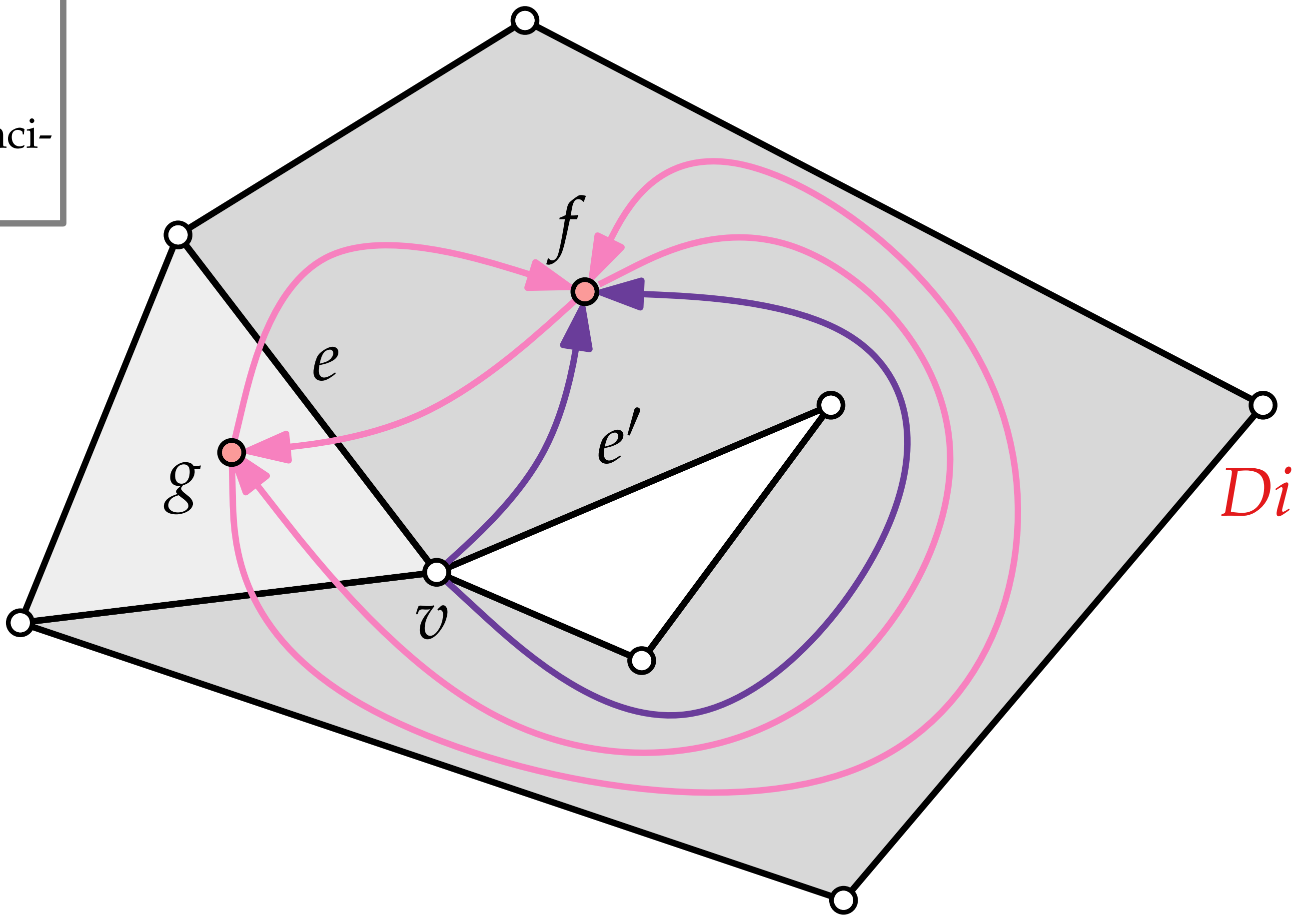
(H3) For each **face** f it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex** v the sum of incident angles is 2π.

Define flow network $N(G) = ((V \cup F, E); b; \ell; u; \text{cost})$:

- $E = \{(v, f)_{ee'} \in V \times F \mid v \text{ between edges } e, e' \text{ of } \partial f\} \cup \{(f, g)_e \in F \times F \mid f, g \text{ have common edge } e\}$



Directed multigraph!

Flow Network for Bend Minimization

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g, sequence δ_1 is reversed and inverted δ_2.

(H3) For each **face** f it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex** v the sum of incident angles is 2π.

Define flow network $N(G) = ((V \cup F, E); b; \ell; u; \text{cost})$:

- $E = \{(v, f)_{ee'} \in V \times F \mid v \text{ between edges } e, e' \text{ of } \partial f\} \cup$
  $\{(f, g)_e \in F \times F \mid f, g \text{ have common edge } e\}$
- $b(v) = 4 \quad \forall v \in V$
- $b(f) = -2\deg_G(f) + \begin{cases} -4 & \text{if } f = f_0, \\ +4 & \text{otherwise} \end{cases}$

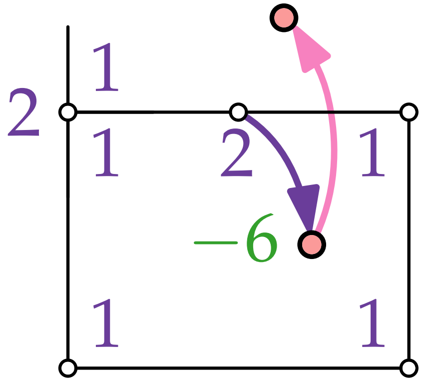
$\Rightarrow \sum_w b(w) = 0$
(Euler)

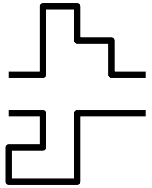
$\forall (v, f) \in E, v \in V, f \in F \qquad \ell(v, f) := 1 \leq \boxed{X(v, f)} \leq 4 =: u(v, f)$

$\text{cost}(v, f) = 0$

$\forall (f, g) \in E, f, g \in F \qquad \ell(f, g) := 0 \leq \boxed{X(f, g)} \leq \infty =: u(f, g)$

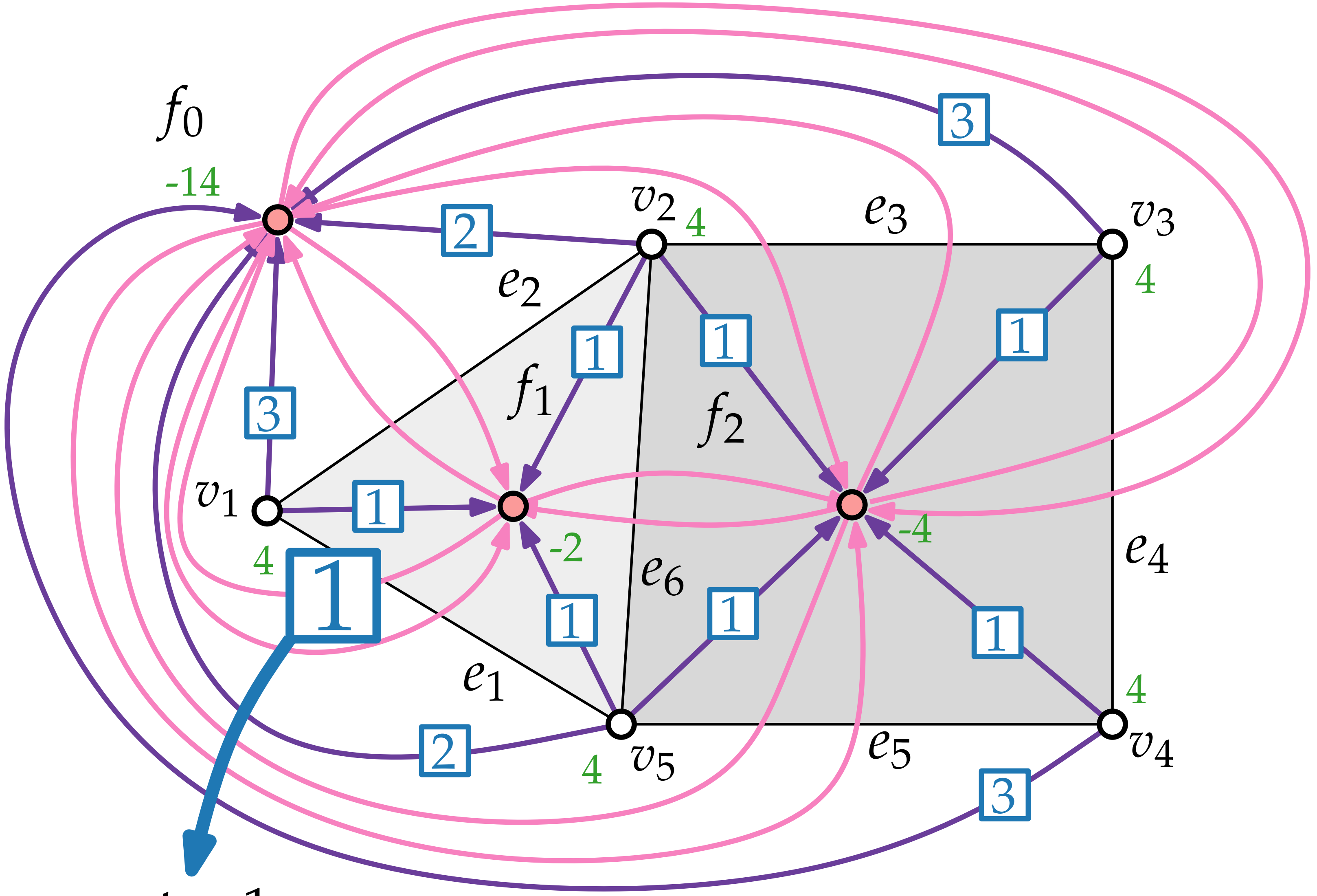
$\text{cost}(f, g) = 1$





We model only the *number* of bends.
Why is it enough?

Flow Network Example



cost = 1
one bend
(outward)

Legend

V ○

F ●

$\ell / u / \mathrm{cost}$

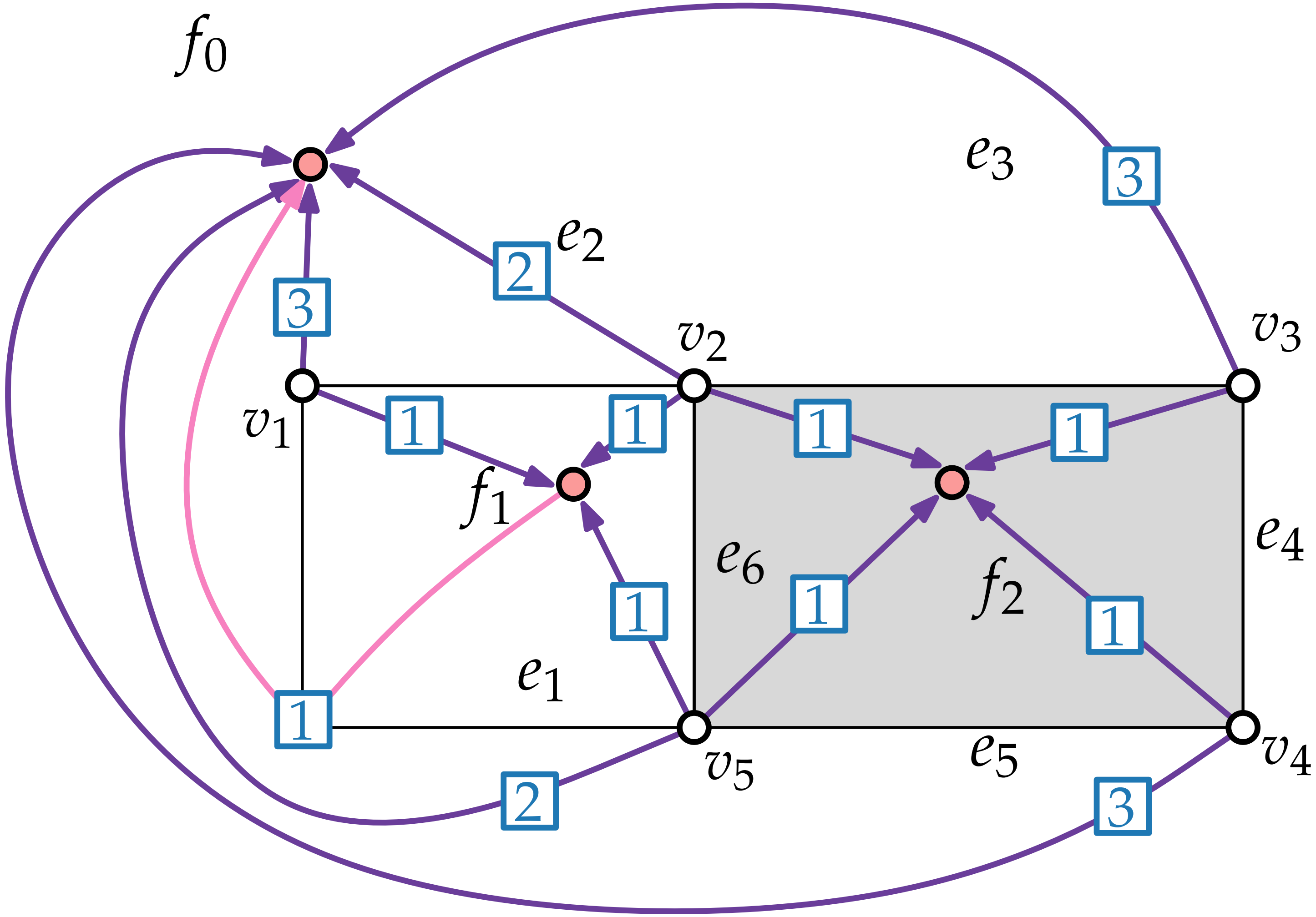
$V \times F \supseteq \xrightarrow{1/4/0}$

$F \times F \supseteq \xrightarrow{0/\infty/1}$

$4 = b$-value

3 flow

Flow Network Example



Legend

V ○

F ●

$\ell/u/\mathrm{cost}$

$V \times F \supseteq \xrightarrow{1/4/0}$

$F \times F \supseteq \xrightarrow{0/\infty/1}$

$4 = b$-value

$\boxed{3}$ flow

Bend Minimization – Result

Theorem. [Tamassia '87]
A plane graph (G, F, f_0) has a valid orthogonal representation $H(G)$ with k bends iff the flow network $N(G)$ has a valid flow X with cost k.

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g, sequence δ_1 is reversed and inverted δ_2.

(H3) For each **face** f it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex** v the sum of incident angles is 2π.

Proof.

$\Leftarrow$ Given valid flow X in $N(G)$ with cost k.
Construct orthogonal representation $H(G)$ with k bends.

- Transform from flow to orthogonal description.
- Show properties (H1)–(H4).

(H1) H(G) matches F, f_0 ✓

(H2) Bend order inverted and reversed on opposite sides ✓

(H3) Angle sum of $f = \pm 4$ ✓ Exercise.

(H4) Total angle at each vertex $= 2\pi$ ✓

Bend Minimization – Result

- $b(v) = 4 \quad \forall v \in V$
- $b(f) = -2\deg_G(f) + \begin{cases} -4 & \text{if } f = f_0, \\ +4 & \text{otherwise} \end{cases}$
- $\ell(v,f) := 1 \leq X(v,f) \leq 4 =: u(v,f)$
  $\text{cost}(v,f) = 0$
  $\ell(f,g) := 0 \leq X(f,g) \leq \infty =: u(f,g)$
  $\text{cost}(f,g) = 1$

Theorem. [Tamassia '87]
A plane graph (G, F, f_0) has a valid orthogonal representation $H(G)$ with k bends iff the flow network $N(G)$ has a valid flow X with cost k.

Proof.

$\Rightarrow$ Given an orthogonal representation $H(G)$ with k bends.
Construct valid flow X in $N(G)$ with cost k.

- Define flow $X \colon E \to \mathbb{R}_0^+$.
- Show that X is a valid flow and has cost k.

(N1) $X(vf) = 1/2/3/4$ ✓

(N2) $X(fg) = |\delta_{fg}|_0$, (e, δ_{fg}, x) describes $e \overset{*}{=} fg$ from f ✓

(N3) capacities, deficit/demand coverage ✓

(N4) cost = k ✓

Bend Minimization – Remarks

- From Theorem follows that the combinatorial orthogonal bend minimization problem for plane graphs can be solved using an algorithm for the Min-Cost-Flow problem.

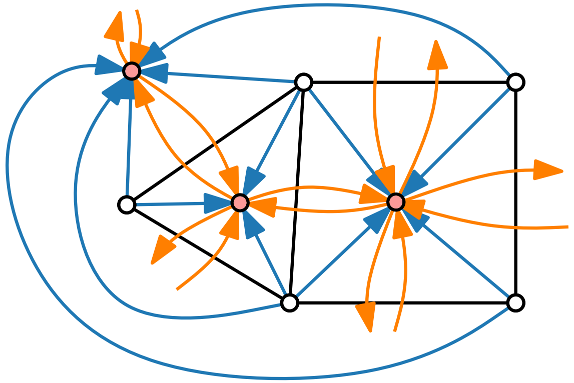
Theorem. [Garg & Tamassia 1996]
The minimum cost flow problem for planar graphs with bounded costs and vertex degrees can be solved in $O(n^{7/4}\sqrt{\log n})$ time.

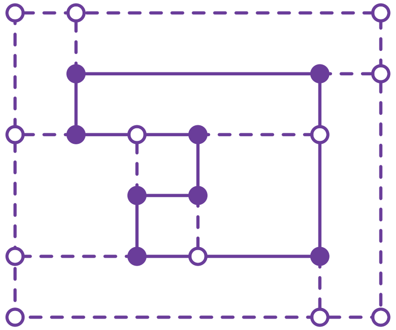
Theorem. [Cornelsen & Karrenbauer 2011]
The minimum cost flow problem for planar graphs with bounded costs and faze sizes can be solved in $O(n^{3/2})$ time.

Theorem. [Garg & Tamassia 2001]
Bend Minimization without a given combinatorial embedding is an NP-hard problem.

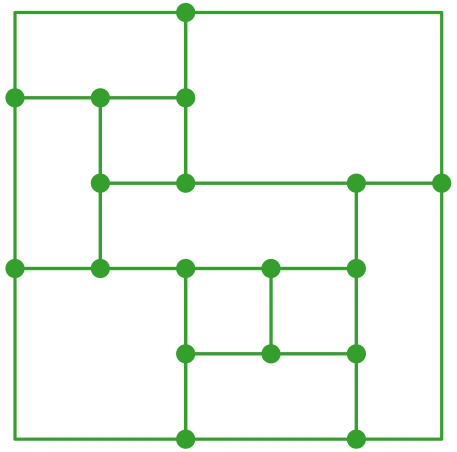
Universität Trier

Visualization of Graphs





Lecture 6:
Orthogonal Layouts



Part V:
Area Minimization

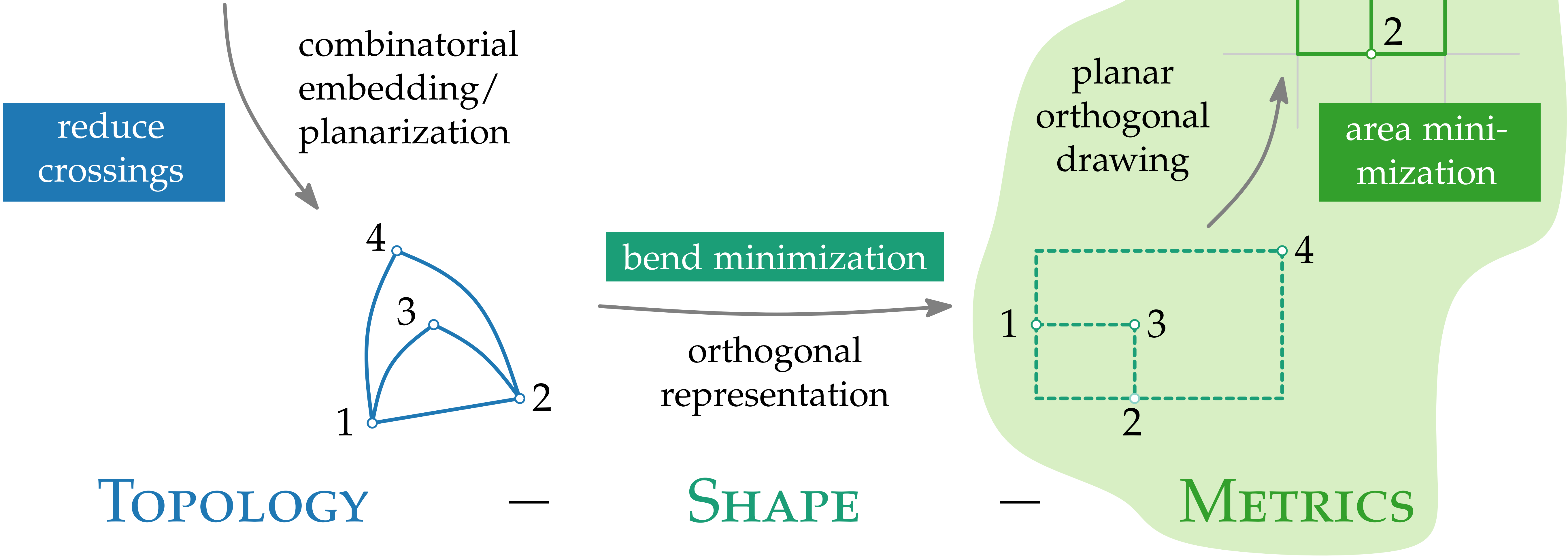
Philipp Kindermann

Topology – Shape – Metrics

Three-step approach:

[Tamassia 1987]

$V = \{v_1, v_2, v_3, v_4\}$
$E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$



combinatorial
embedding/
planarization

reduce crossings

bend minimization

orthogonal
representation

planar
orthogonal
drawing

area minimization

Topology — Shape — Metrics

Compaction

Compaction problem.

Given:
- Plane graph $G = (V, E)$ with maximum degree 4
- Orthogonal representation $H(G)$

Find: Compact orthogonal layout of G that realizes $H(G)$

Special case.

All faces are rectangles.

→ Guarantees possible
- minimum total edge length
- minimum area

Properties.

- bends only on the outer face
- opposite sides of a face have the same length

Idea.

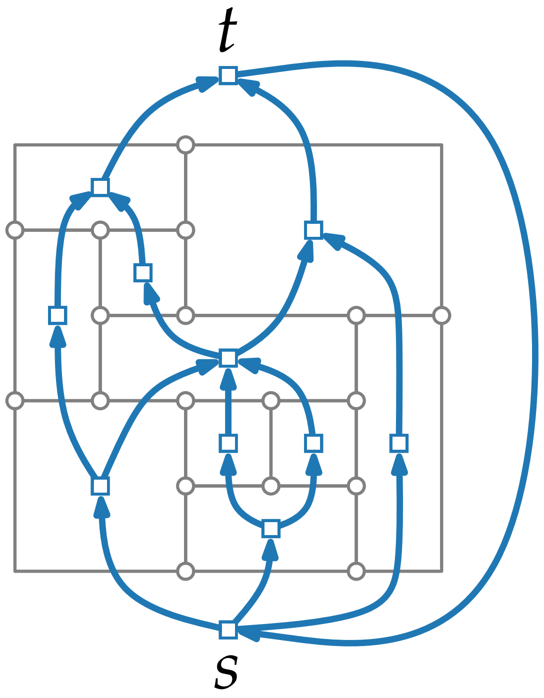
- Formulate flow network for horizontal/vertical compaction

Flow Network for Edge Length Assignment

Definition.

Flow Network $N_{\text{hor}} = ((W_{\text{hor}}, E_{\text{hor}}); b; \ell; u; \text{cost})$

- $W_{\text{hor}} = F \setminus \{f_0\} \cup \{s, t\}$ □
- $E_{\text{hor}} = \{(f, g) \mid f, g$ share a *horizontal* segment and f lies *below* $g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in E_{\text{hor}}$
- $u(a) = \infty \quad \forall a \in E_{\text{hor}}$
- $\text{cost}(a) = 1 \quad \forall a \in E_{\text{hor}}$
- $b(f) = 0 \quad \forall f \in W_{\text{hor}}$

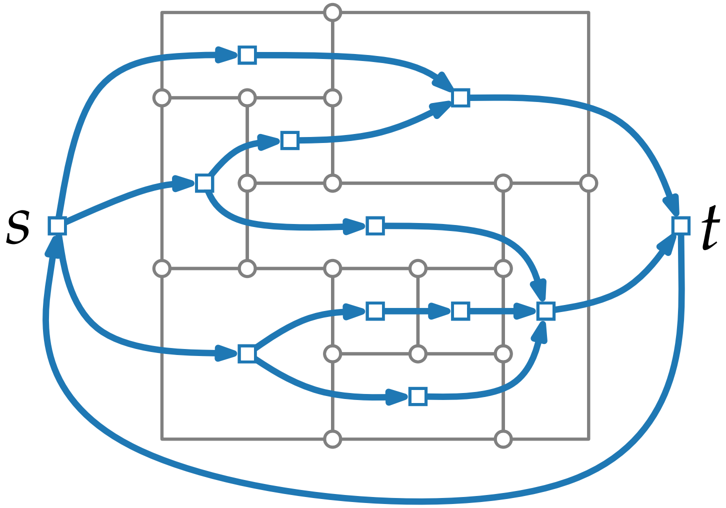


Flow Network for Edge Length Assignment

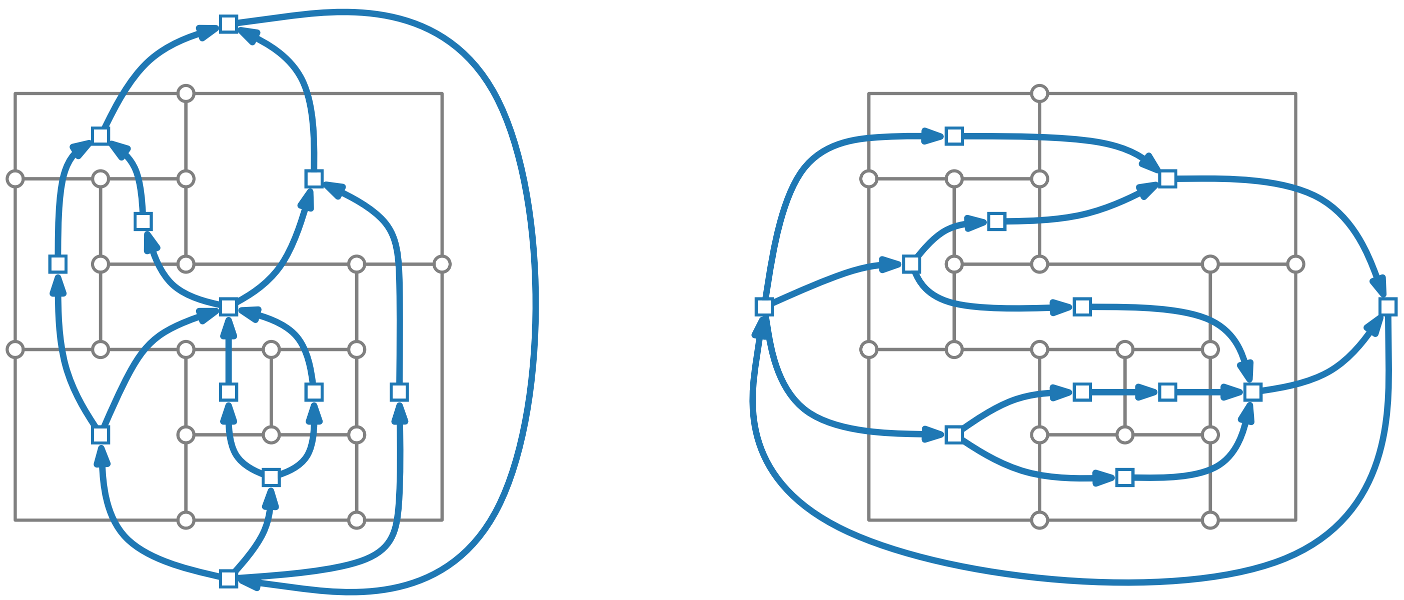
Definition.

Flow Network $N_{\text{ver}} = ((W_{\text{ver}}, E_{\text{ver}}); b; \ell; u; \text{cost})$

- $W_{\text{ver}} = F \setminus \{f_0\} \cup \{s, t\}$ □
- $E_{\text{ver}} = \{(f, g) \mid f, g$ share a *vertical* segment and f lies to the *left* of $g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in E_{\text{ver}}$
- $u(a) = \infty \quad \forall a \in E_{\text{ver}}$
- $\text{cost}(a) = 1 \quad \forall a \in E_{\text{ver}}$
- $b(f) = 0 \quad \forall f \in W_{\text{ver}}$



Compaction – Result



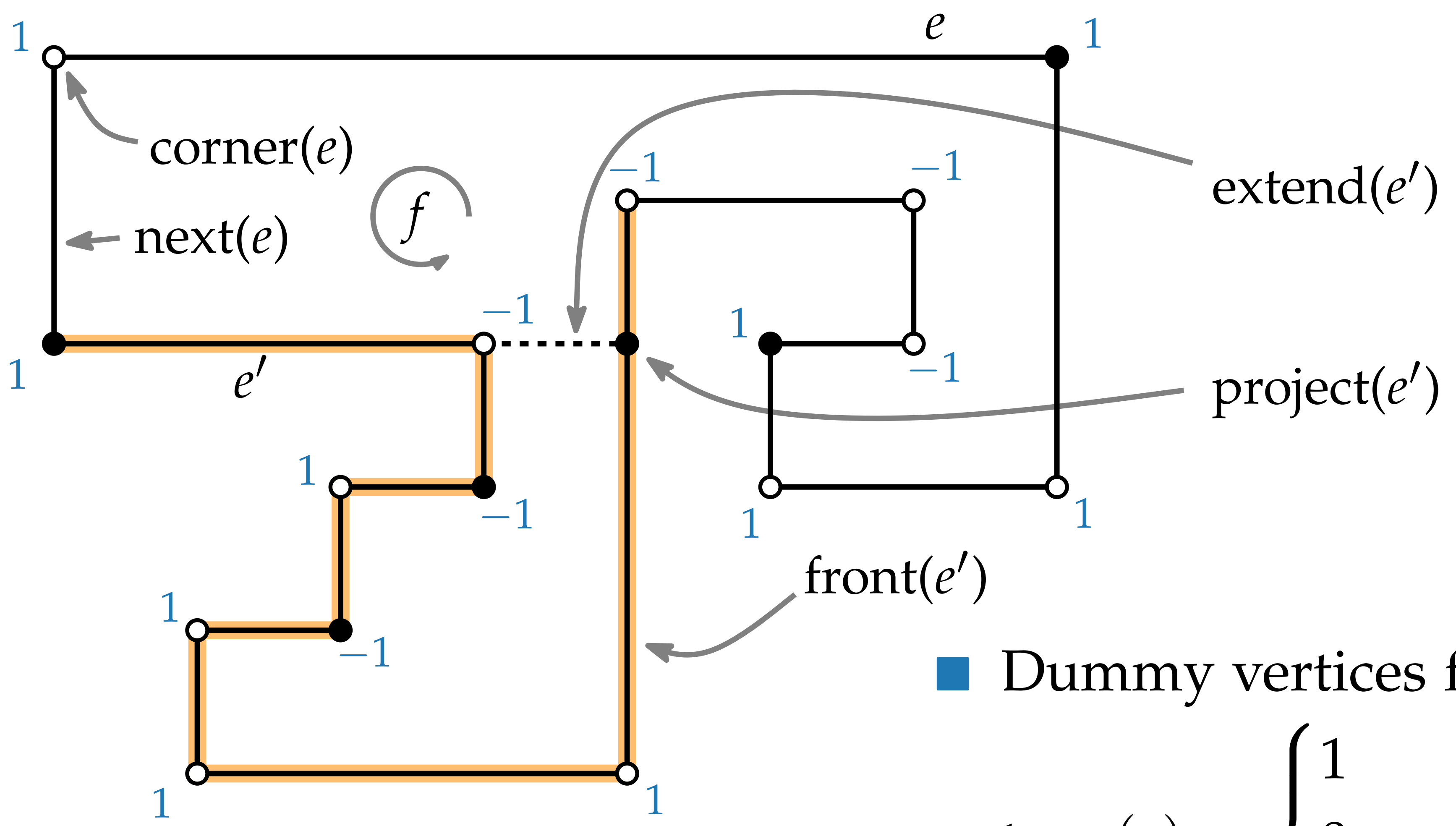
What if not all faces rectangular?

Theorem.
Valid min-cost-flows for N_{hor} and N_{ver} exists iff corresponding edge lenghts induce orthogonal drawing.

What values of the drawing represent the following?

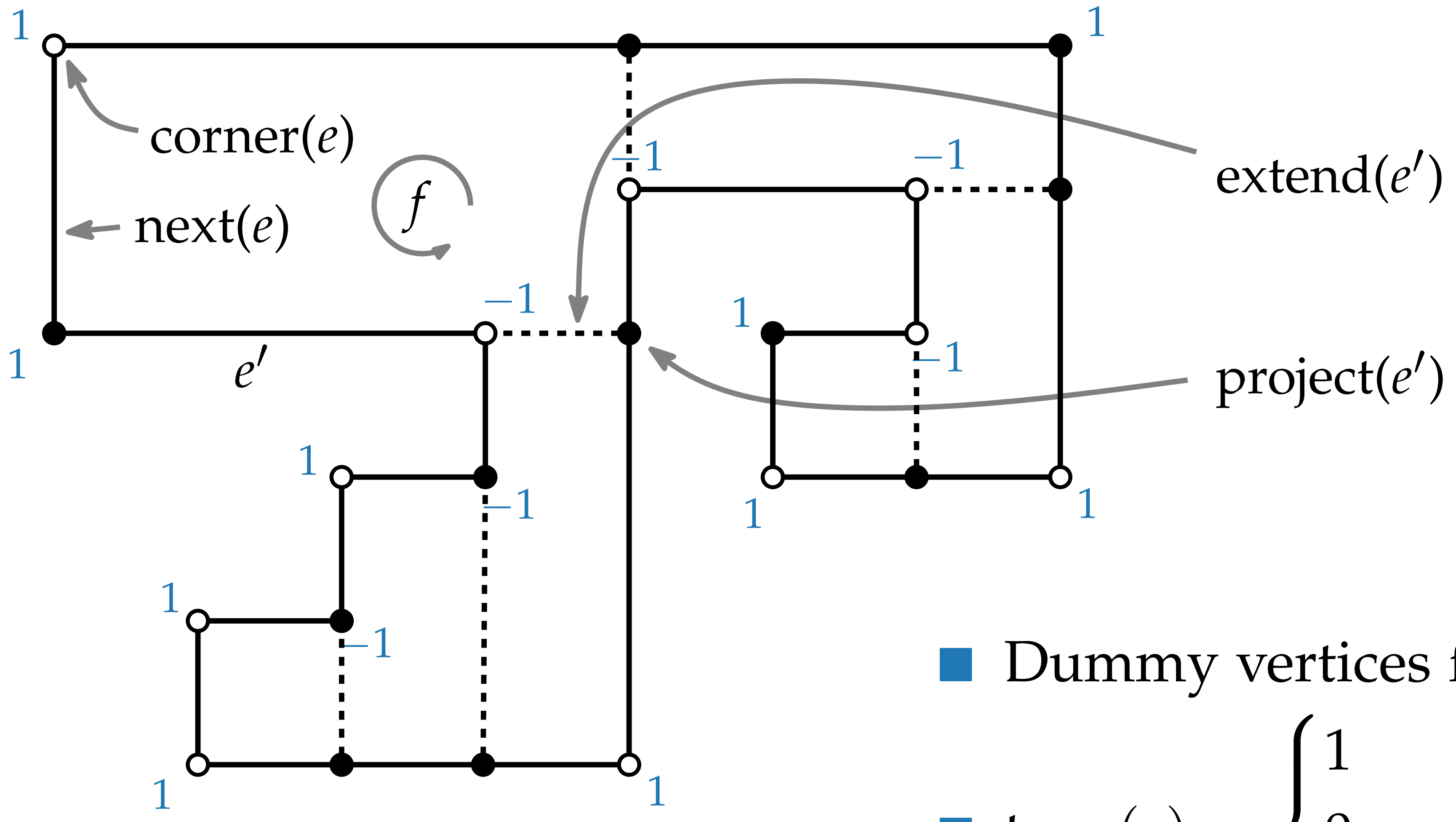
- $|X_{\text{hor}}(t,s)|$ and $|X_{\text{ver}}(t,s)|$? width and height of drawing
- $\sum_{e \in E_{\text{hor}}} X_{\text{hor}}(e) + \sum_{e \in E_{\text{ver}}} X_{\text{ver}}(e)$ total edge length

Refinement of (G, H) – Inner Face



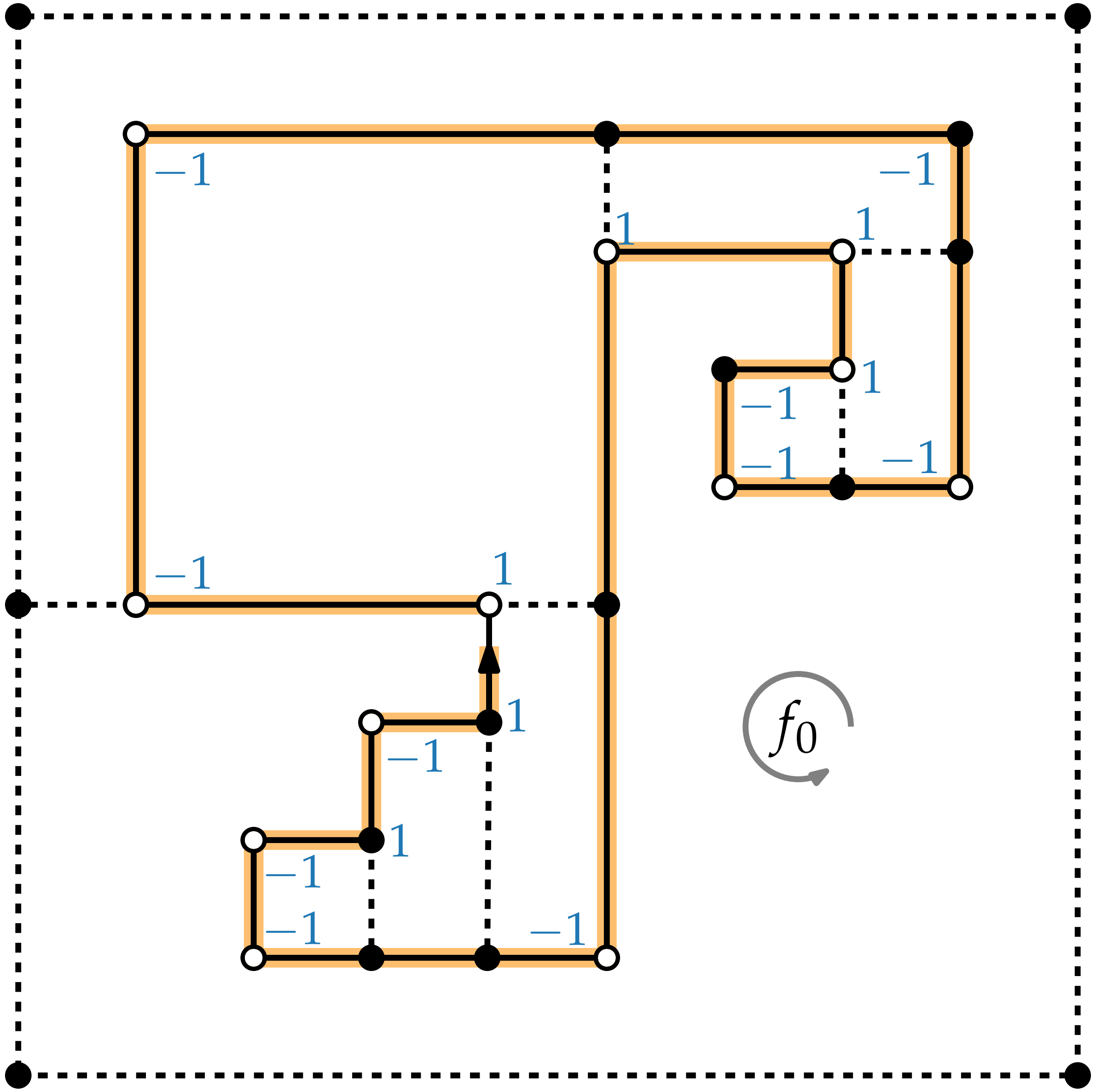
- Dummy vertices for bends
- $\text{turn}(e) = \begin{cases} 1 & \text{left turn} \\ 0 & \text{no turn} \\ -1 & \text{right turn} \end{cases}$

Refinement of (G, H) – Inner Face

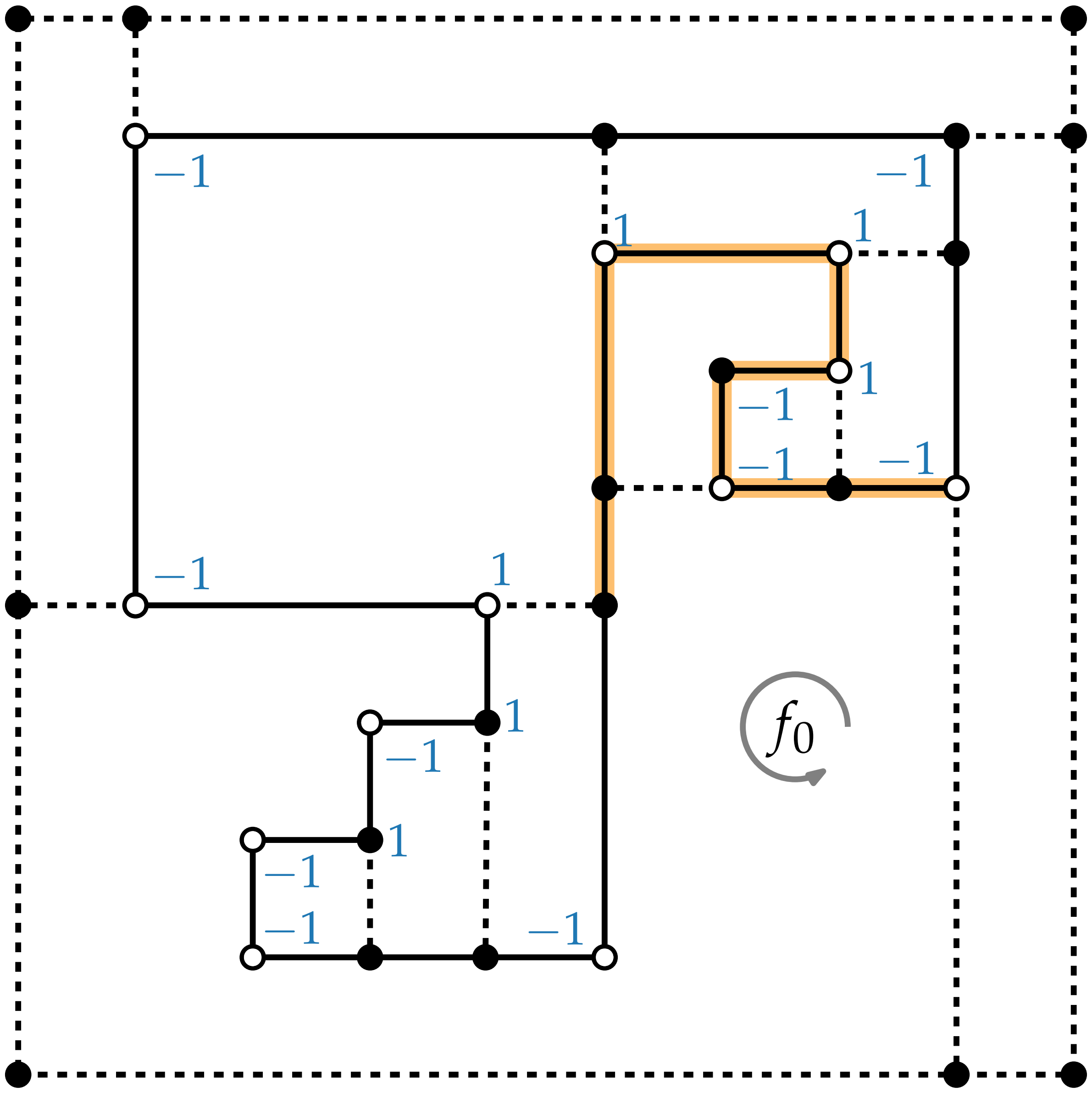


- Dummy vertices for bends
- $\text{turn}(e) = \begin{cases} 1 & \text{left turn} \\ 0 & \text{no turn} \\ -1 & \text{right turn} \end{cases}$

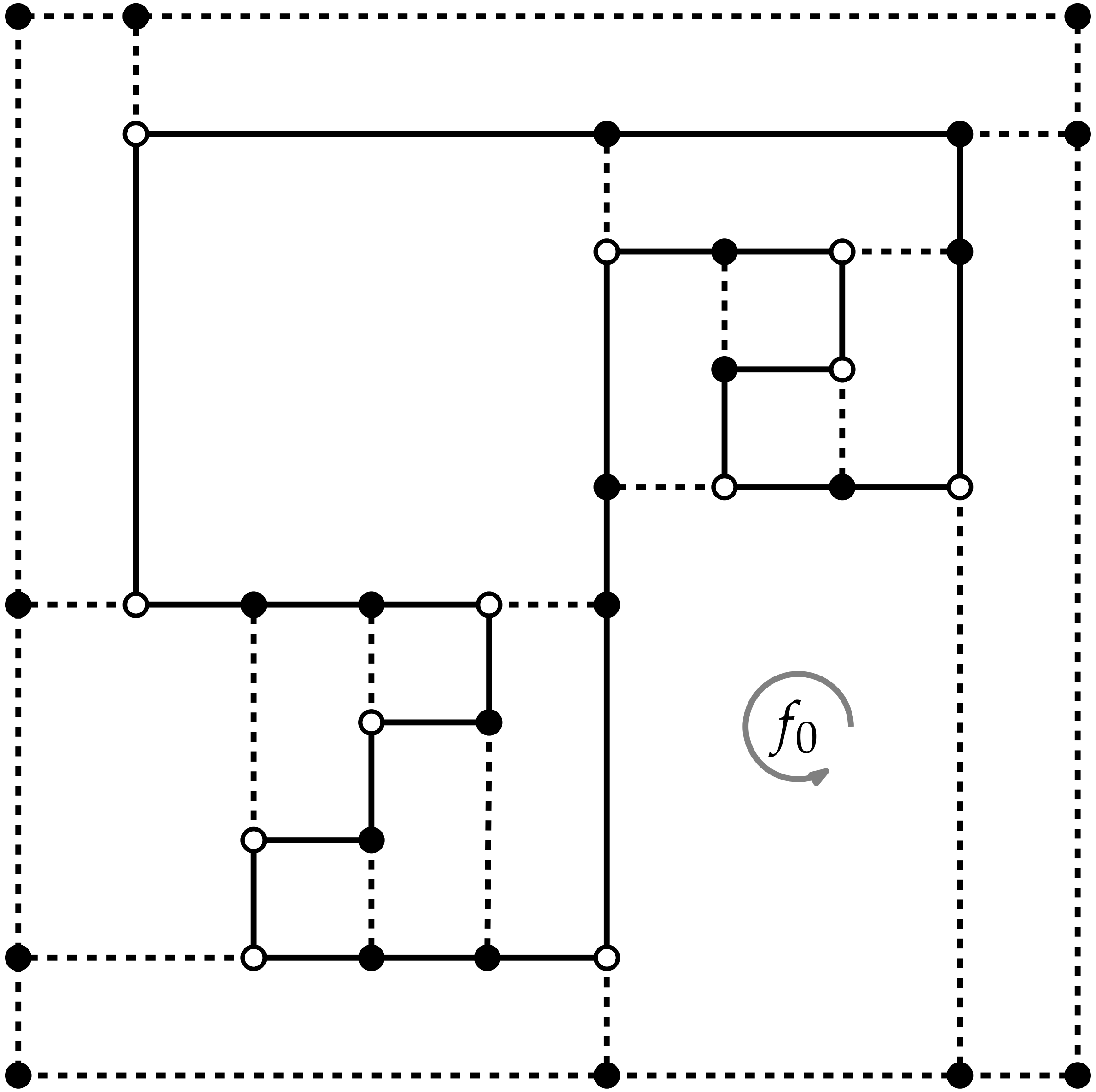
Refinement of (G, H) – Outer Face



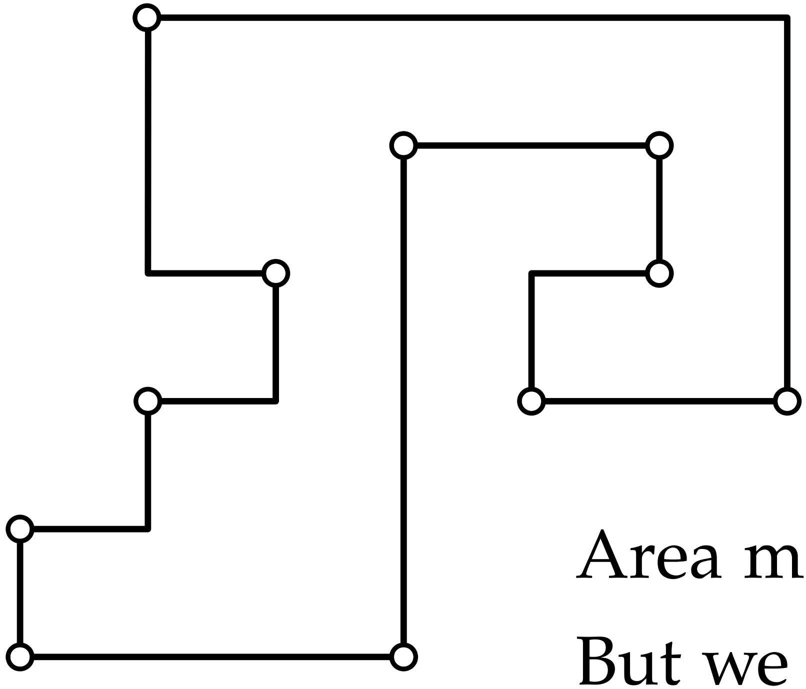
Refinement of (G, H) – Outer Face



Refinement of (G, H) – Outer Face



Refinement of (G, H) – Outer Face



Area minimized? No!

But we get bound $O((n+b)^2)$ on the area.

Theorem. [Patrignani 2001]
Compaction for given orthogonal representation is in general NP-hard.