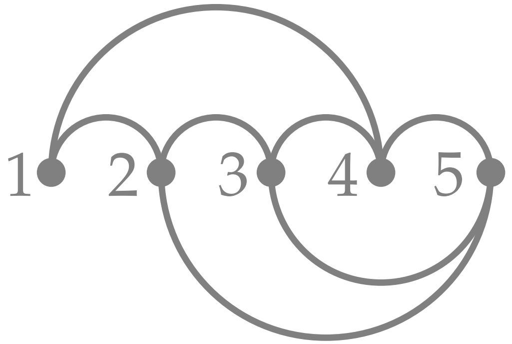
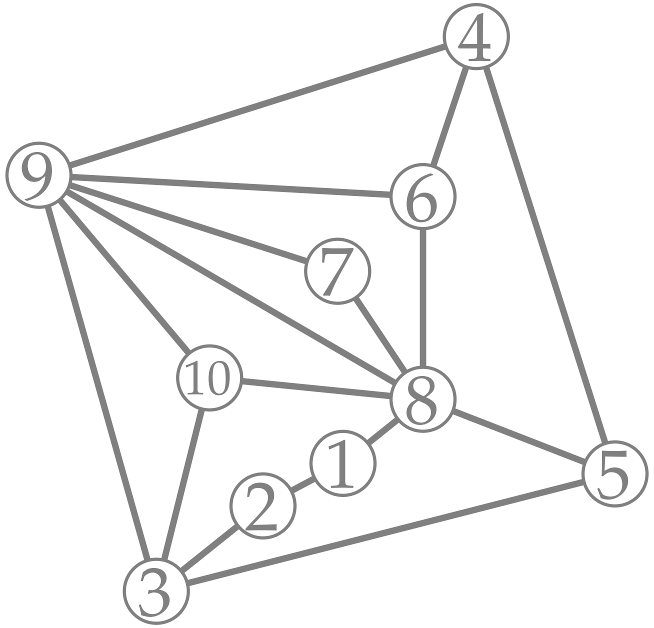
Universität Trier

Visualization of Graphs



Lecture 1:
The Graph Visualization Problem



Part I:
Organizational & Overview

Philipp Kindermann

Organizational

Lectures:

- Pre-recorded videos (as you see here)

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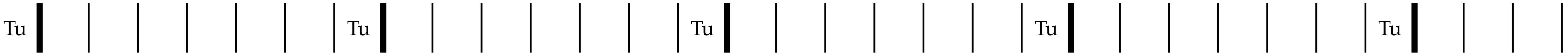

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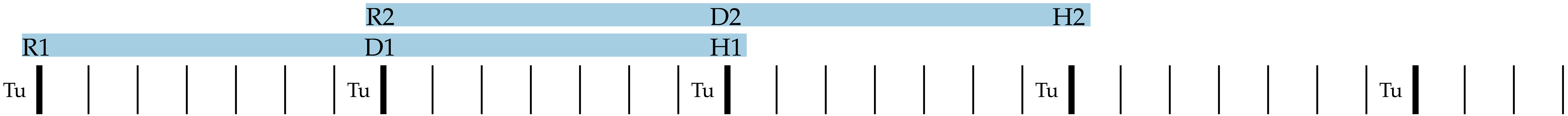

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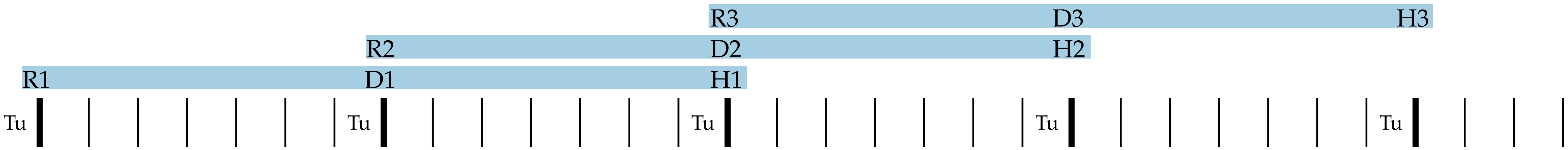

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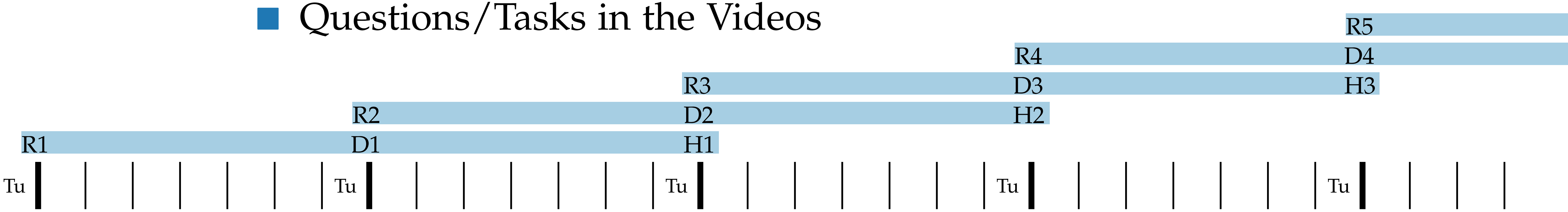

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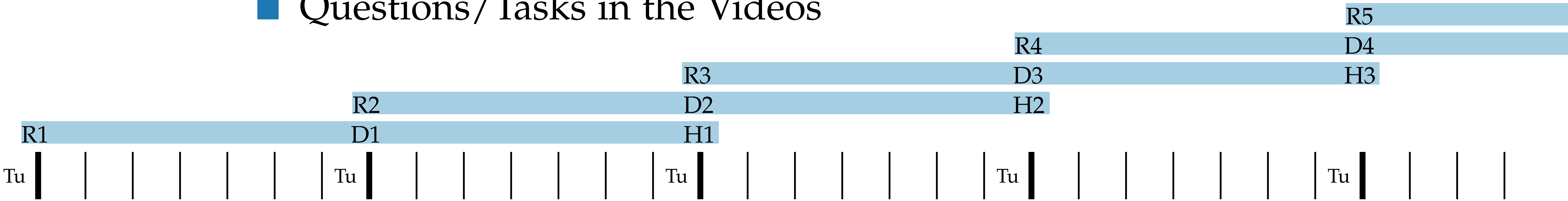

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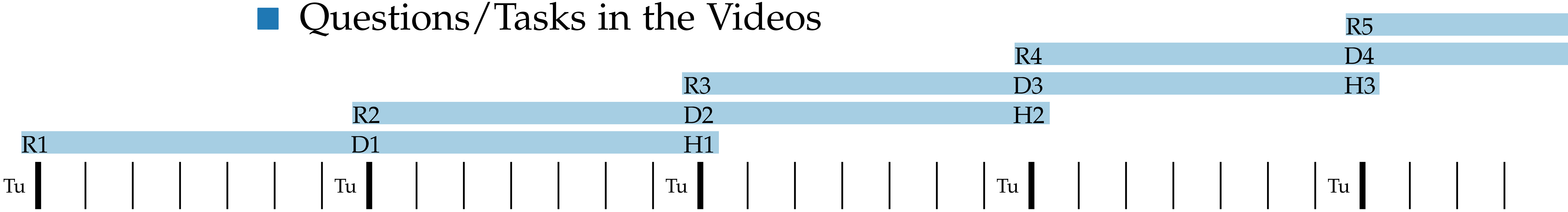

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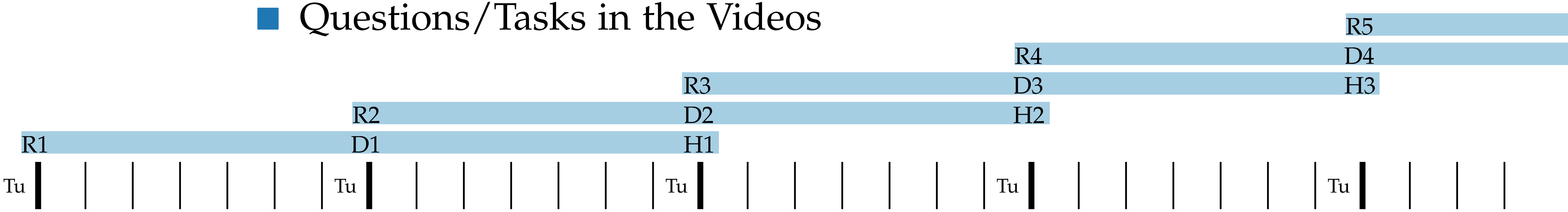

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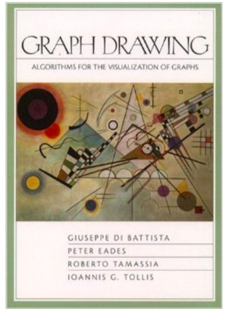
- Discussion and Solutions in BigBlueButton (Date: ?)

Books



G. Di Battista, P. Eades, R. Tamassia, I. Tollis:
Graph Drawing: Algorithms for the Visualization of Graphs
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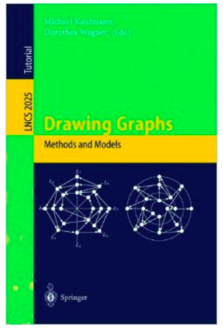

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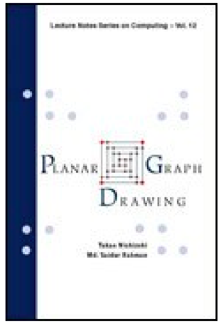

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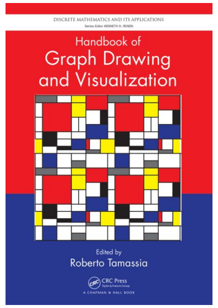
World Scientific, 2004



R. Tamassia:
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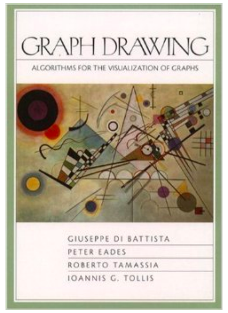
[GD]



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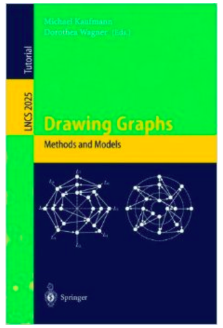

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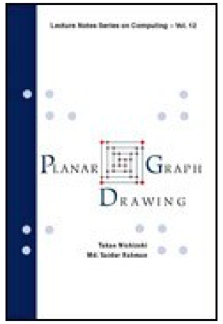

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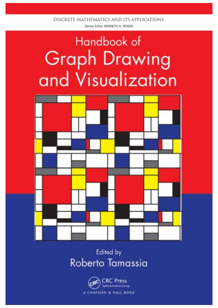

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- force-based algorithm

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- proofs

What is this course about?

Topics

- Drawing Trees and Series-Parallel Graphs
- Straight-Line Drawings of Planar Graphs
- Orthogonal Grid Drawings
- Octilinear Drawings for Metro Maps
- Upwards Planar Drawings
- Hierarchical Layouts of Directed Graphs
- Contact Representations
- Visibility Representations
- The Crossing Lemma
- Beyond Planarity

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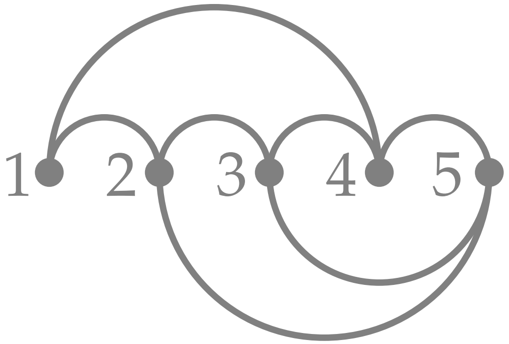

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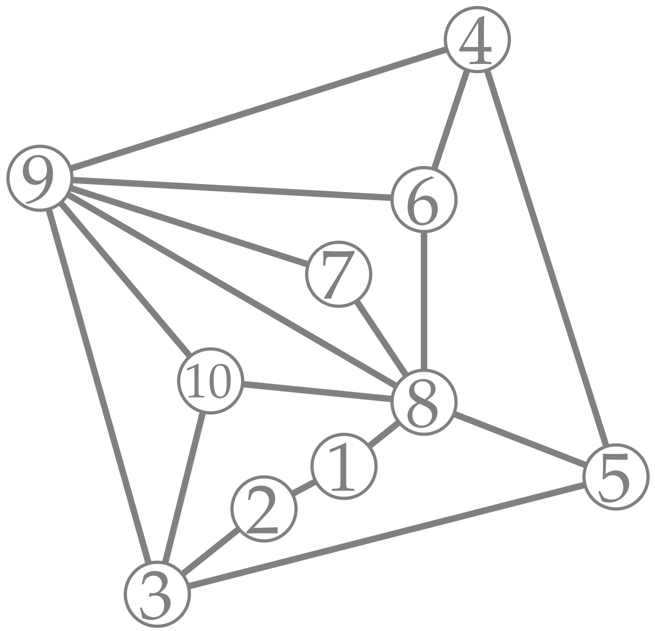

Part II: The Layout Problem

Philipp Kindermann

Graphs and their representations

What is a graph?

- graph $G = (V, E)$
- vertices $V = \{v_1, v_2, \ldots, v_n\}$
- edge $E = \{e_1, e_2, \ldots, e_m\}$

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Representation?

- Set notation

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$$
$$E = \{\{v_1, v_2\}, \{v_1, v_8\}, \{v_2, v_3\}, \{v_3, v_5\}, \{v_3, v_9\}, \{v_3, v_{10}\}, \{v_4, v_5\}, \{v_4, v_6\}, \{v_4, v_9\}, \{v_5, v_8\}, \{v_6, v_8\}, \{v_6, v_9\}, \{v_7, v_8\}, \{v_7, v_9\}, \{v_8, v_{10}\}, \{v_9, v_{10}\}\}$$

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\end{aligned}
$$

- Adjacency list

v_1: v_2, v_8
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v_3: v_2, v_5, v_9, v_{10}
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v_5: v_3, v_4, v_8
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- Adjacency matrix

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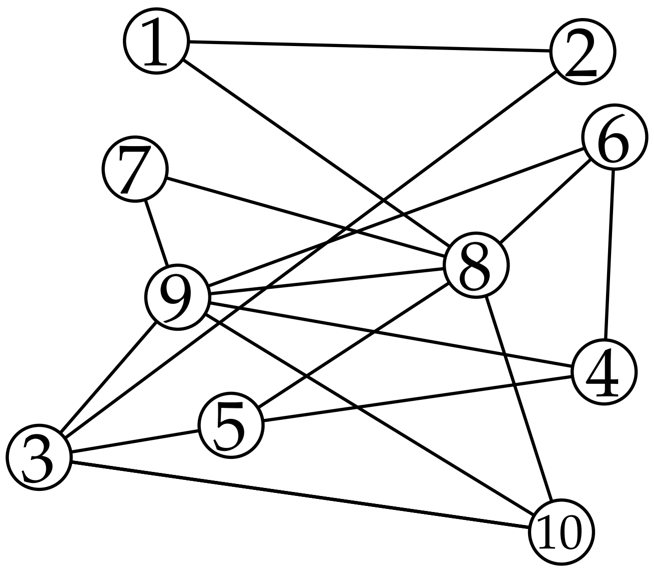
\end{pmatrix}$$

- Drawing



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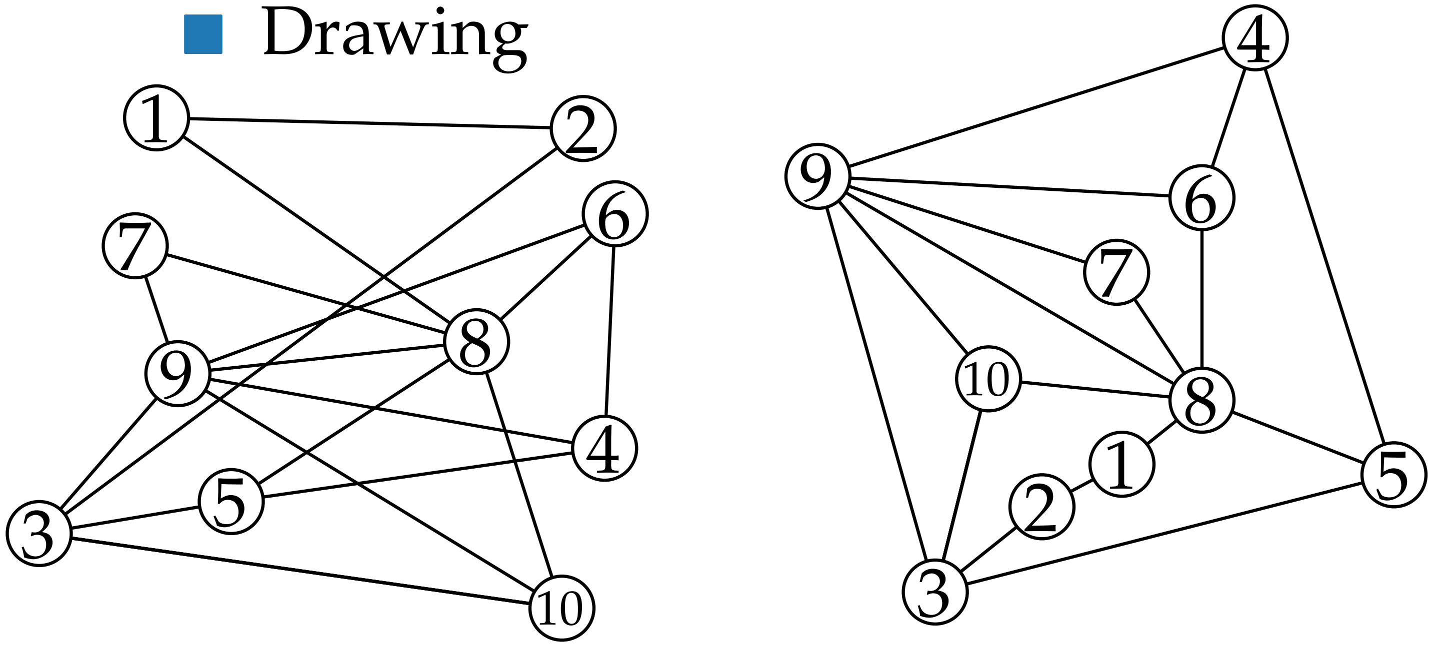

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Abstract networks

- Social networks
- Communication networks
- Phylogenetic networks
- Metabolic networks
- Class/Object Relation Digraphs (UML)
- …

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Physical networks

- Metro systems
- Road networks
- Power grids
- Telecommunication networks
- Integrated circuits
- ...

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We need algorithms that draw graphs automatically to make networks more accessible to humans.

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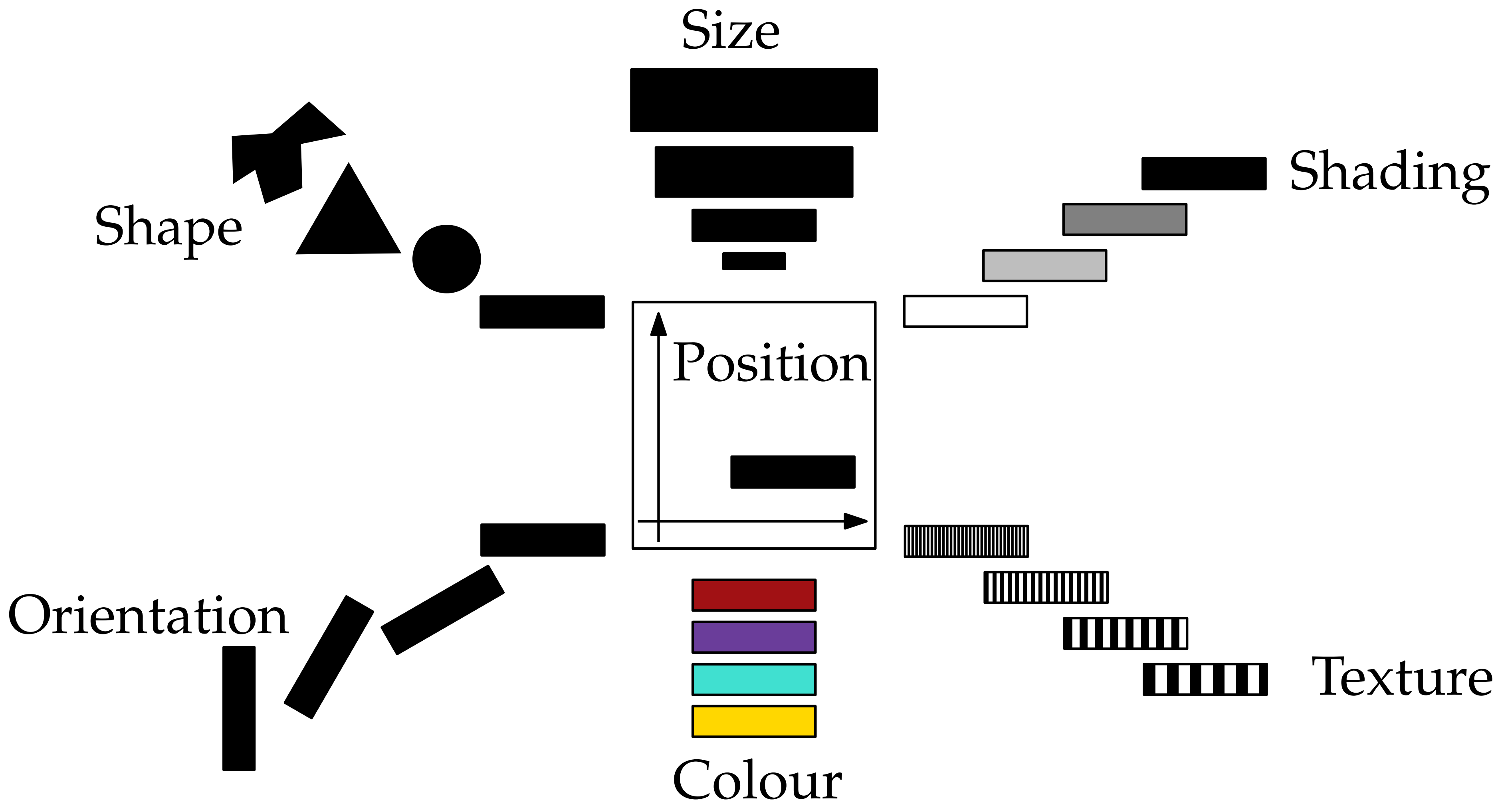

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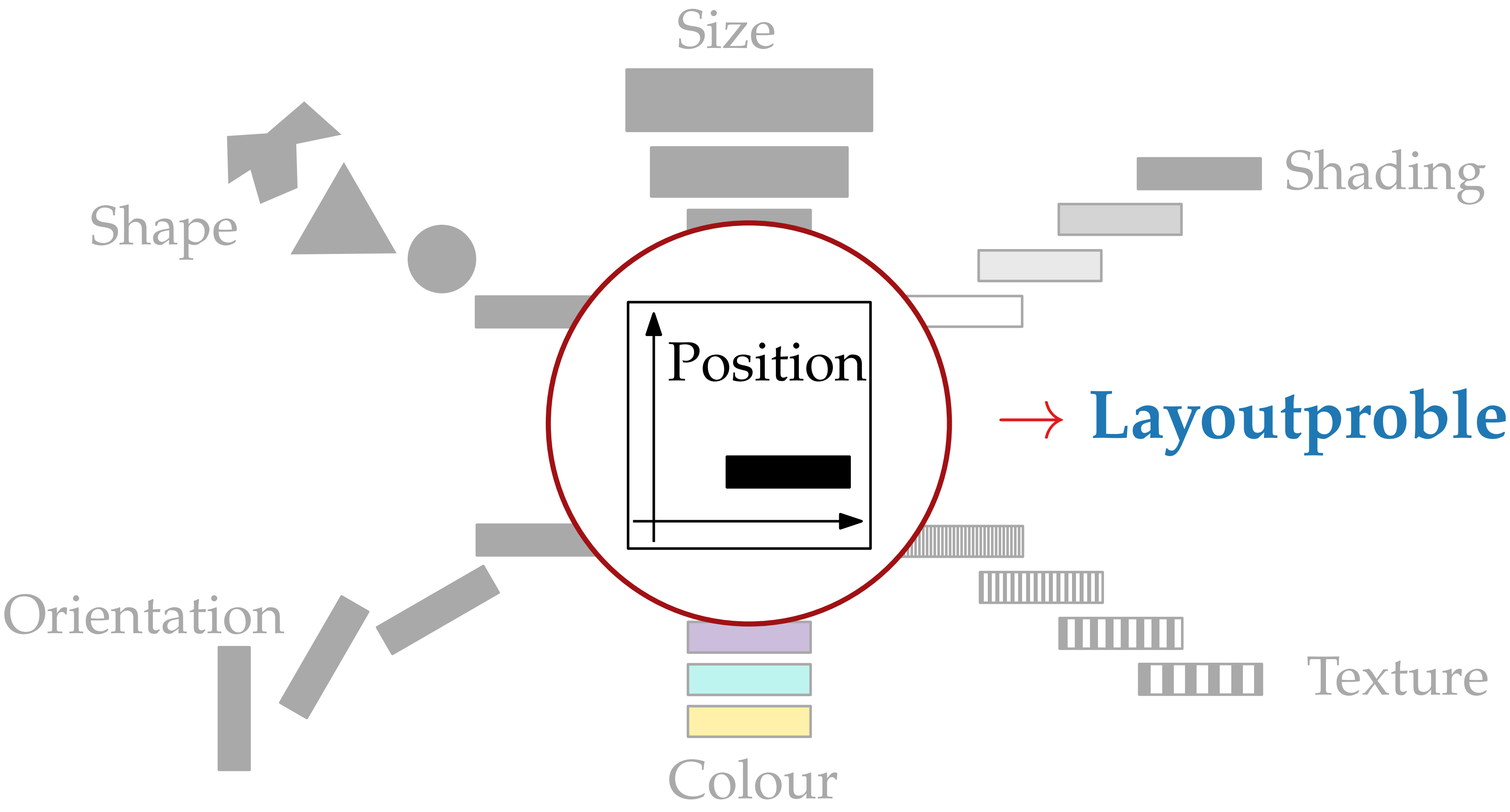

→ **Layoutproblem**

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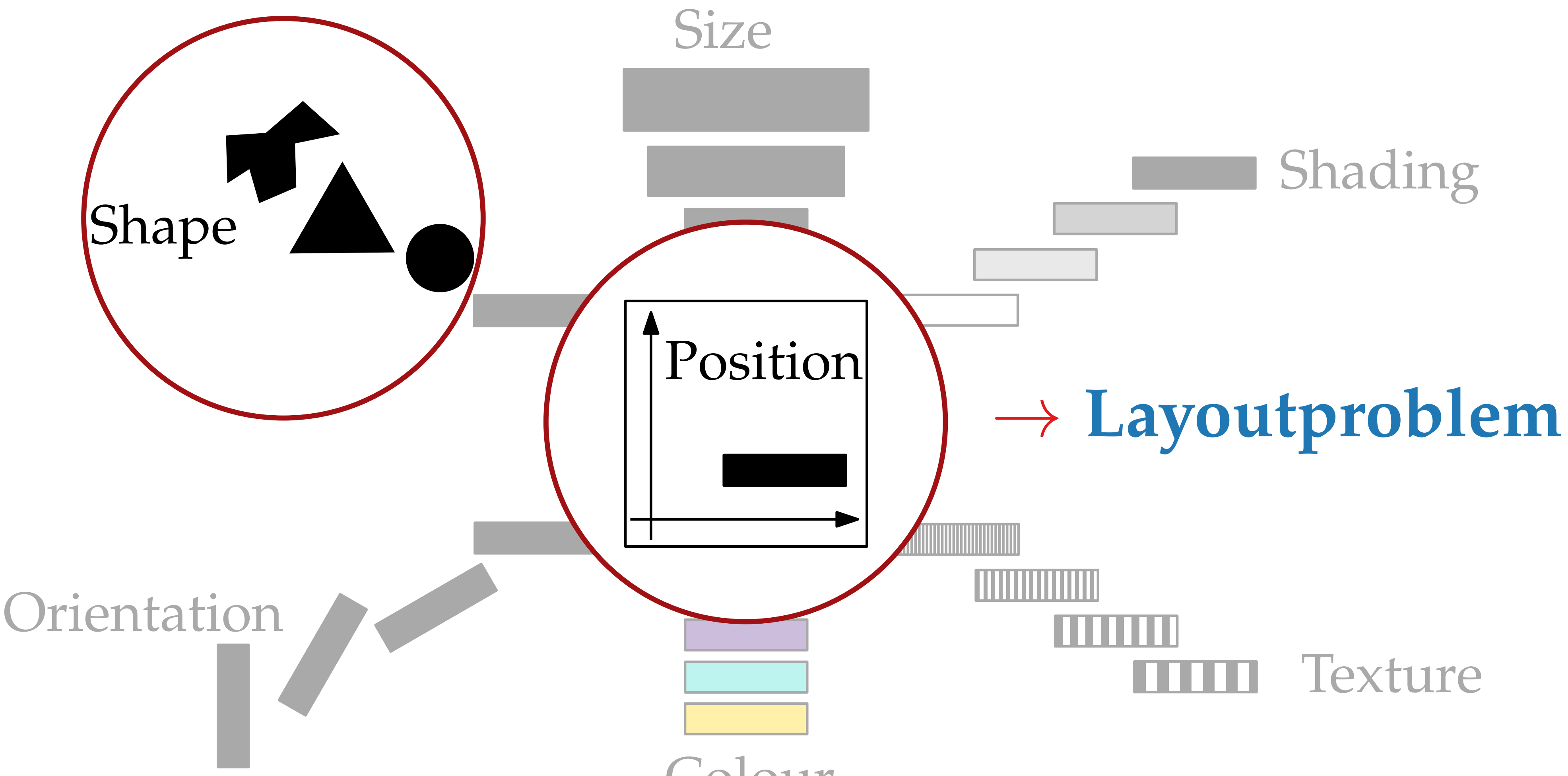
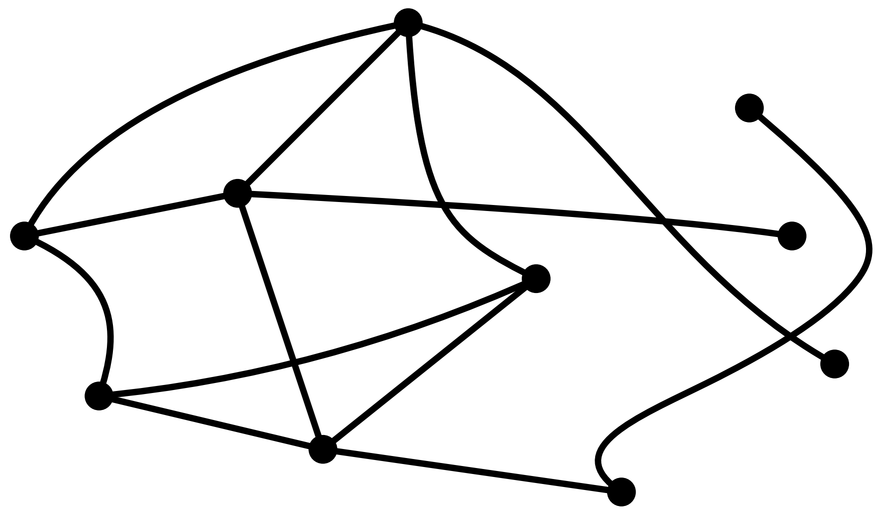


The layout problem

- *Here* restricted to the **standard representation**, so-called node-link diagrams.



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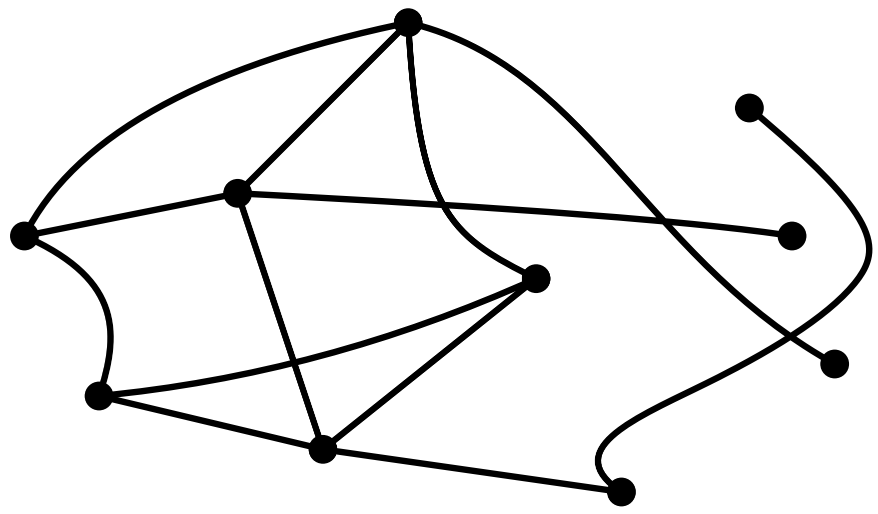

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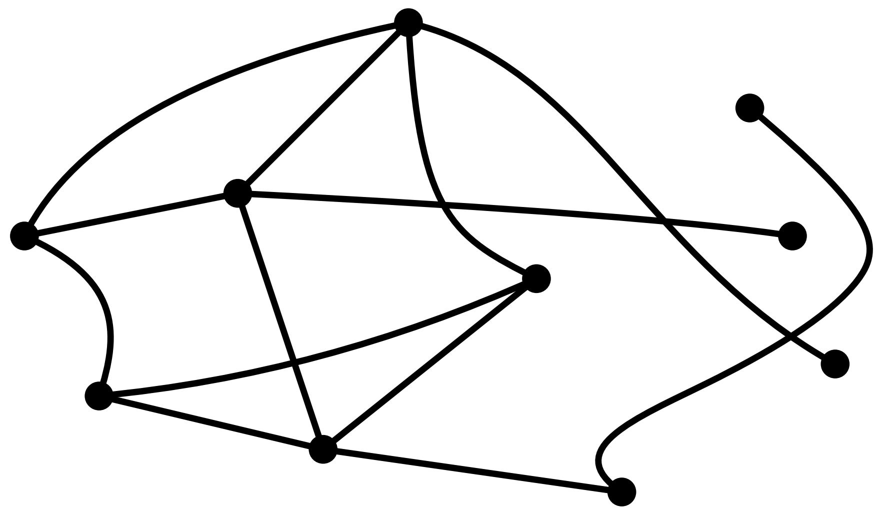
Graph Visualization Problem

in: Graph $G = (V, E)$

out:

The layout problem



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- $\Gamma \colon V \to \mathbb{R}^2$, vertex $v \mapsto$ point $\Gamma(v)$
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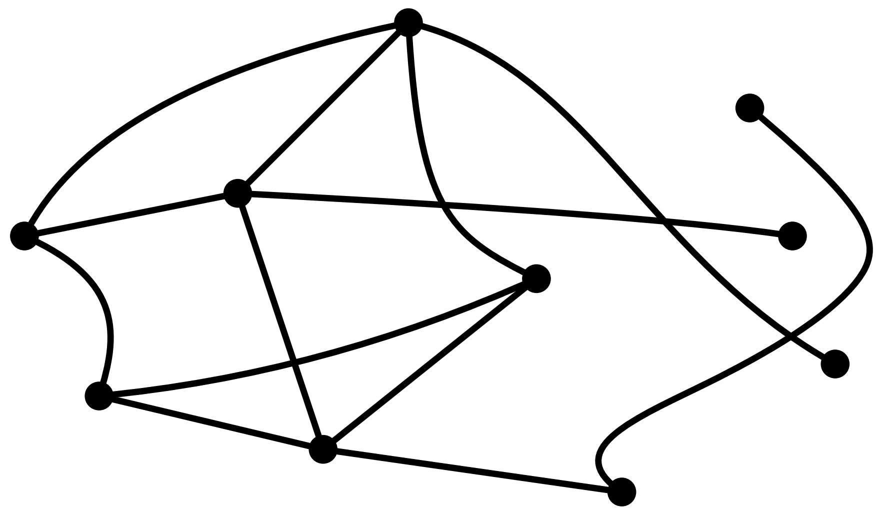

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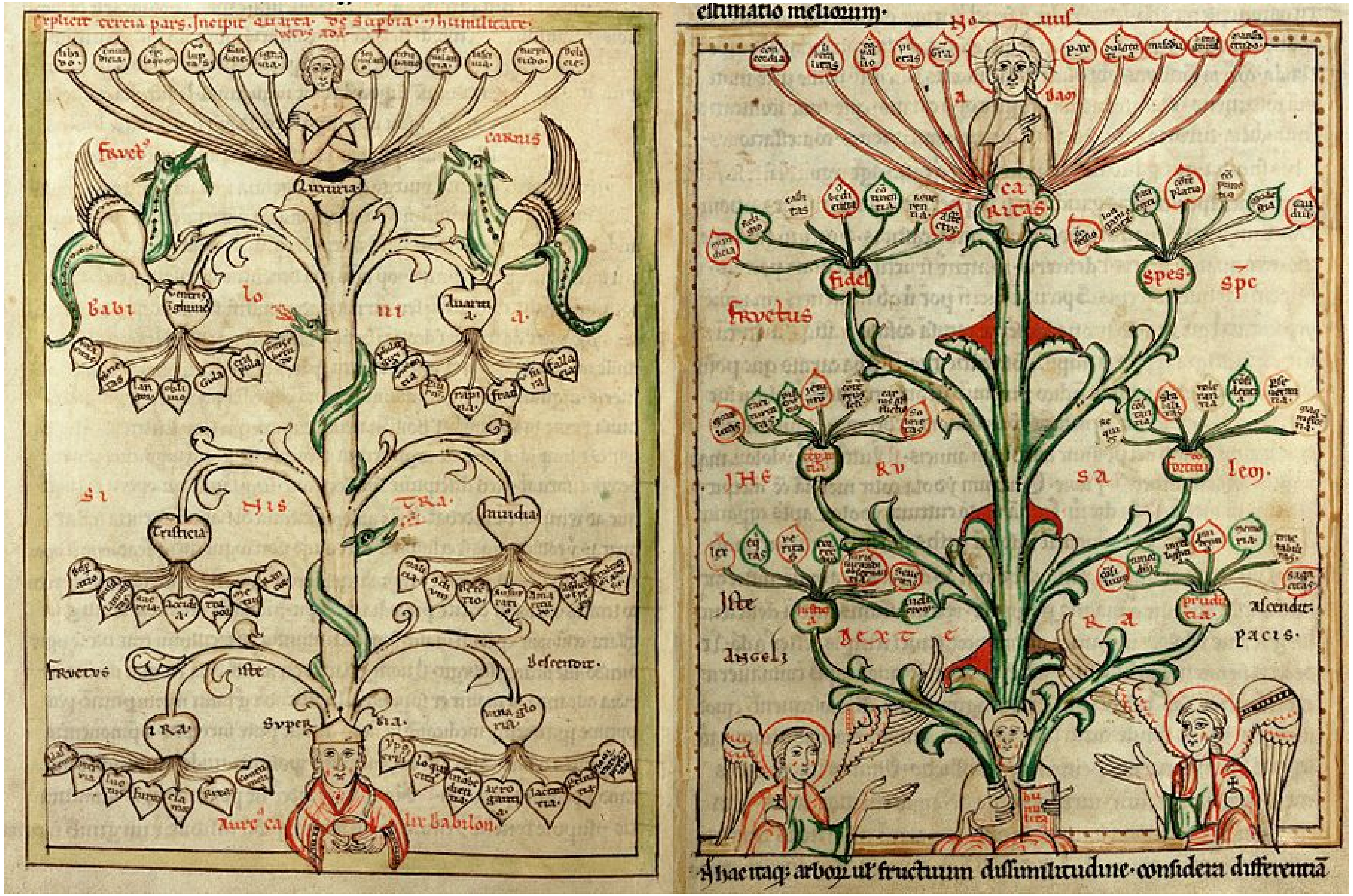
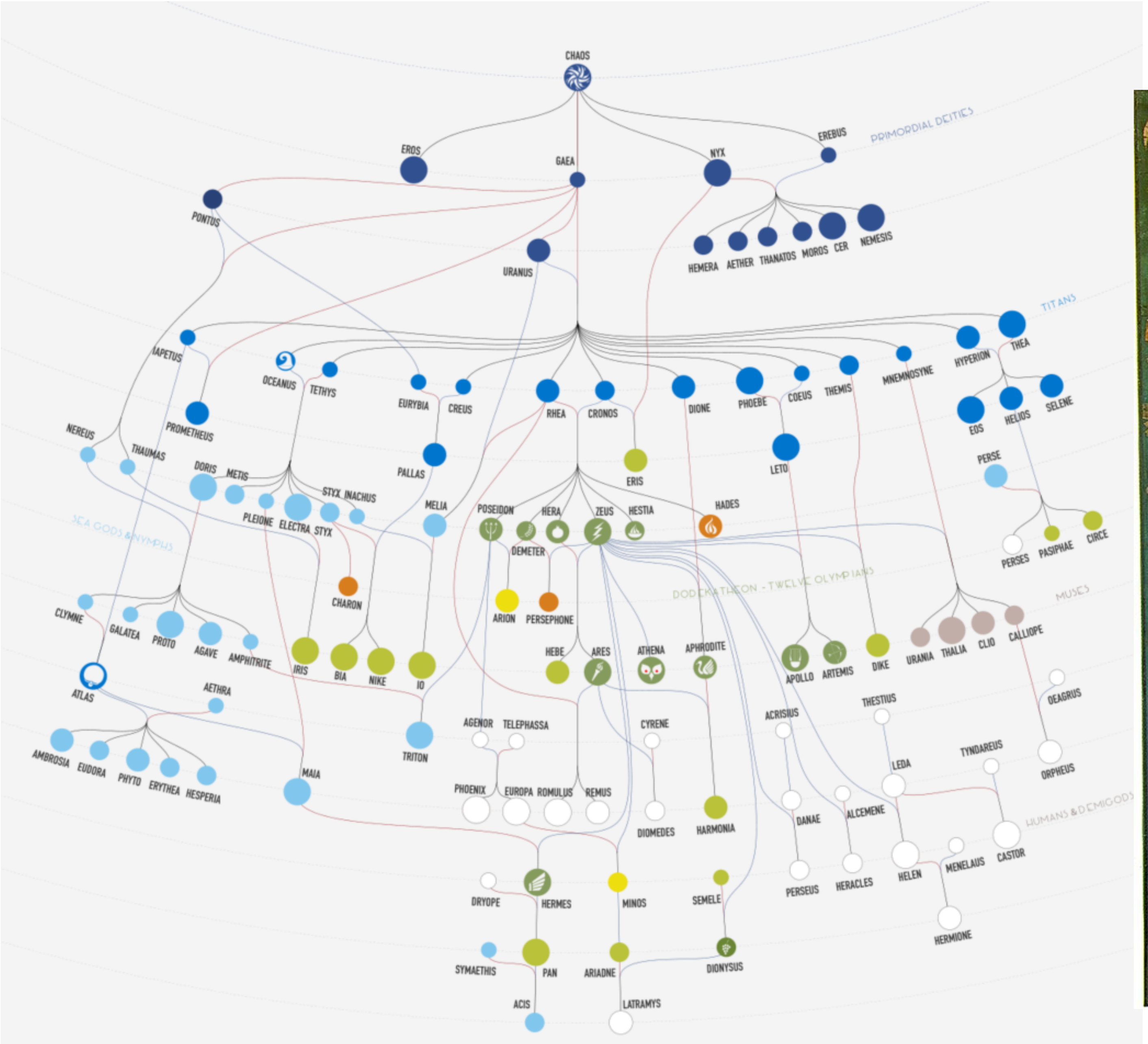
But what is a **nice** drawing?

Tree of virtues and tree of vices

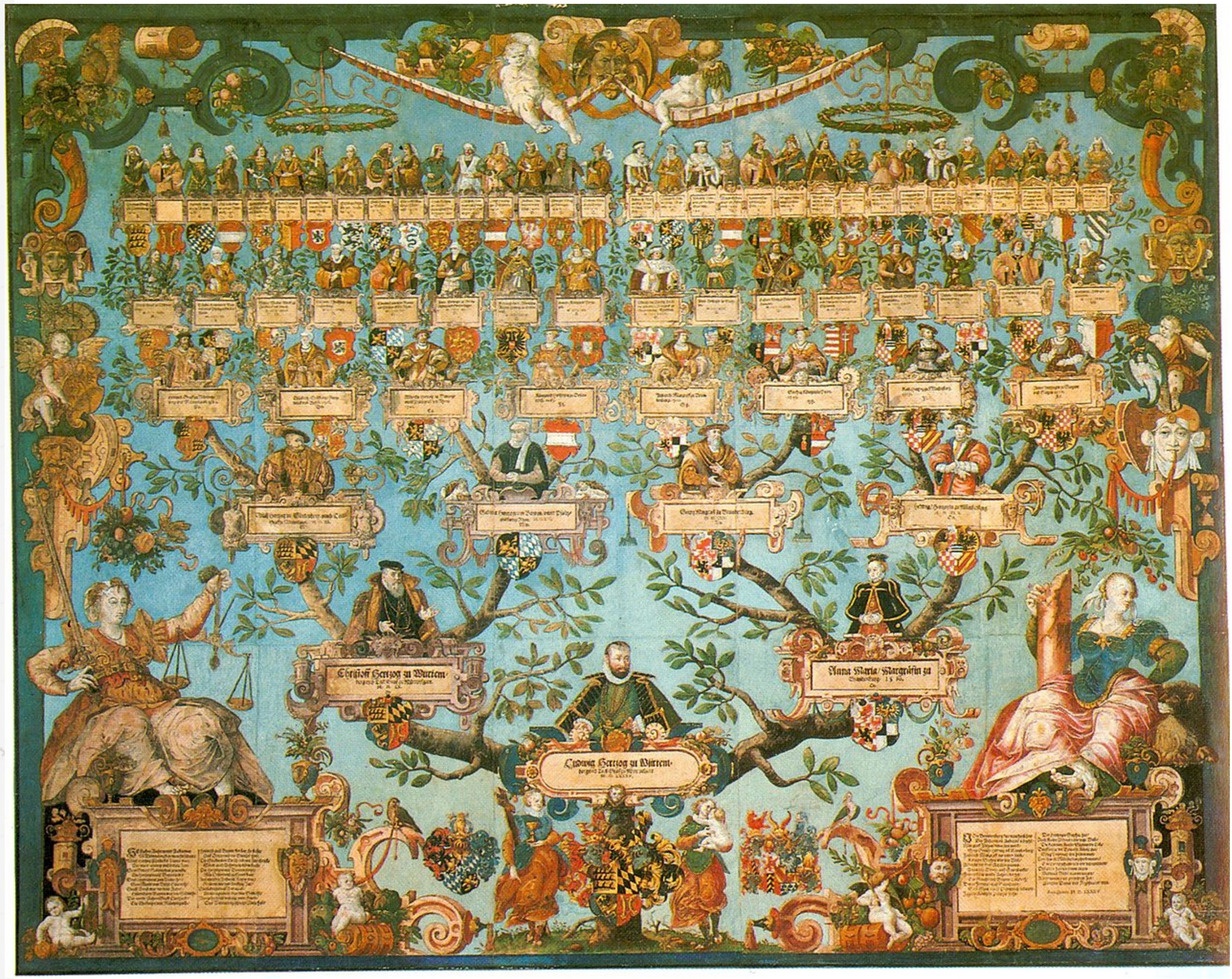
ca. 1200



Social networks - family trees

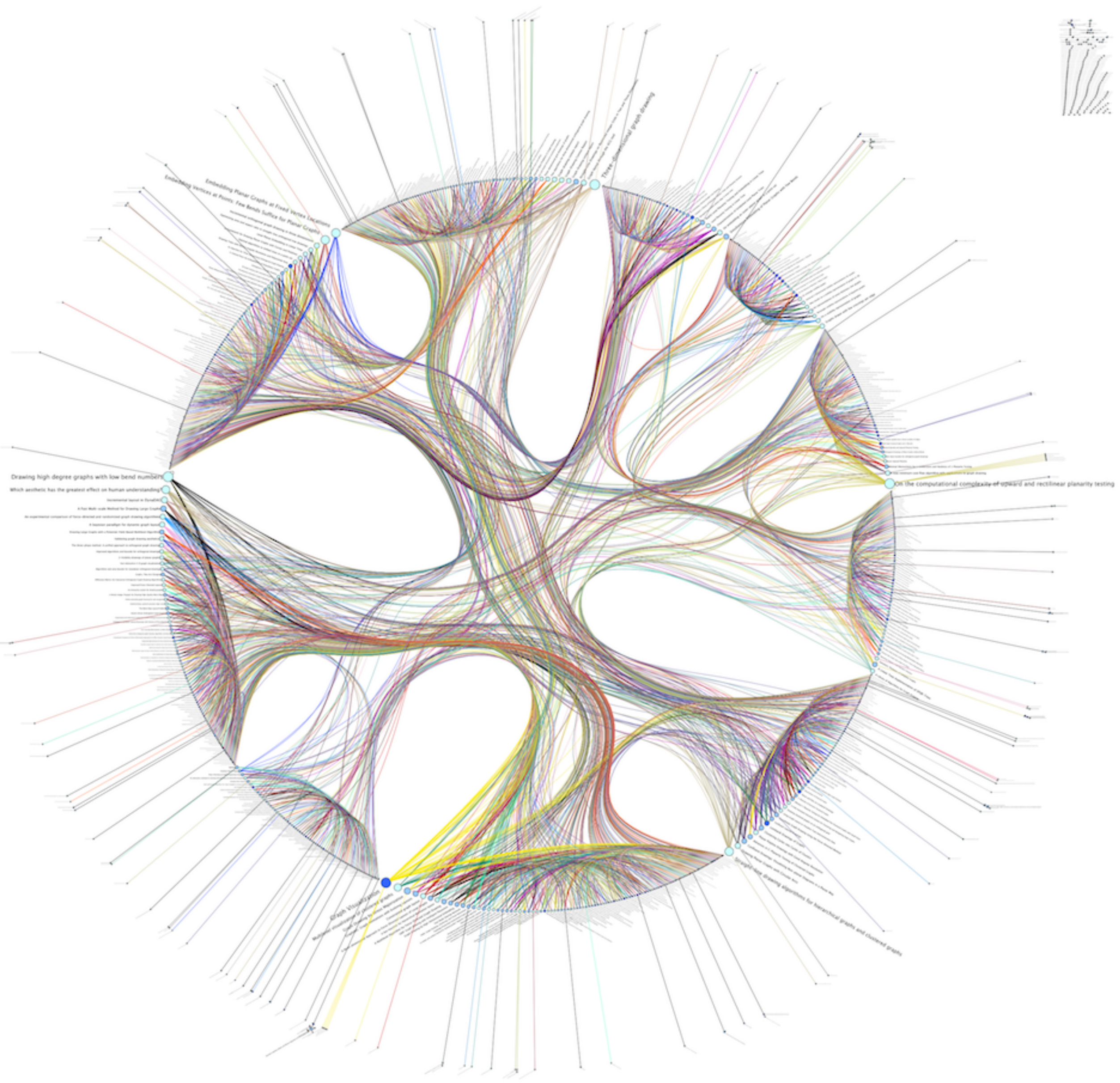


J. Klawitter, T. Mchedlidze, *Link:* go.uniwue.de/myth-poster



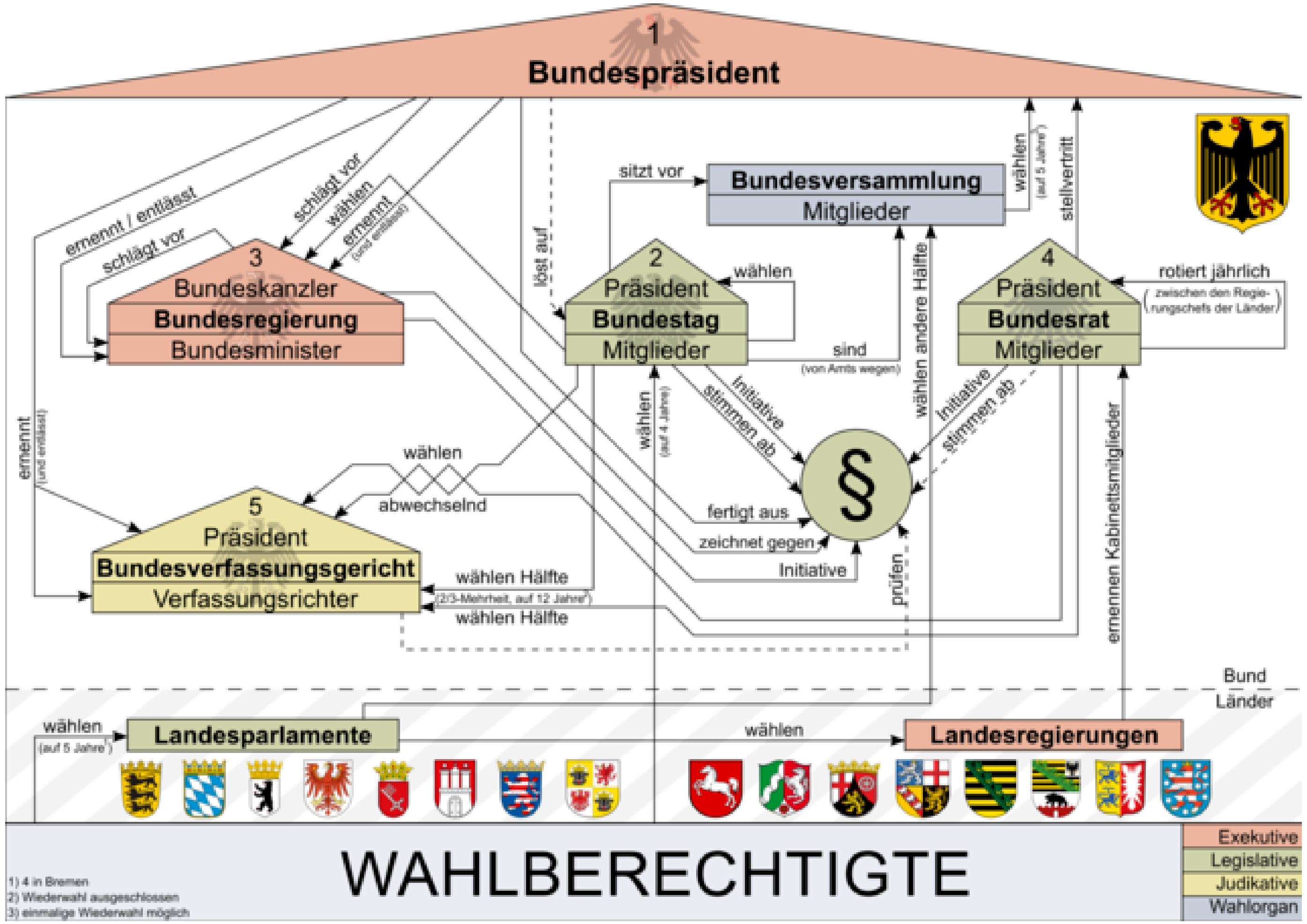
Ahnentafel Herzog Ludwig von Württemberg, 1585

Social network – citation graph

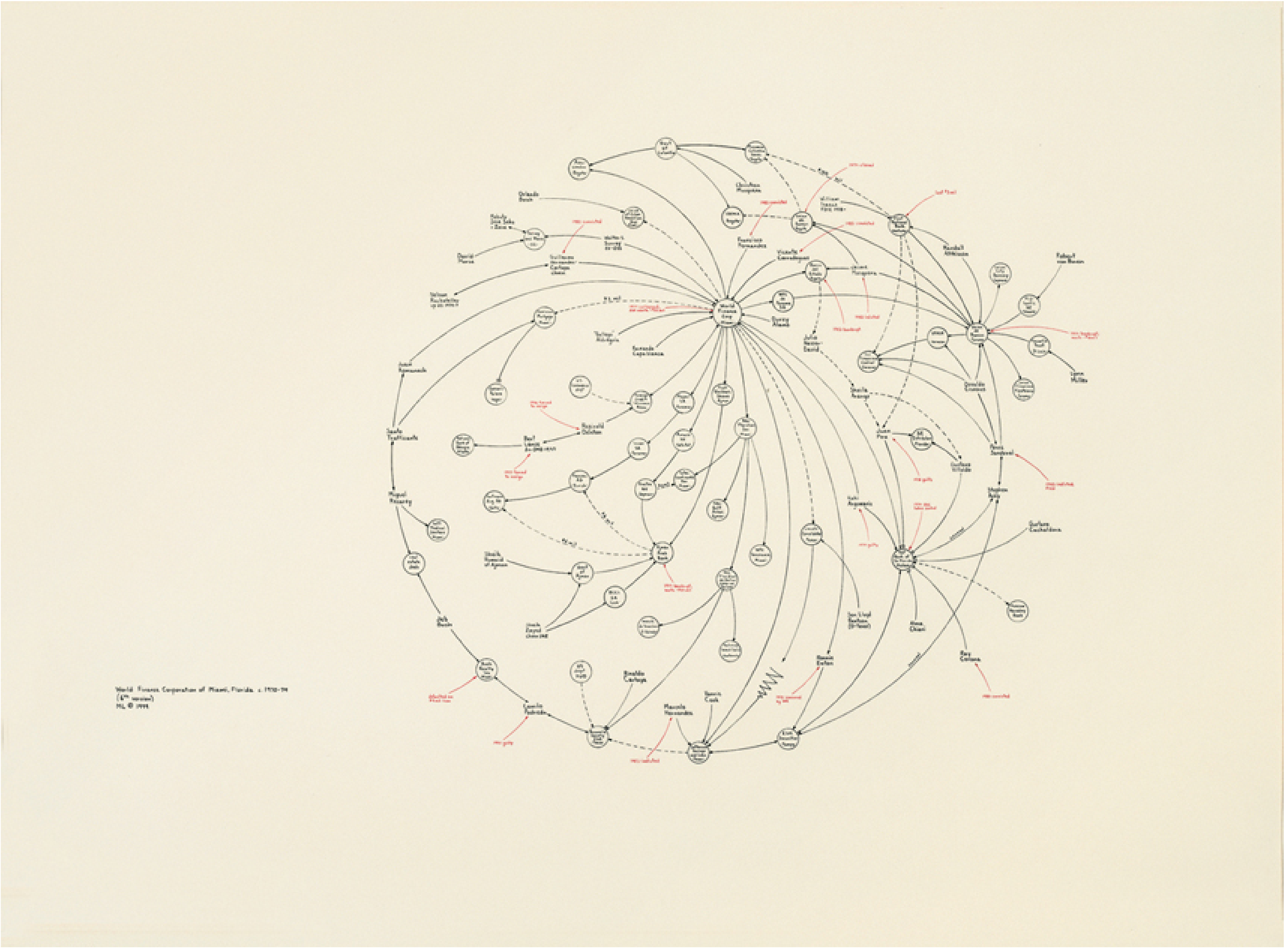


Da Ye, *Link:* https://go.uniwue.de/citation-graph

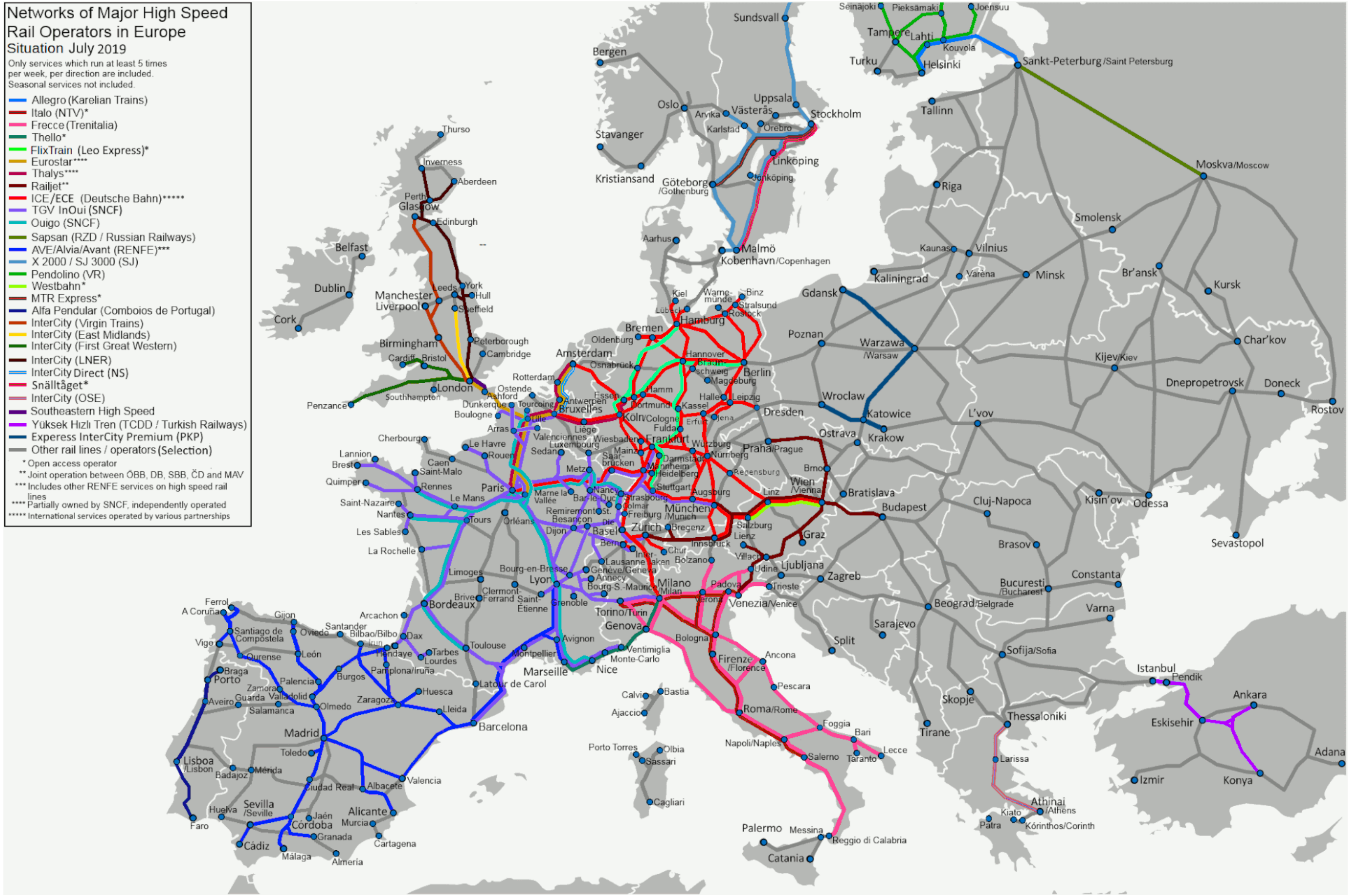
Social network - organisational chart



Social network - world finance corporation



Transportation network – European high speed railroads

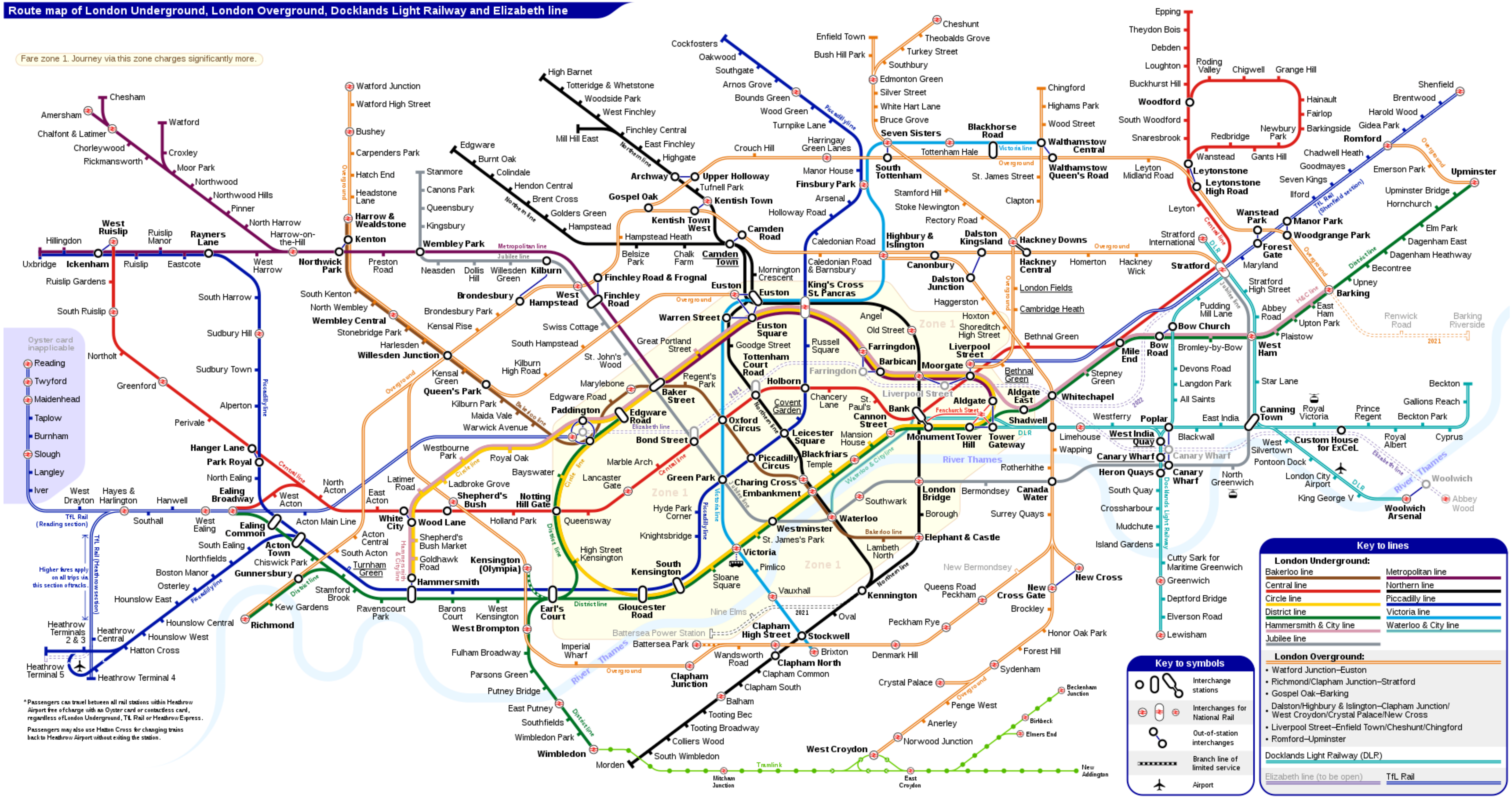


Source: Wiki Commons: Networks of Major High Speed Rail Operators in Europe - CC BY-SA 3.0

Transportation network – London Underground

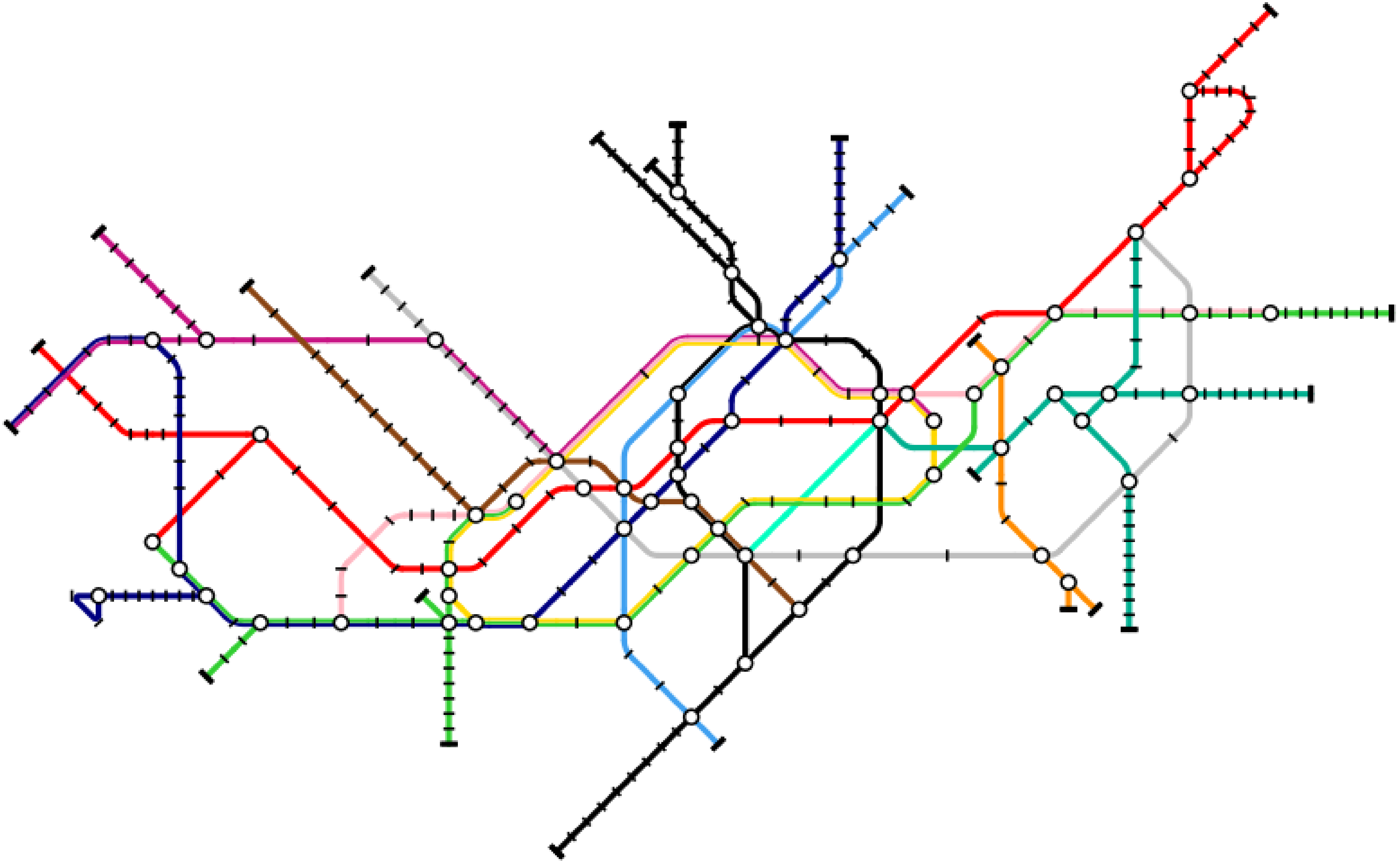
Source: Wiki Commons: London Underground full map - CC BY-SA 3.0

Transportation network – London Underground

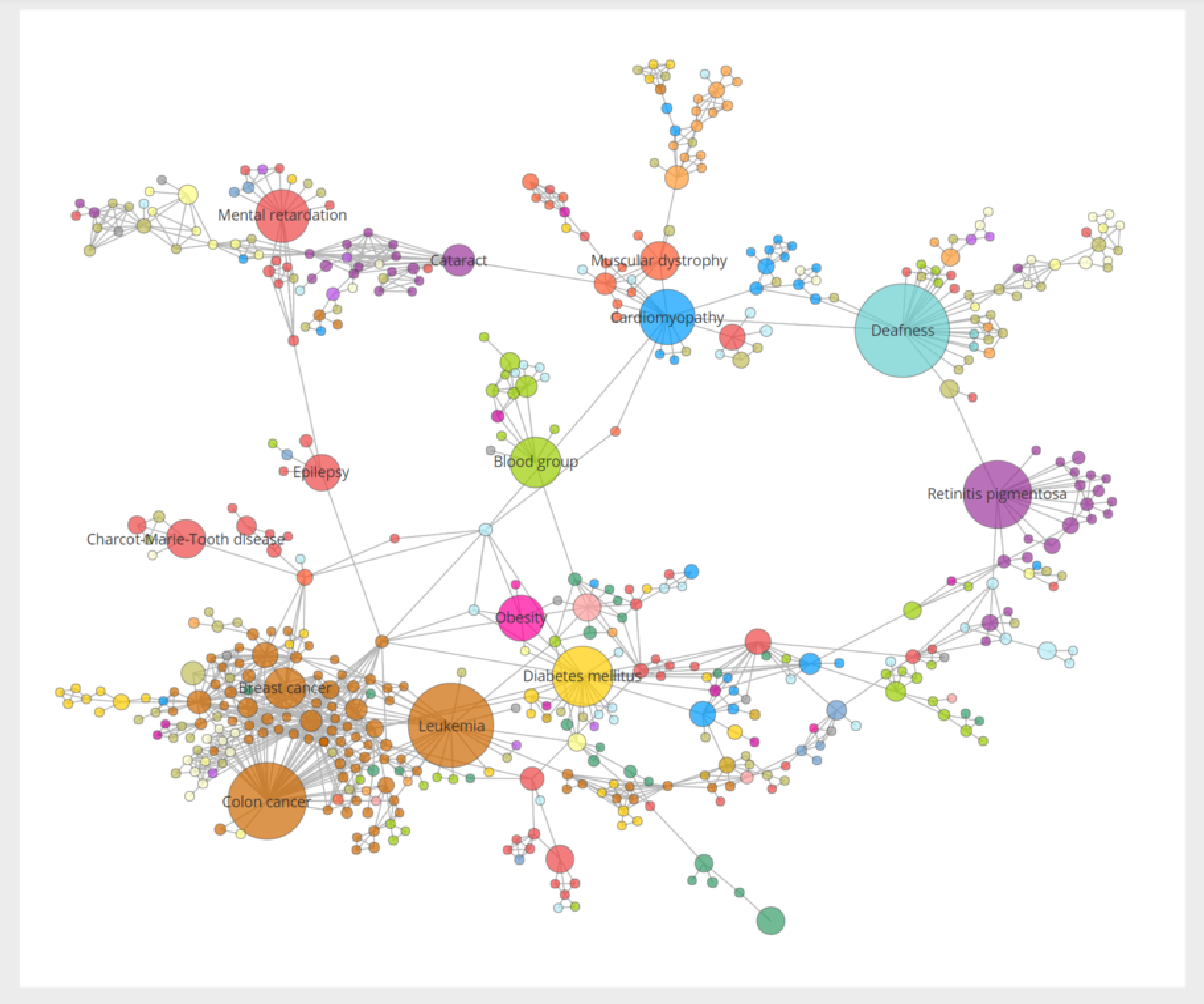


Source: Wiki Commons: London Underground Overground DLR Crossrail map - CC BY-SA 4.0

Transportation network – London Underground

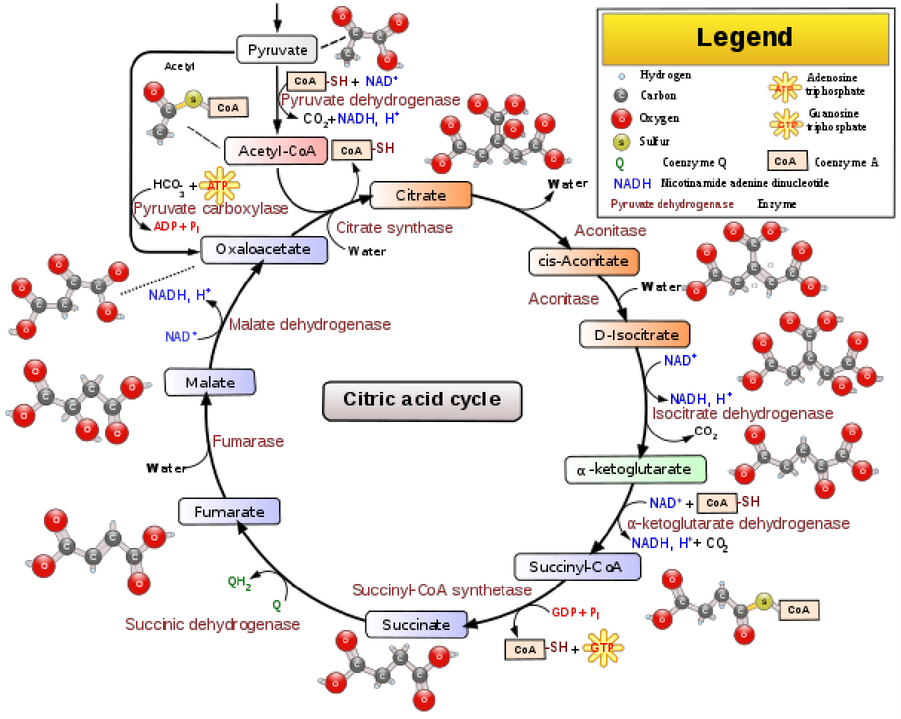


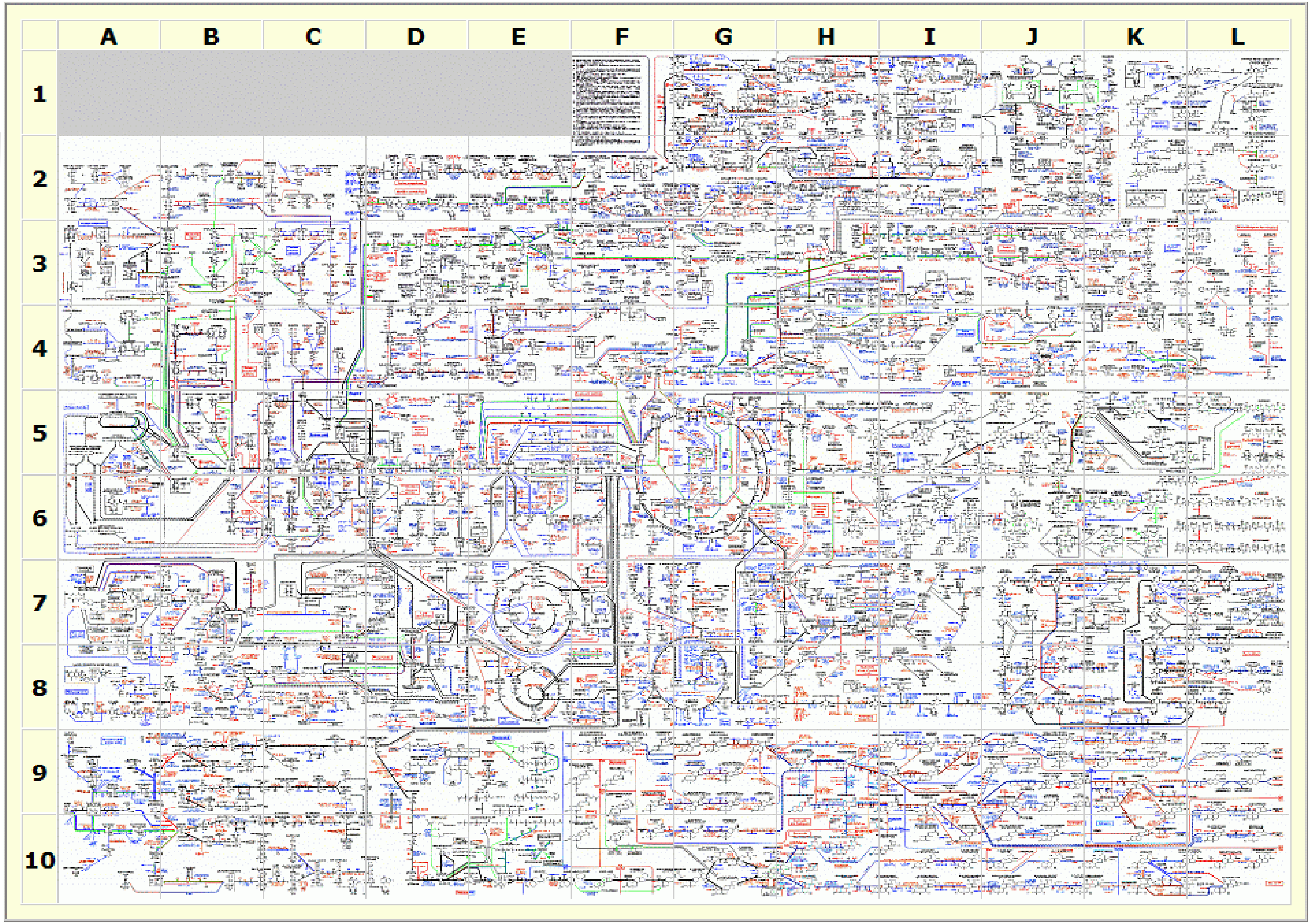
Bioinformatics – disease interaction



Source: Wiki Commons: Human disease network - CC BY-SA 4.0

Bioinformatics – molecular metabolic network

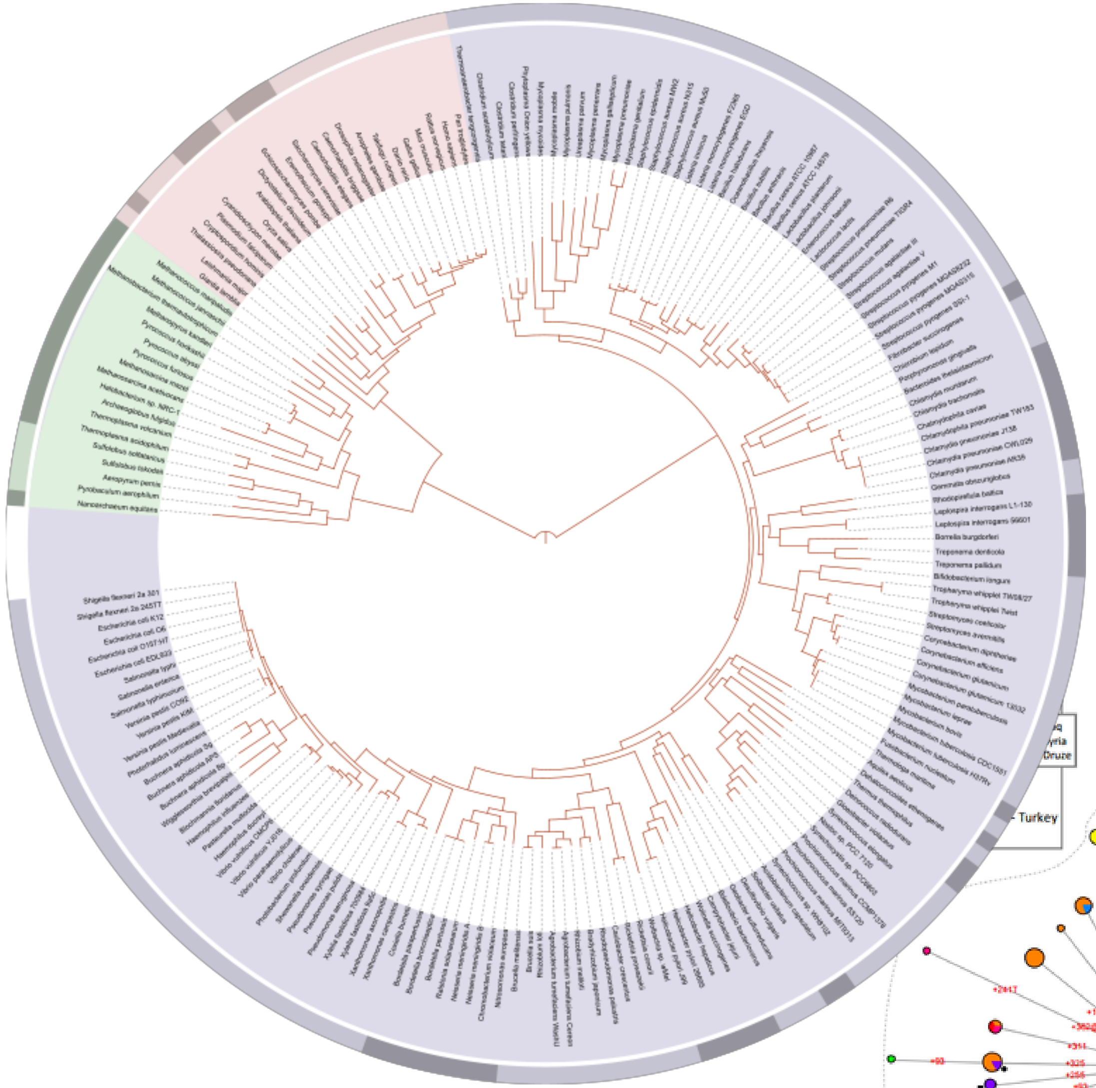


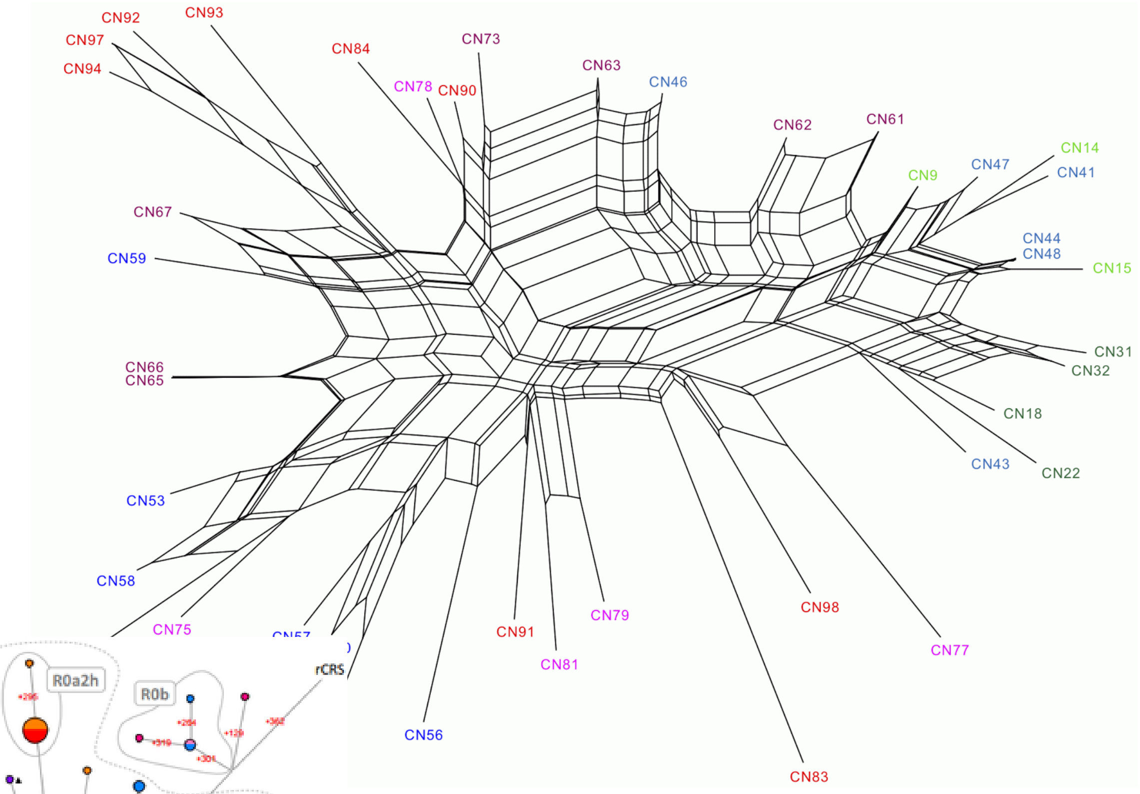


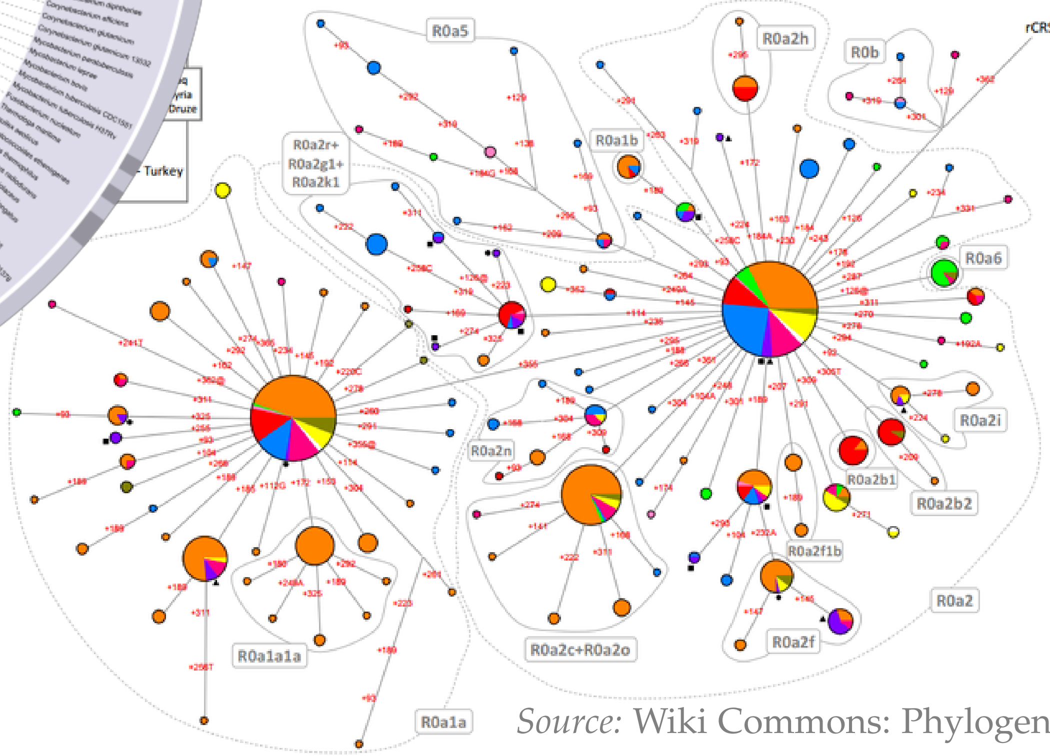
Source: Wiki Commons: Citric acid cycle with aconitate 2 - CC BY-SA 3.0

Source: Thiele et al., Nature Biotechnology 31, 419–425 (2013)

Bioinformatics – phylogenetic trees & networks

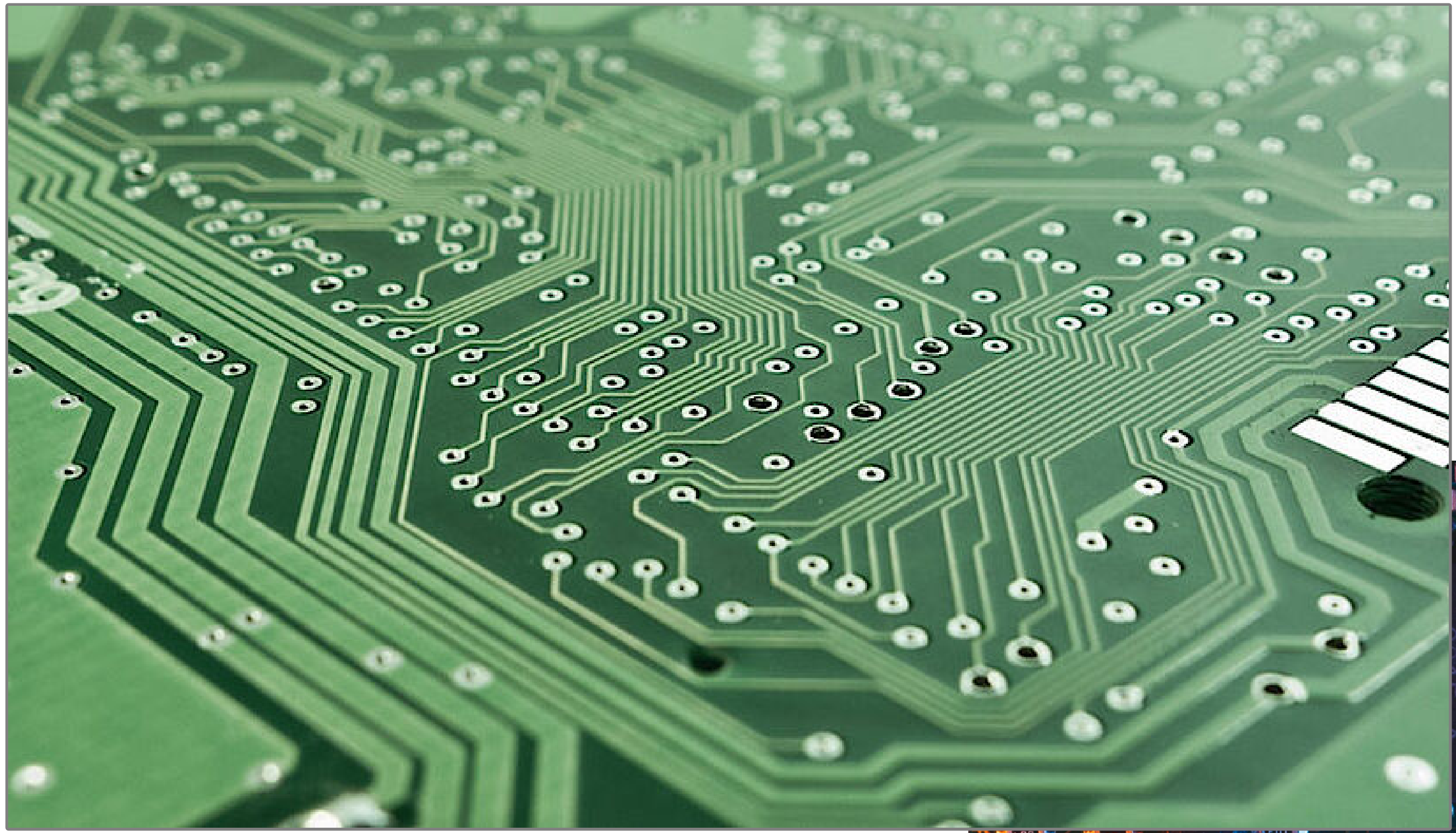




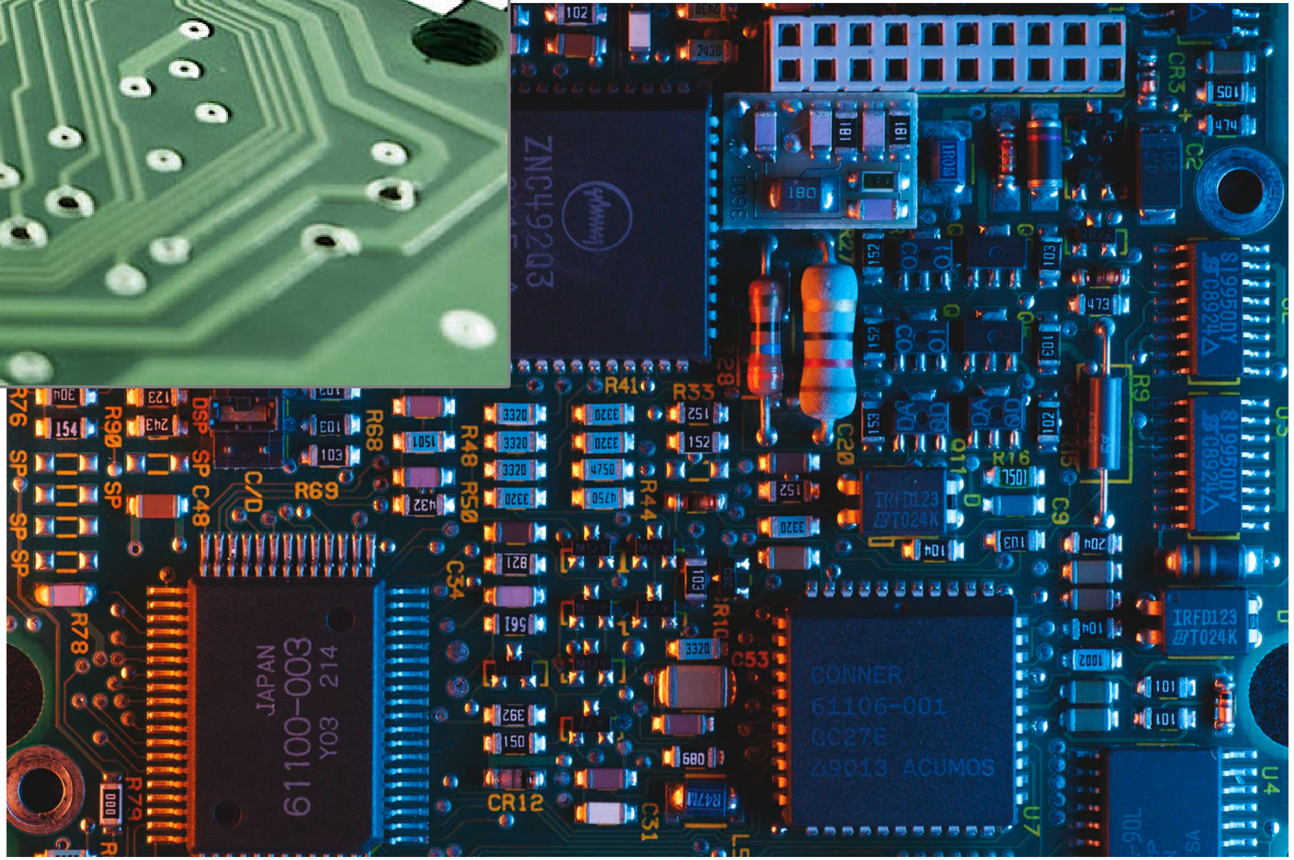


Source: Wiki Commons: Phylogenetic network of HVS-I variation - CC BY 4.0

Technical network – very large-scale integration (VLSI)

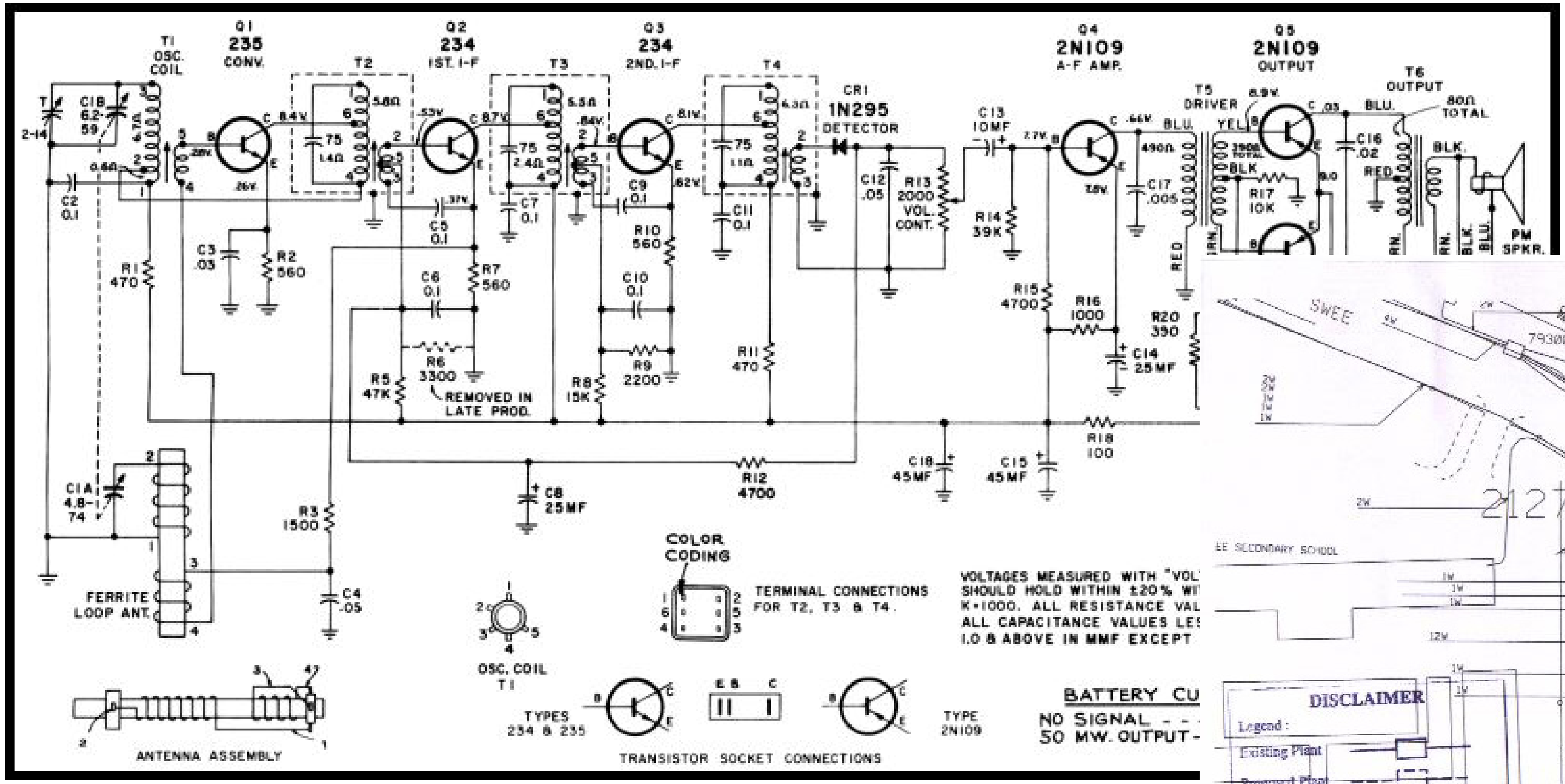


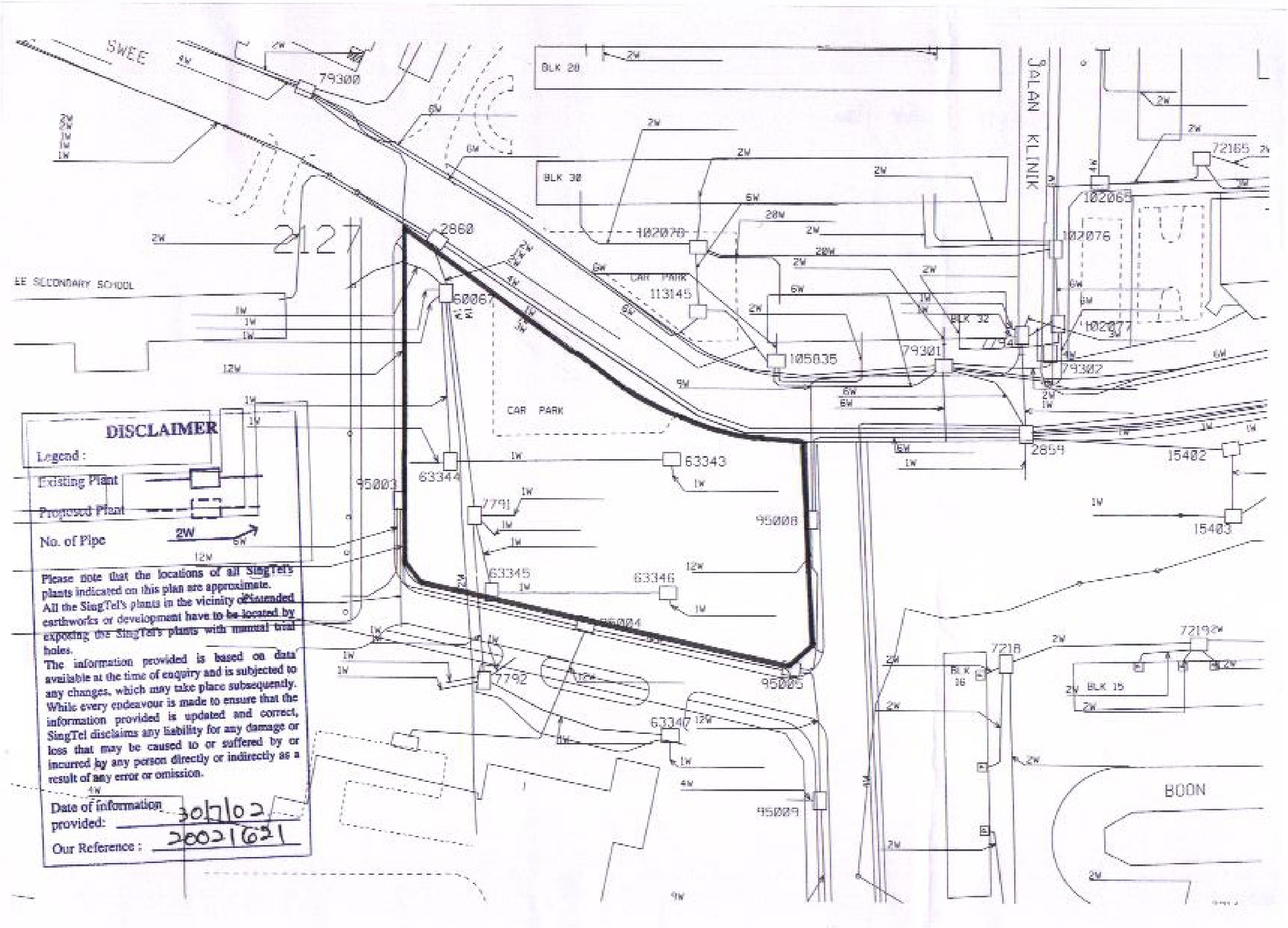
Source: Pixabay



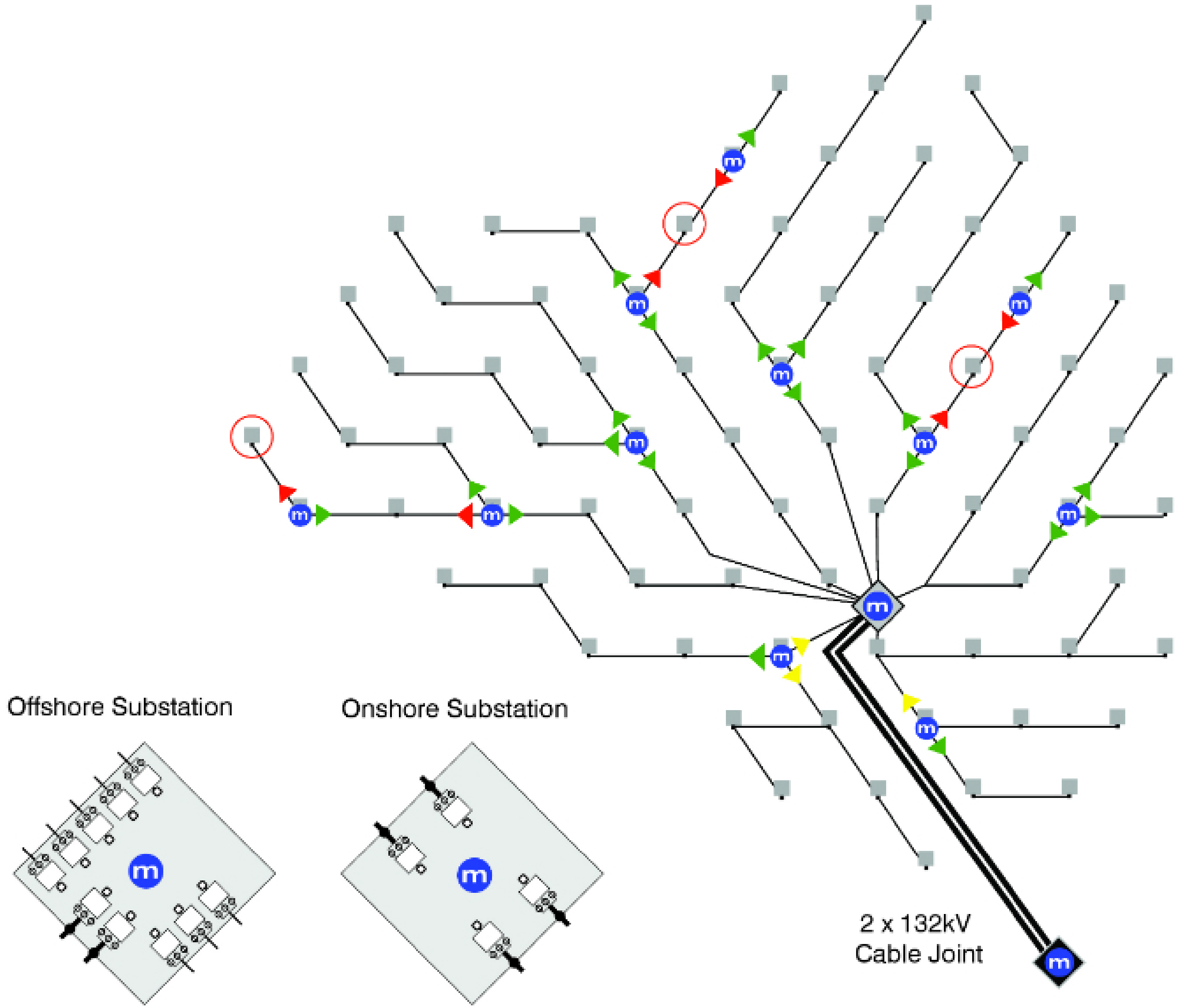
Source: Wiki Commons: Diopsis - CC BY-SA 3.0

Technical network – transistor diagram, wiring



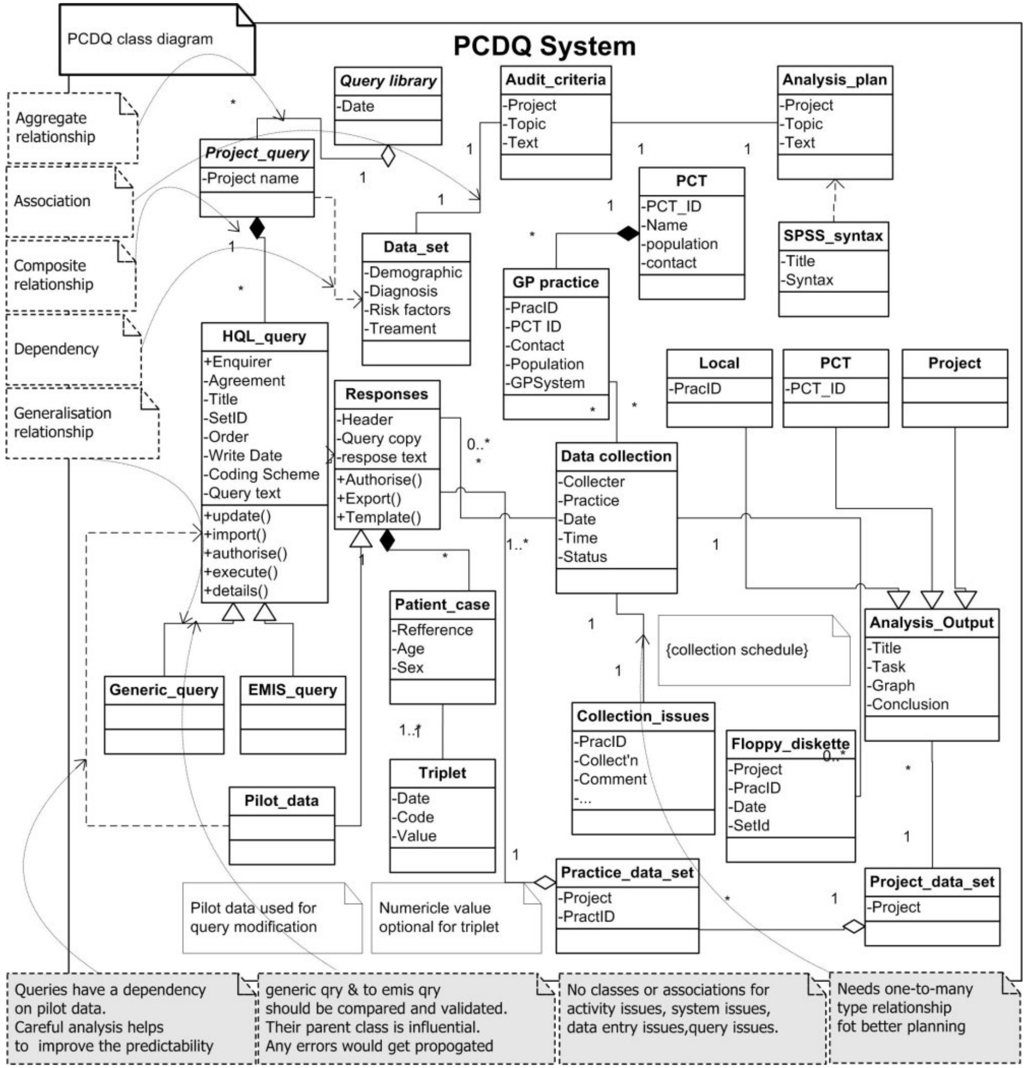


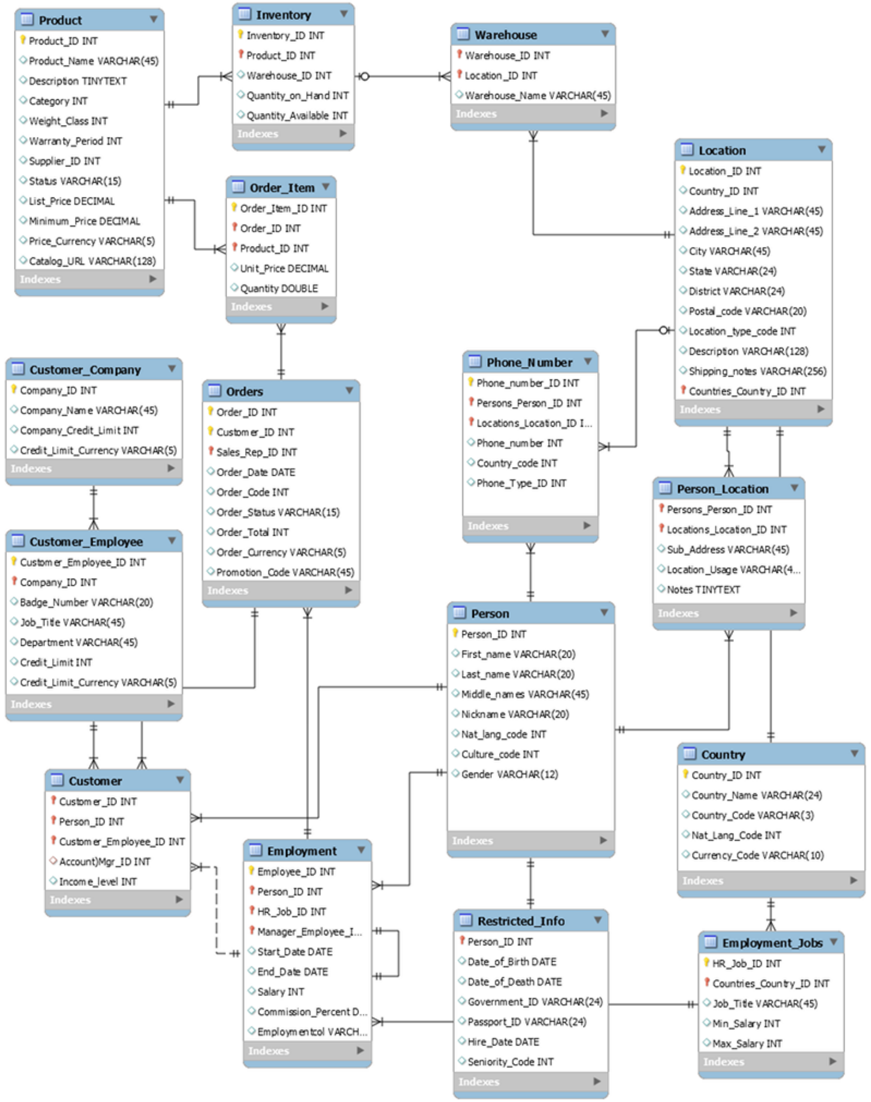
Technical networks – offshore wind farms



Source: Wiki Commons: Alpha Ventus Windmills - CC BY-SA 3.0

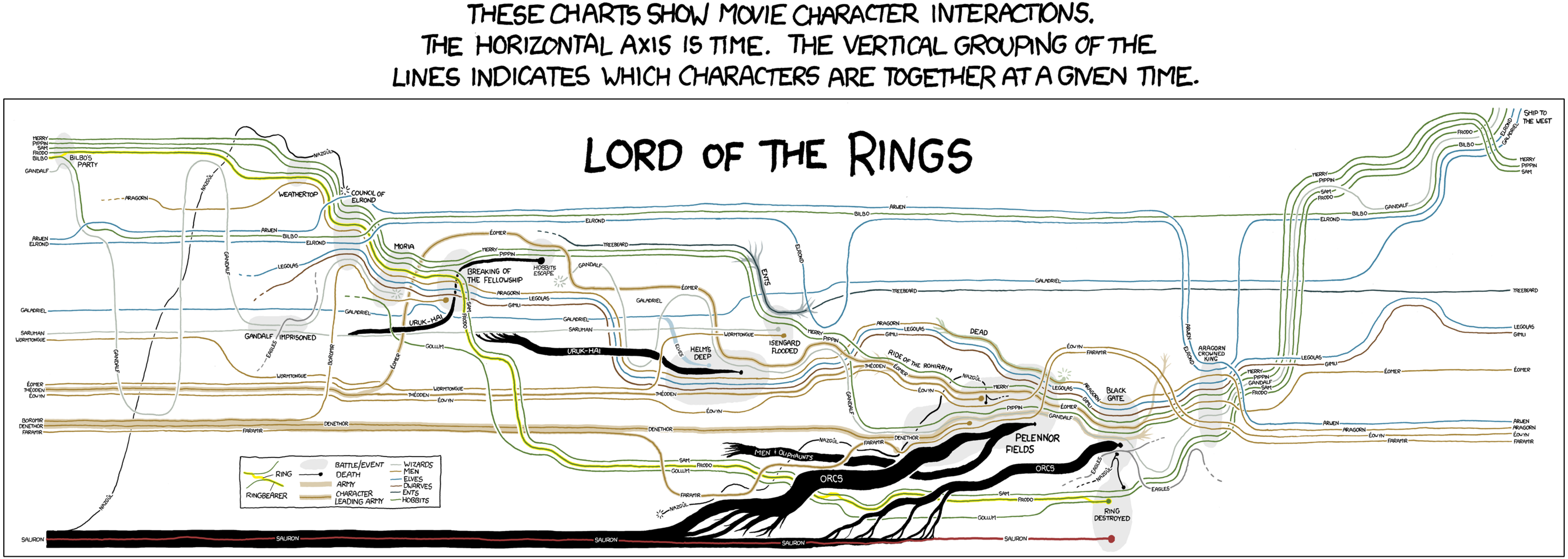
Technical network – UML diagram





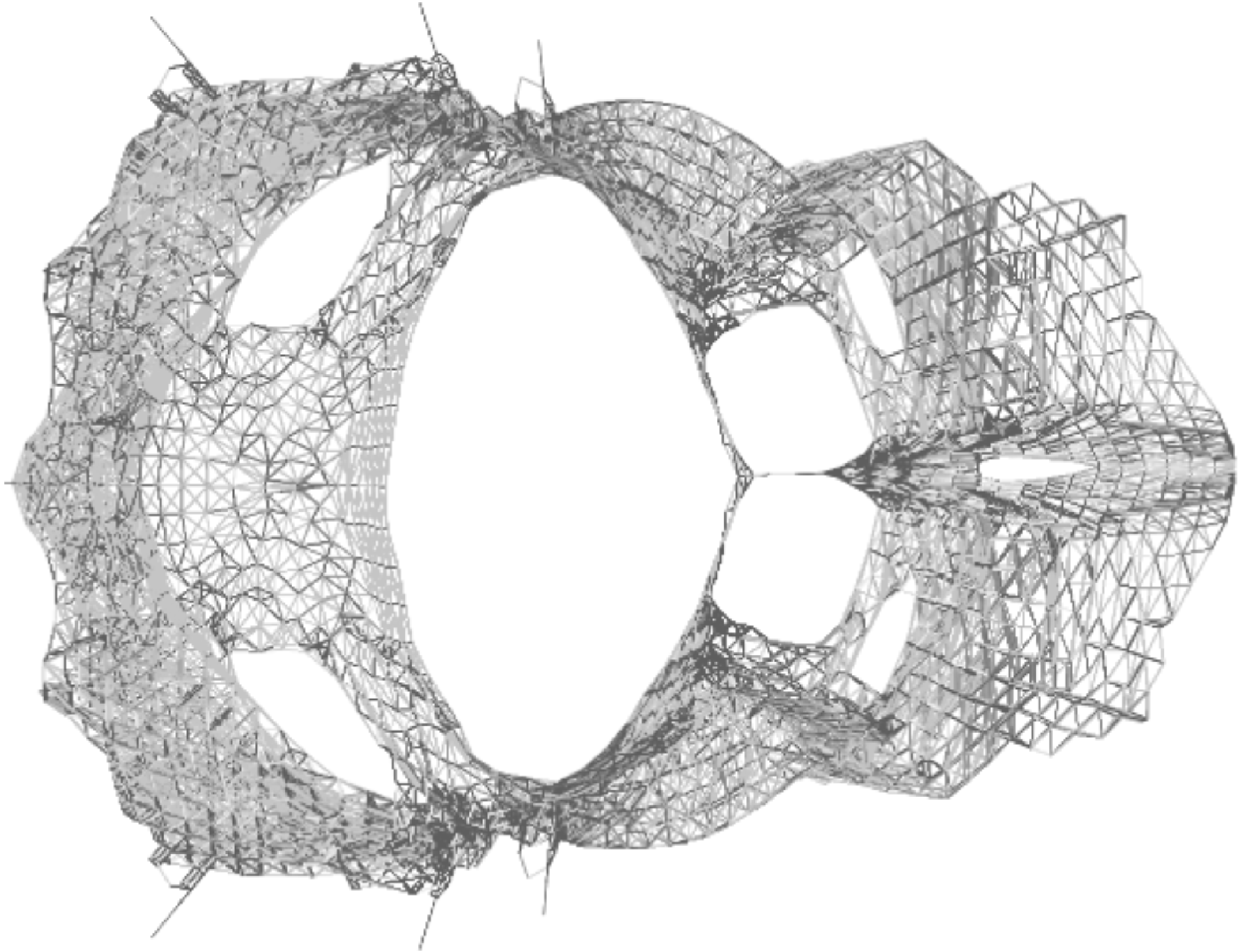
© AWS

Temporal graph layout – storylines

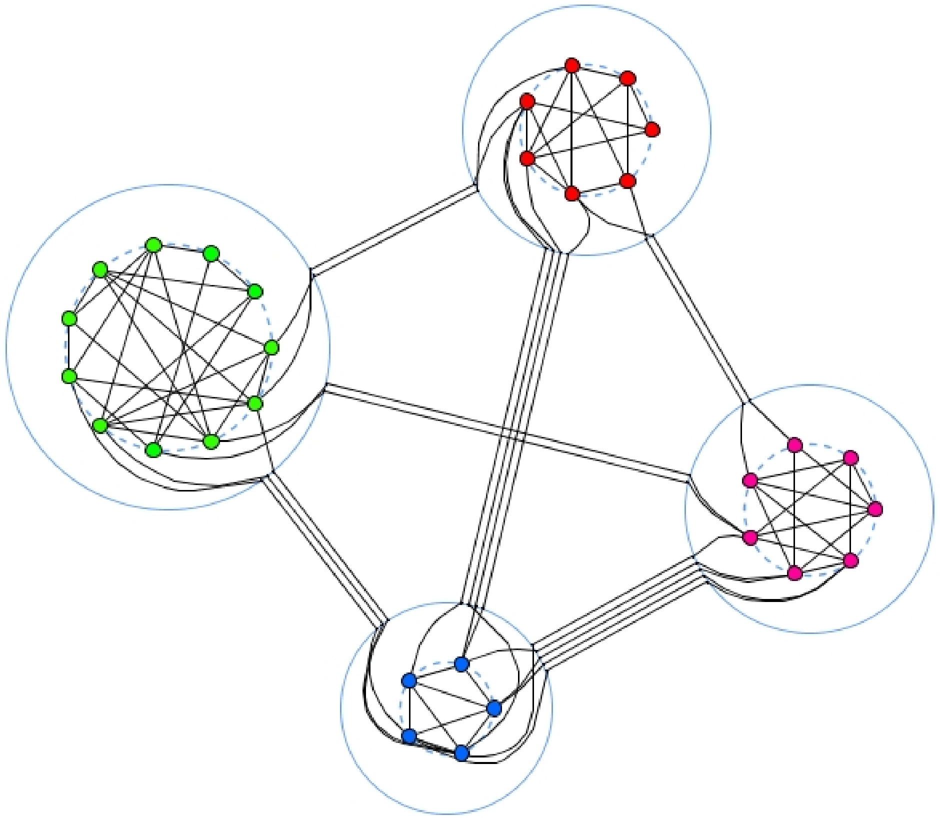


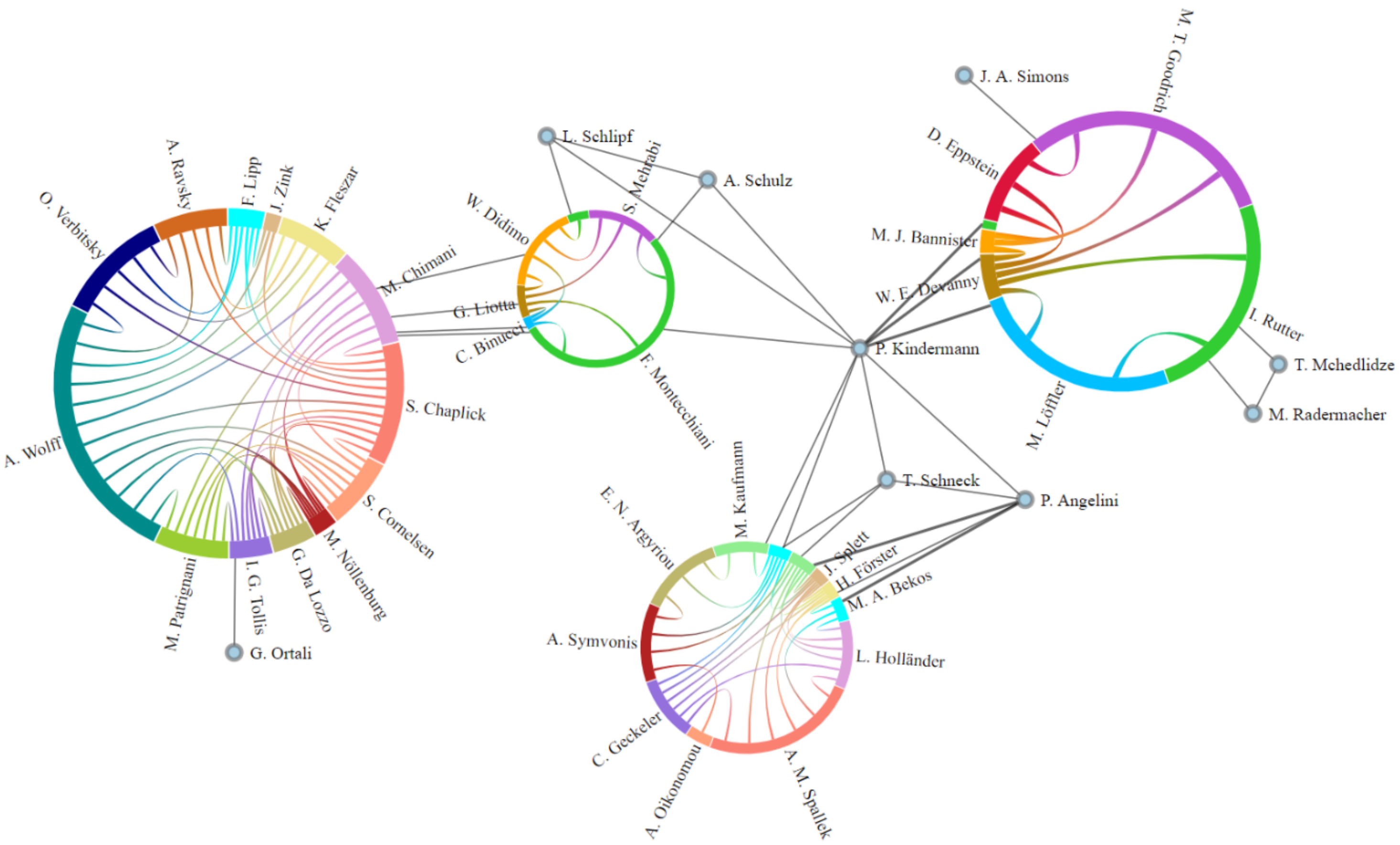
Source and more: xkcd Comic 657 – xkcd.com/657/

Large graphs – object mesh



General graphs – micro-macro layout

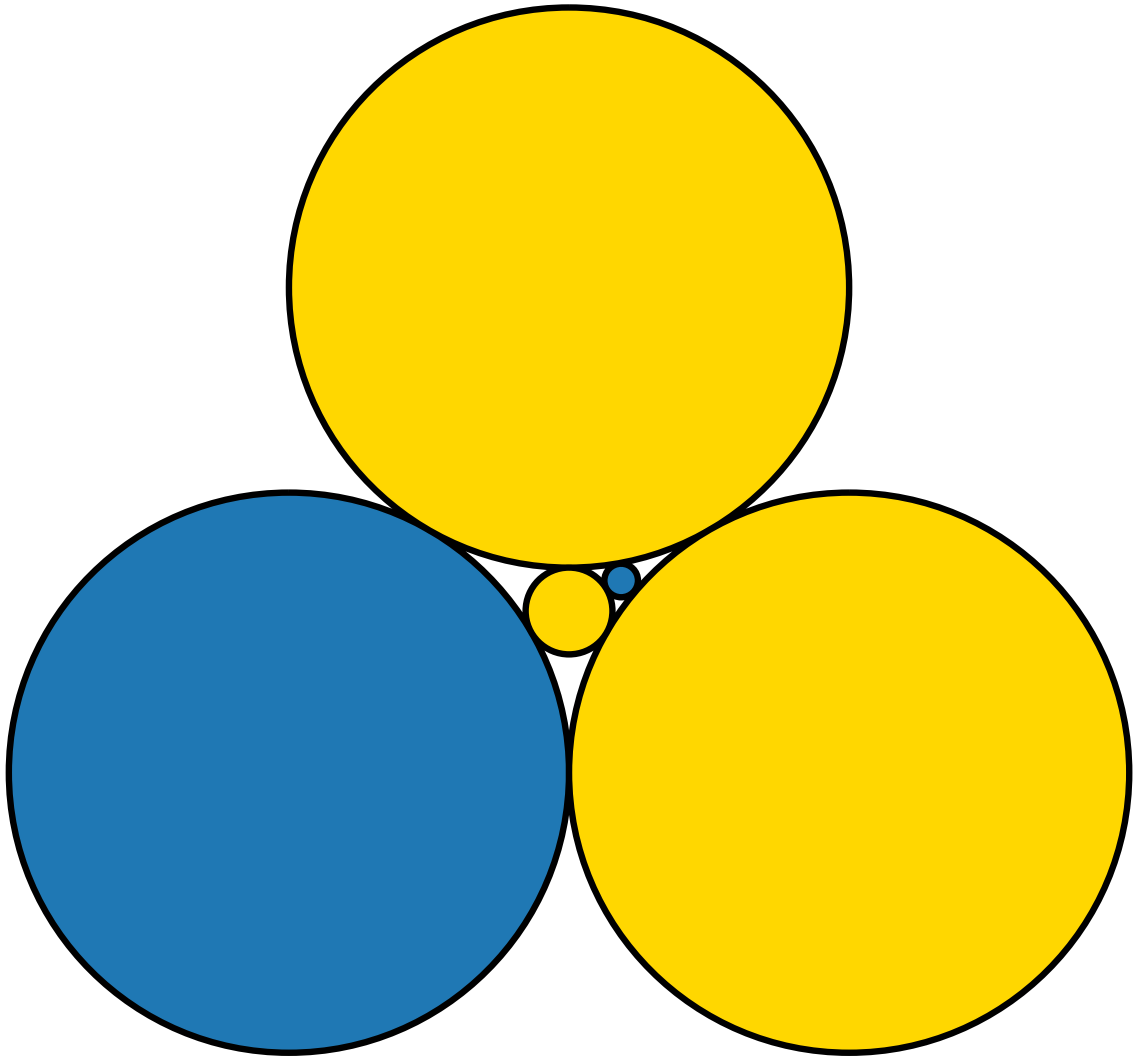


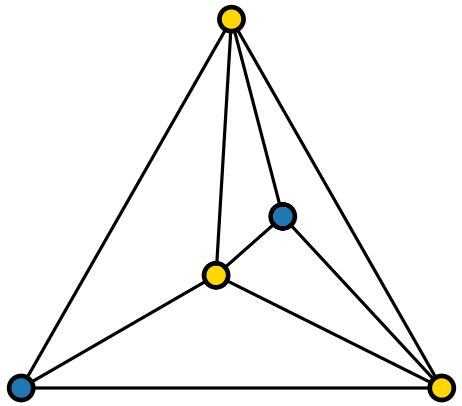


Source: Angori et al., ChordLink: A New Hybrid Visualization Model, GD’19 (2019)

Alternative representations – treemap

Alternative representations – contact graphs





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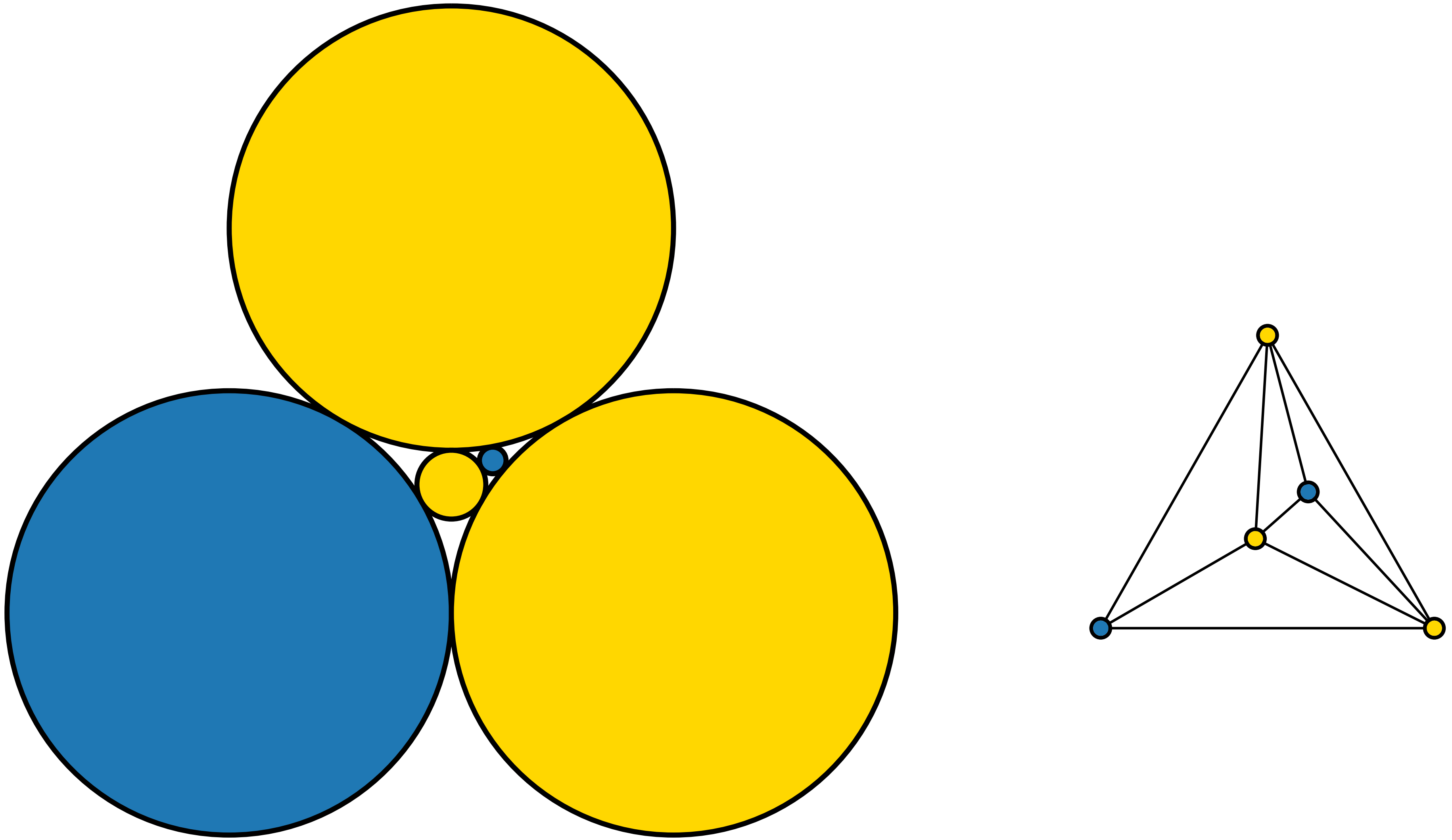
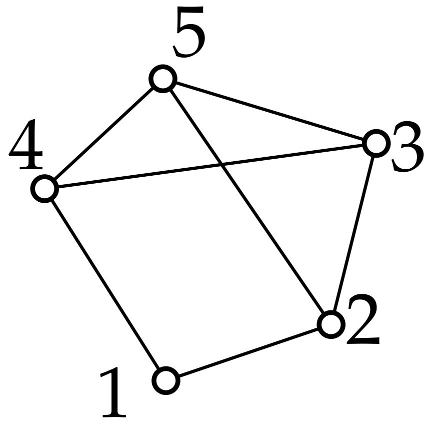


- For more examples see `visualcomplexity.com`

Requirements of a graph layout

1. Drawing conventions and requirements, e.g.,

Requirements of a graph layout



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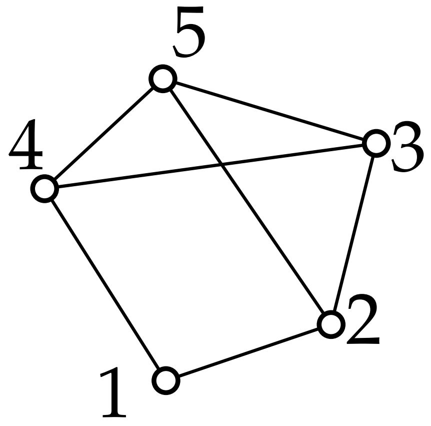

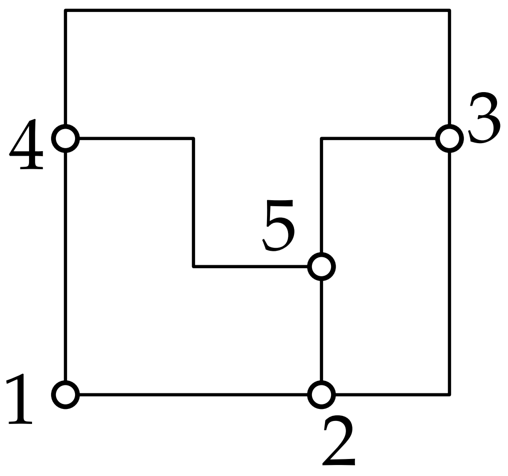

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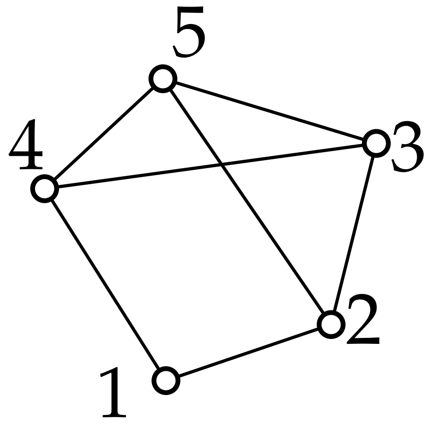

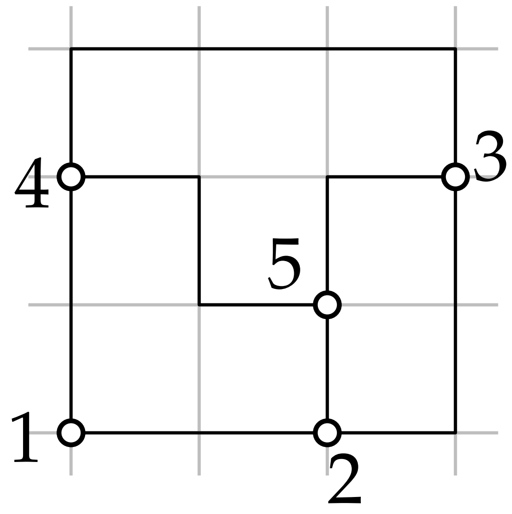

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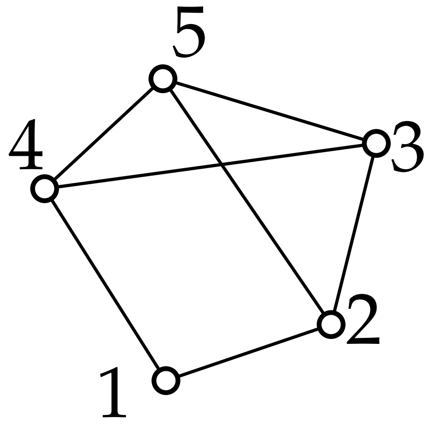

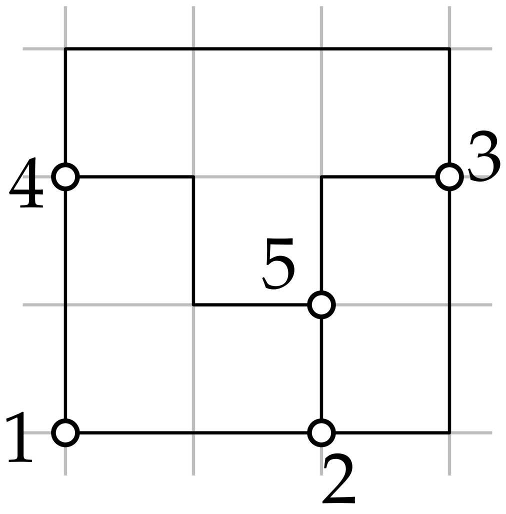

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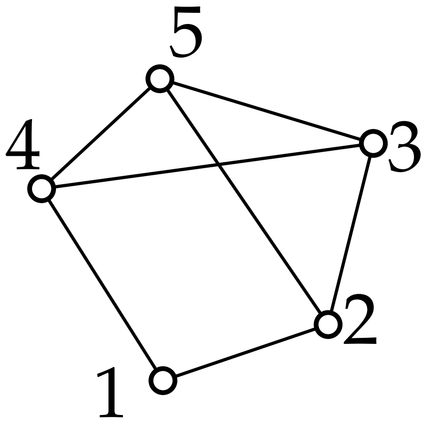

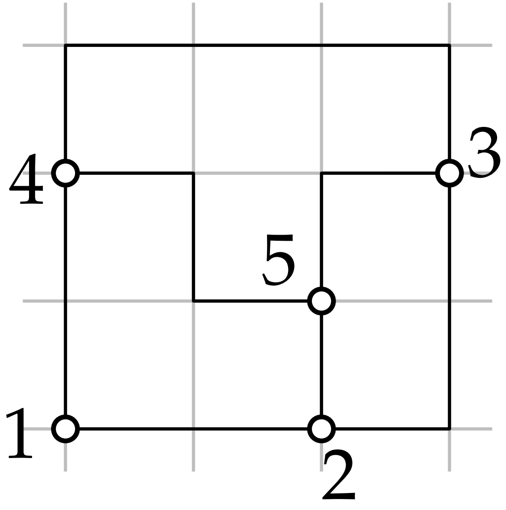

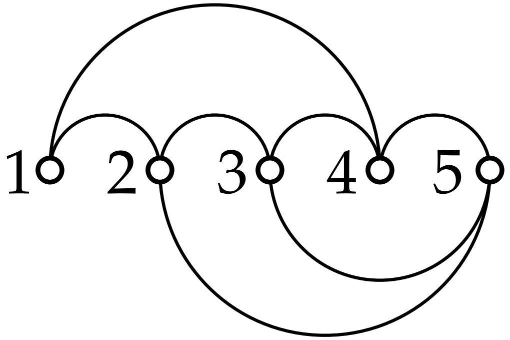

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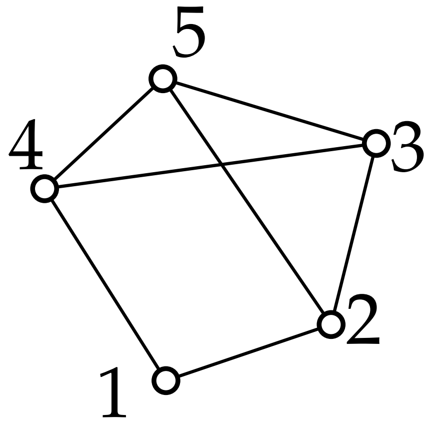

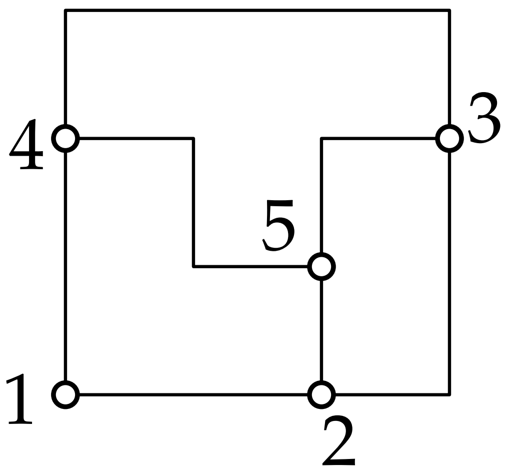

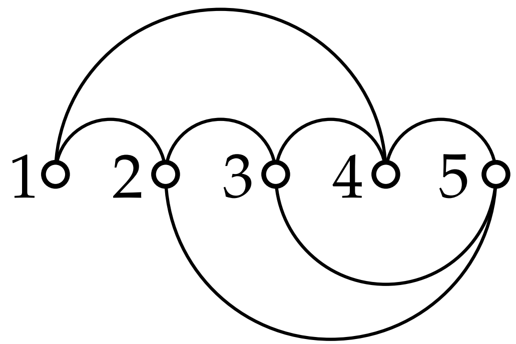

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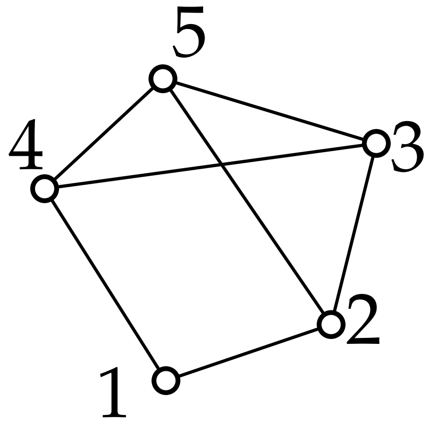

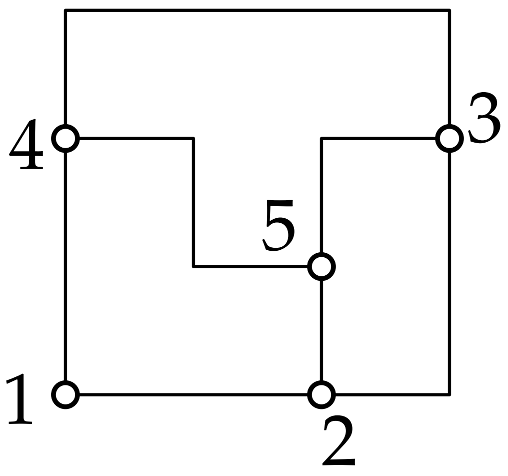

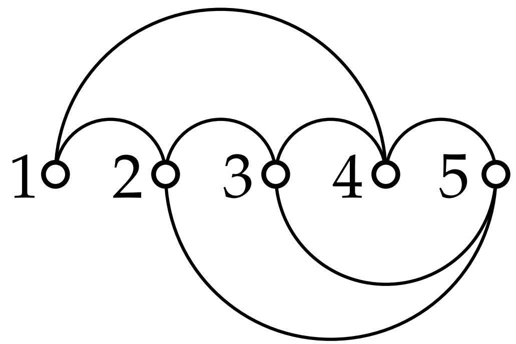

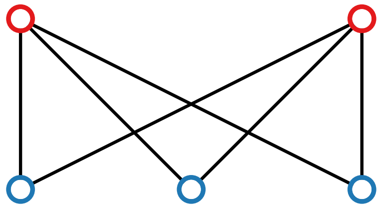

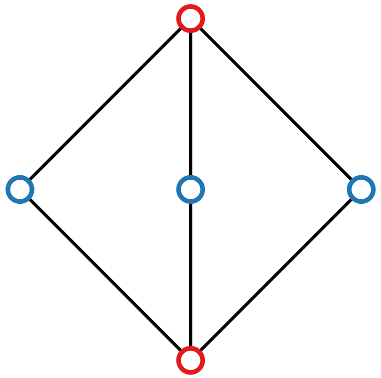

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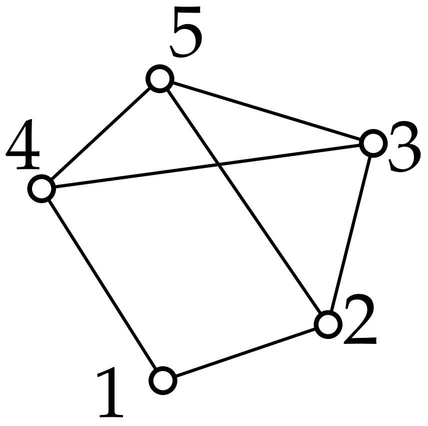

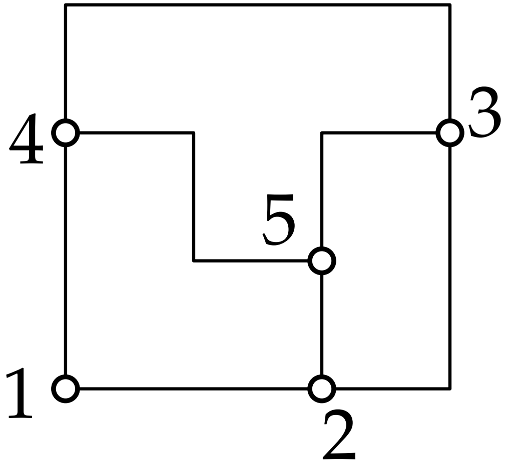

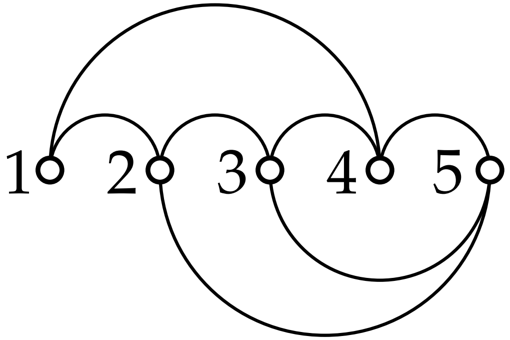

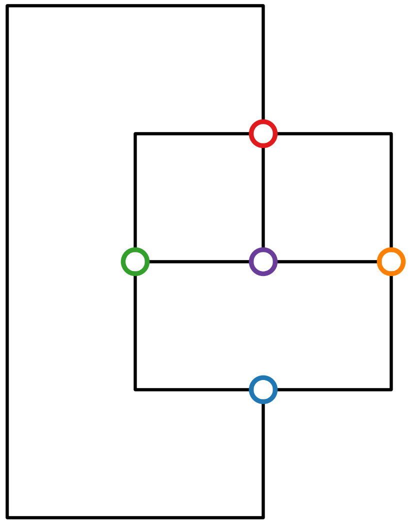

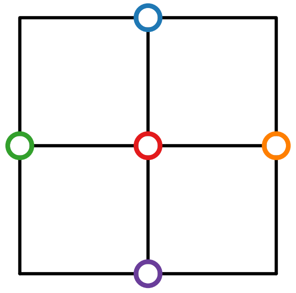

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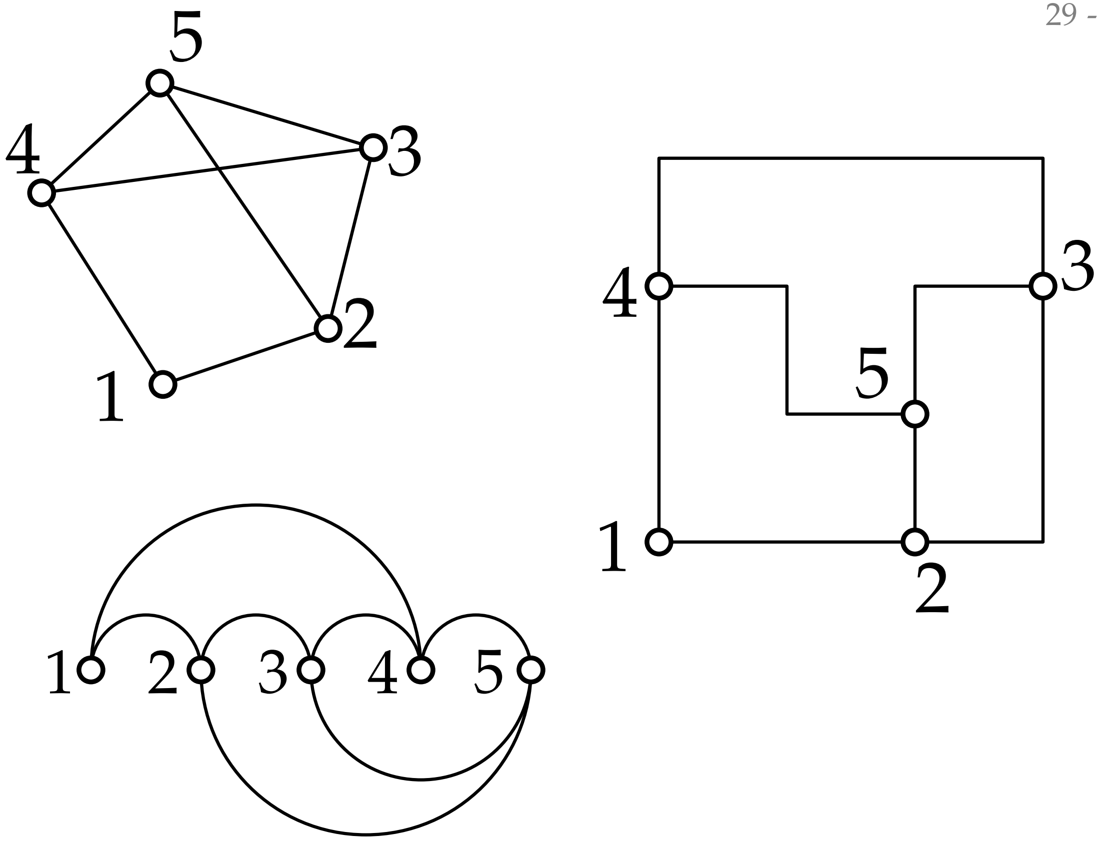

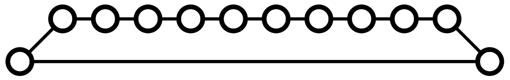

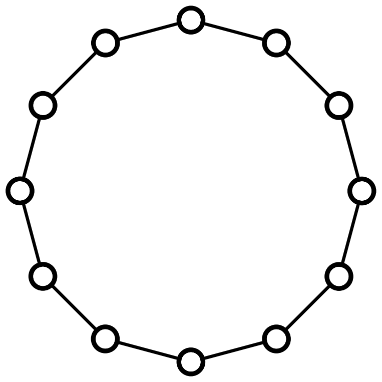

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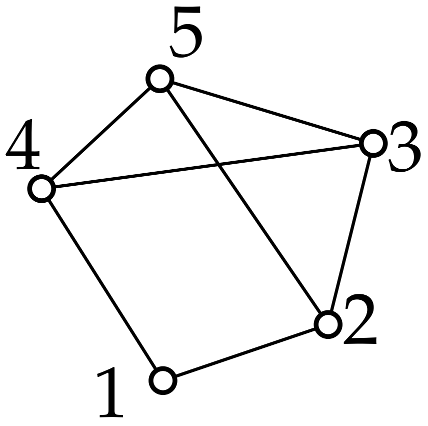
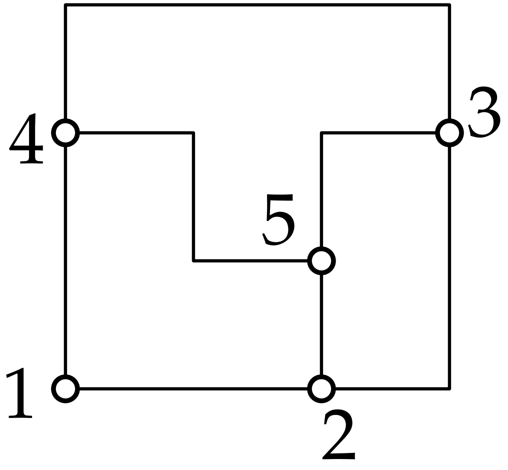
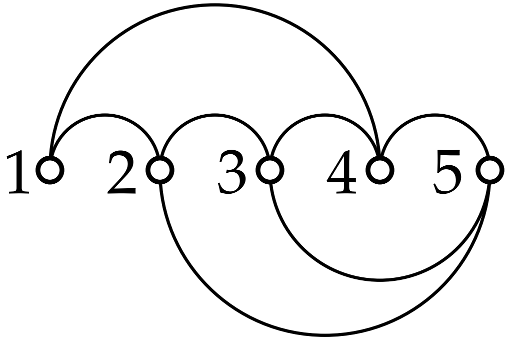
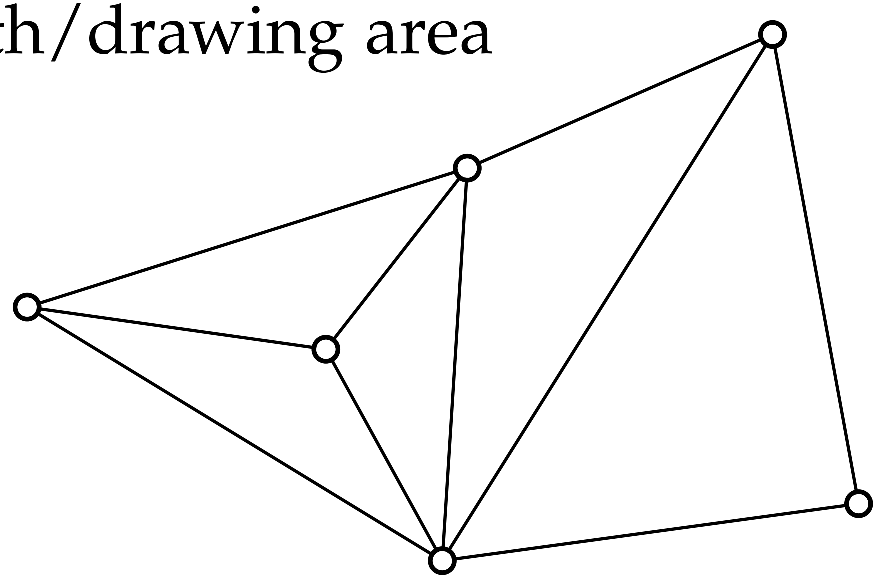
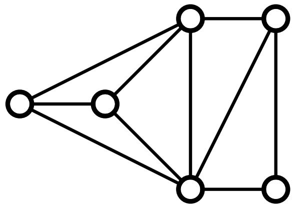
   - edge length uniformity
   - minimizing total edge length/drawing area











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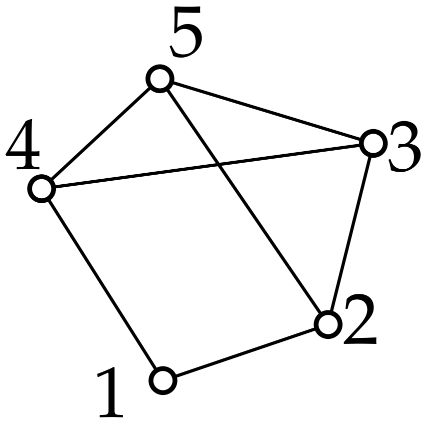

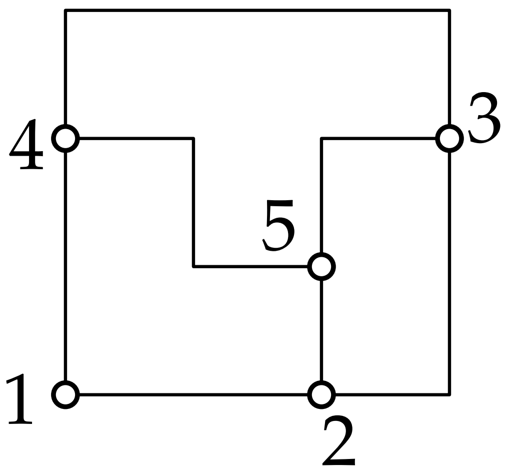

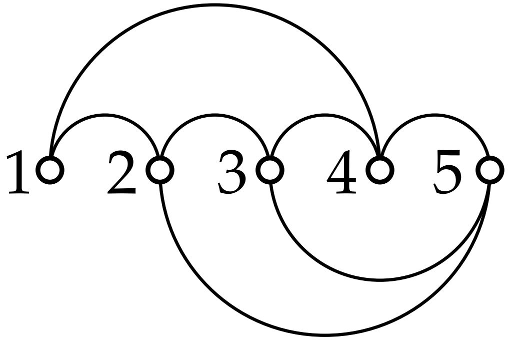

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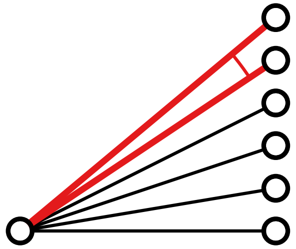

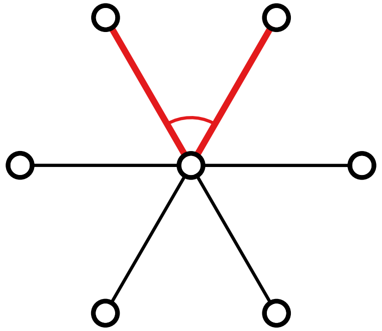

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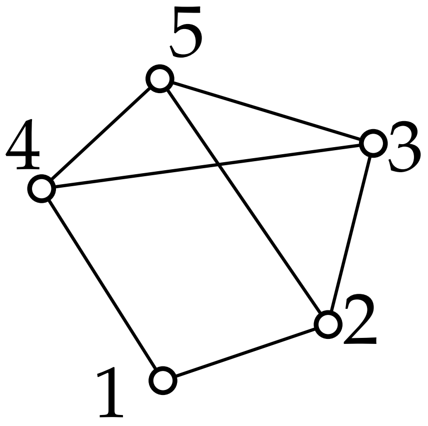

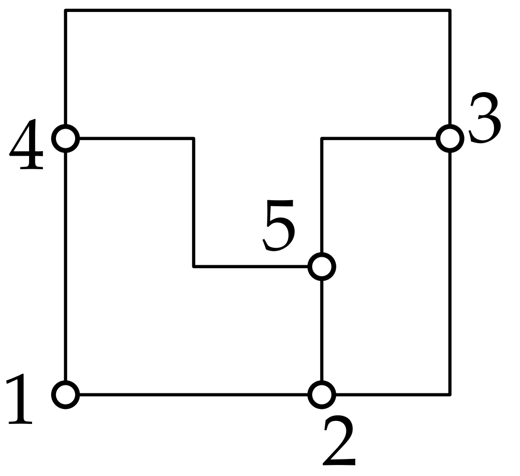

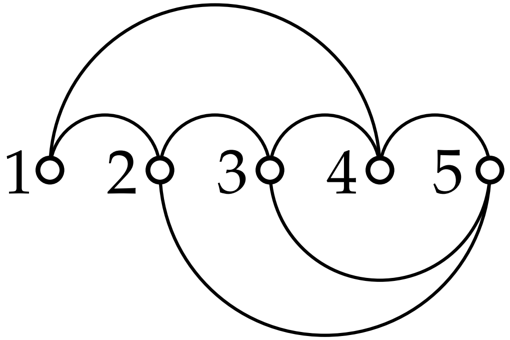

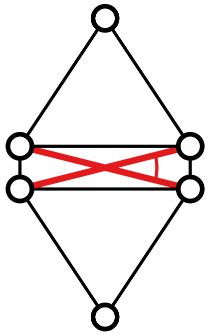

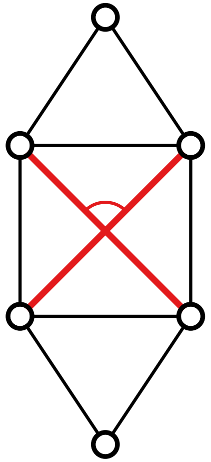

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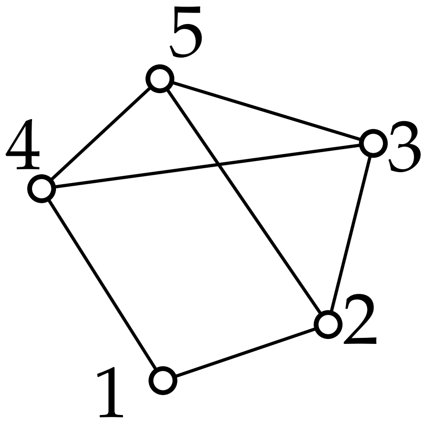

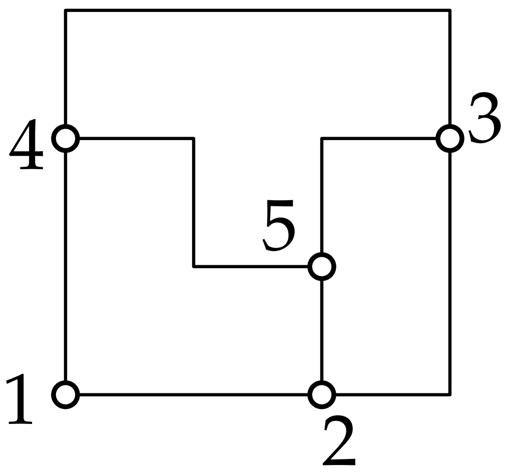

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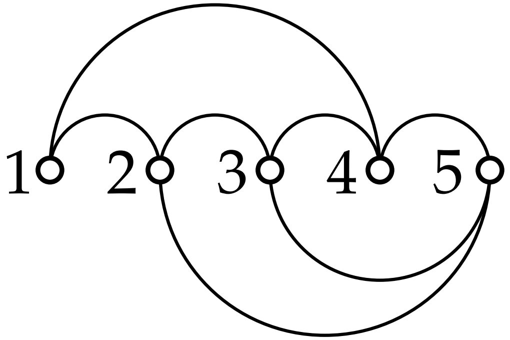

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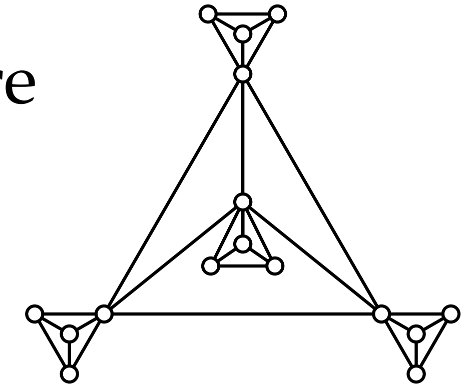

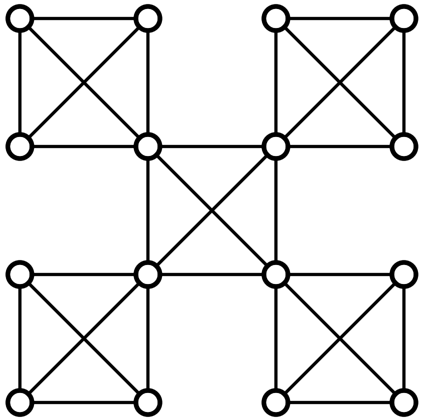

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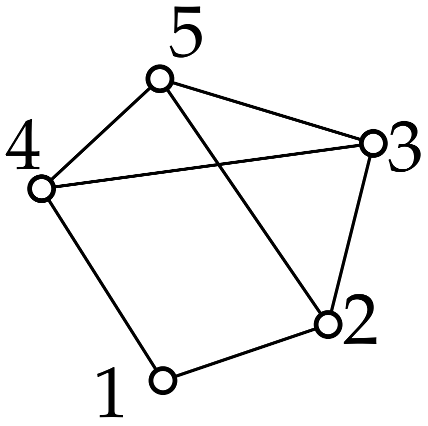

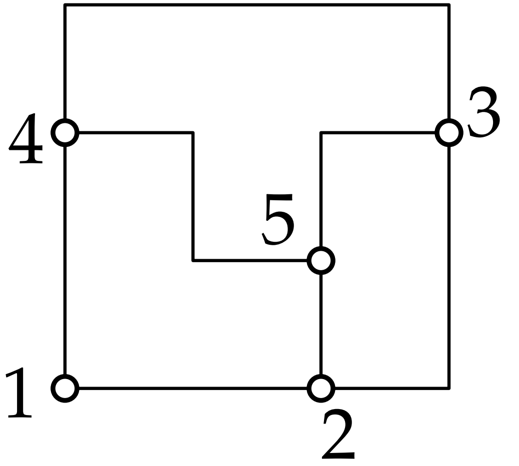

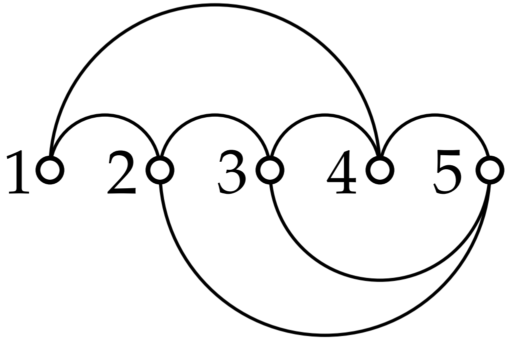

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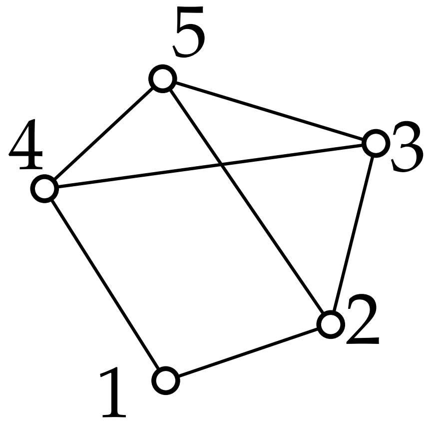

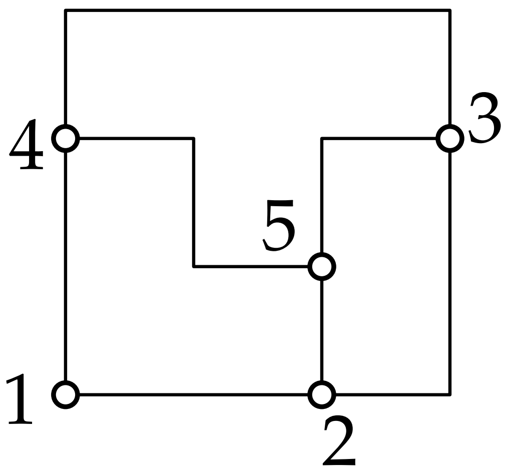

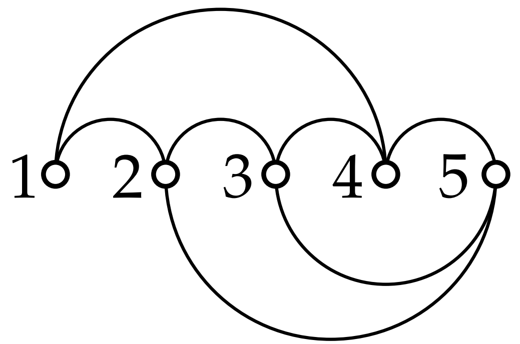

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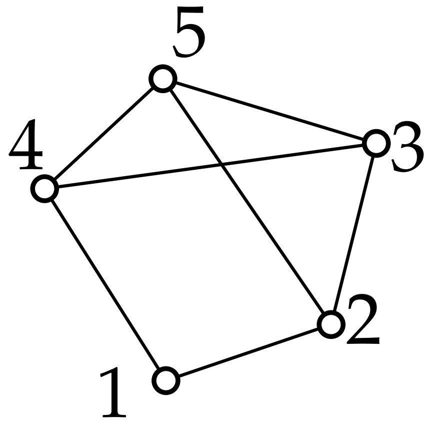

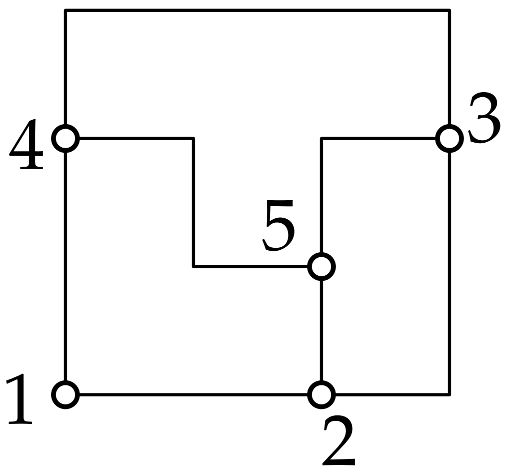

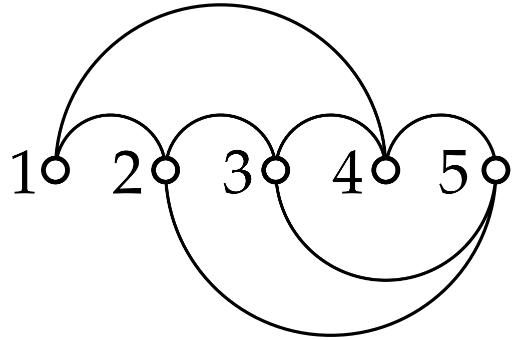

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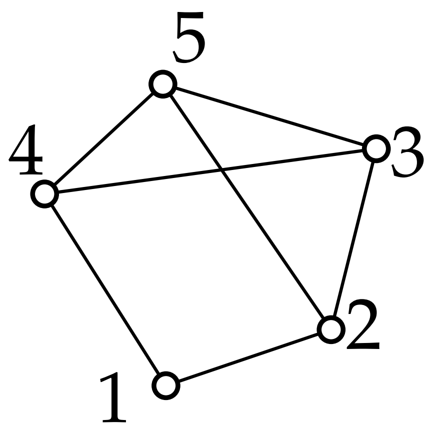
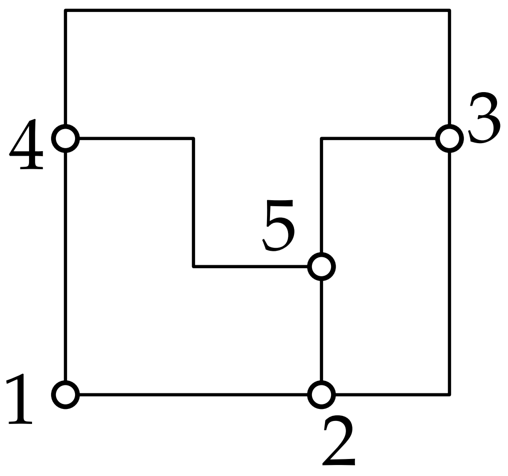
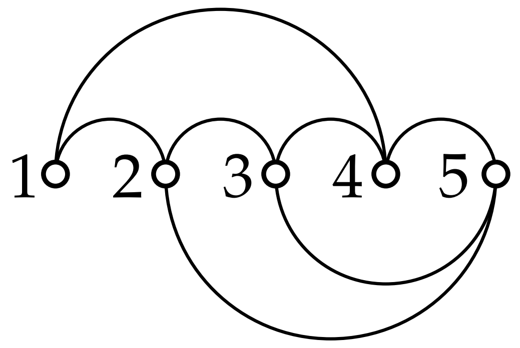
3. Local Constraints, e.g.
   - restrictions on neighboring vertices (e.g., “upward”).
   - restrictions on groups of vertices/edges (e.g., “clustered”).







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The layout problem

Graph Visualization Problem

in: Graph $G = (V, E)$

out: Drawing Γ of G such that

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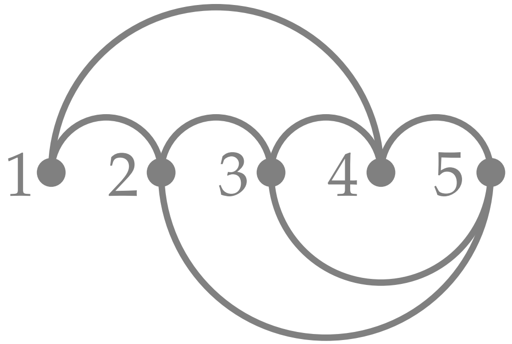
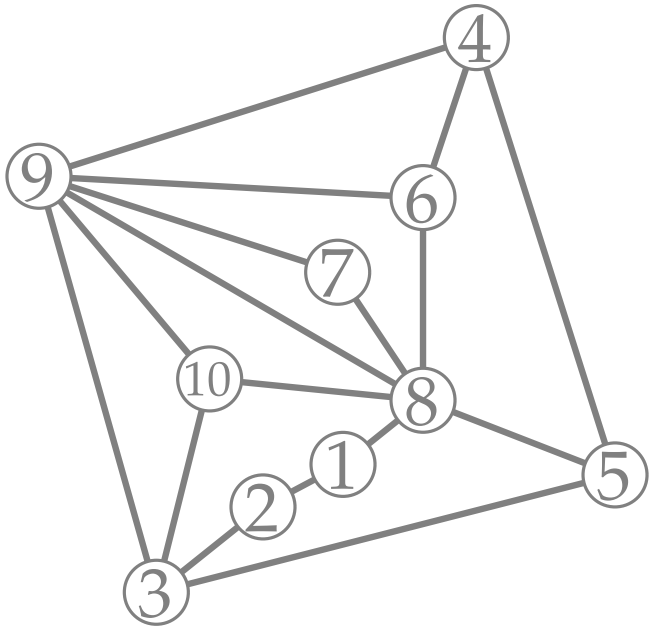
Universität Trier

Visualization of Graphs



Lecture 1:
The Graph Visualization Problem

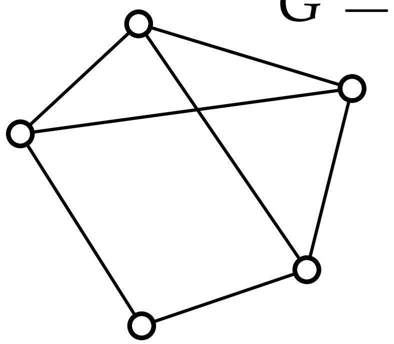


Part III:
Basics

Philipp Kindermann

Summer Semester 2021

Basic Definitions



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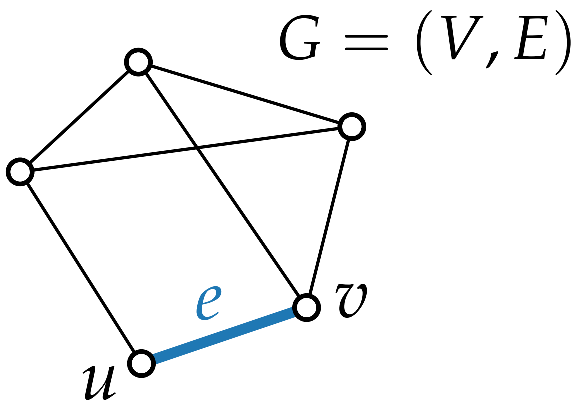
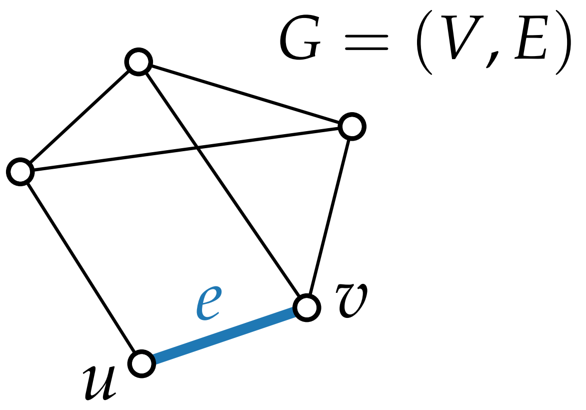


Edge $e = \{u, v\} \in E$:

Basic Definitions



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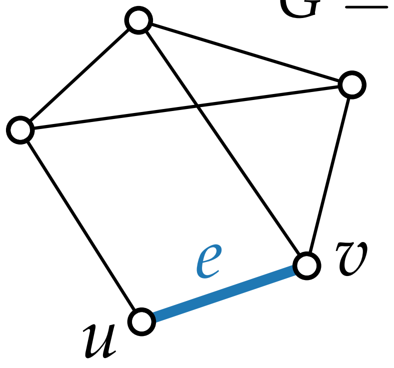

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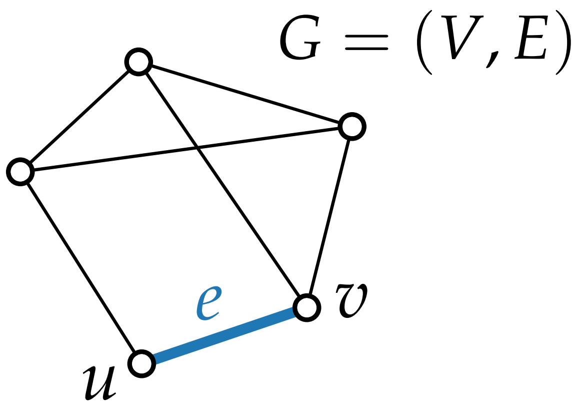

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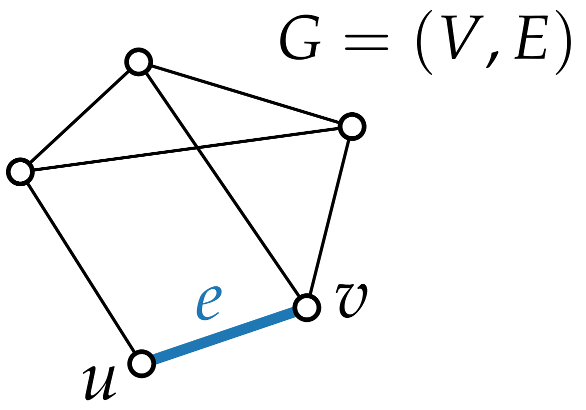

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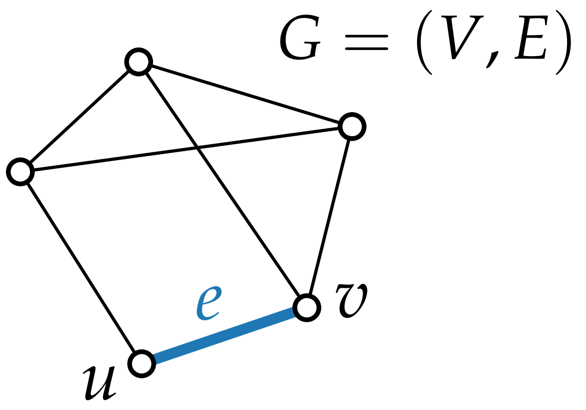

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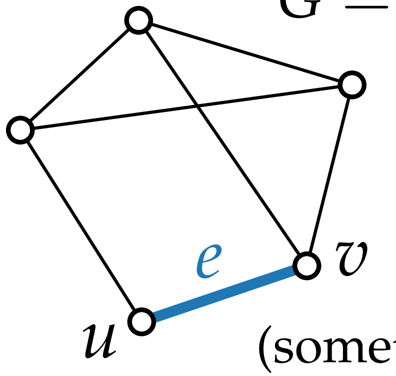

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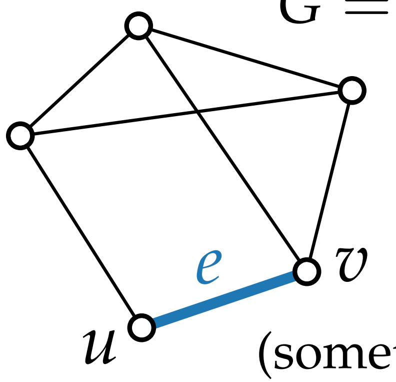

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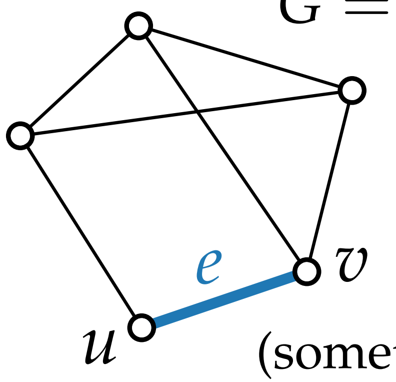

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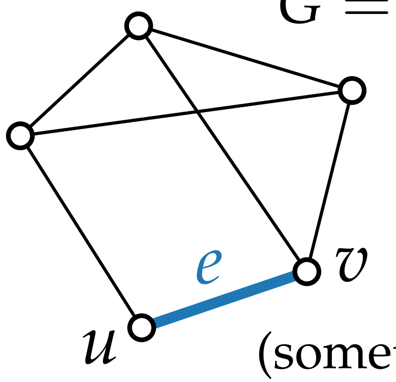

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Corollary.
The number of odd-degree vertices is even.

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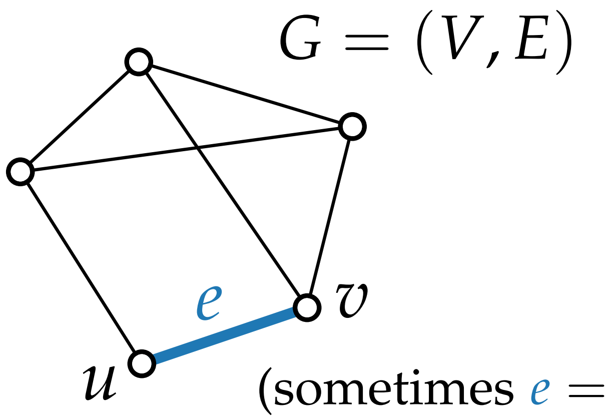

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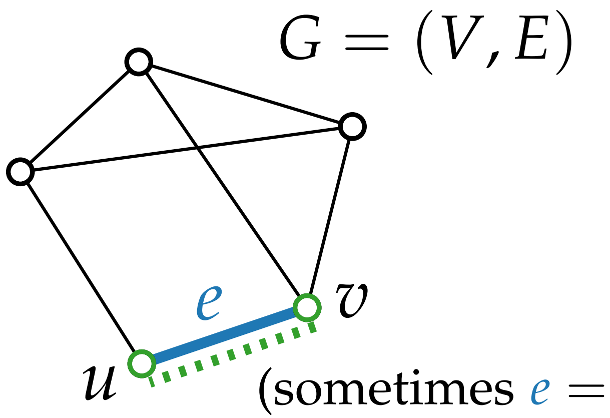

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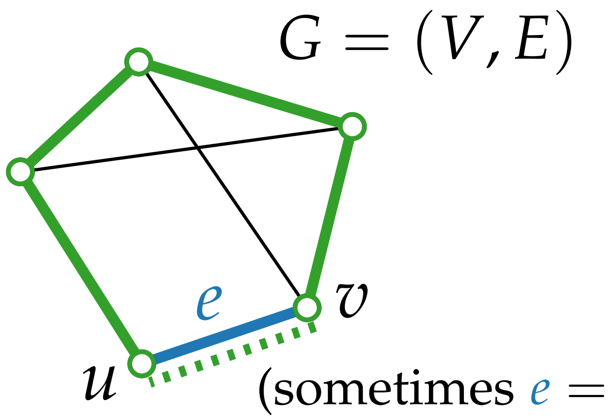

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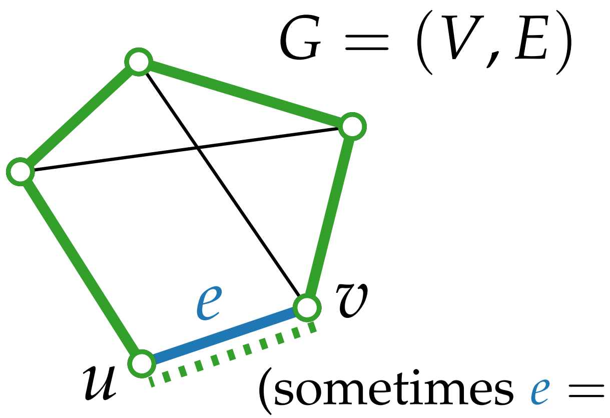

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simple cycle: u-u-path

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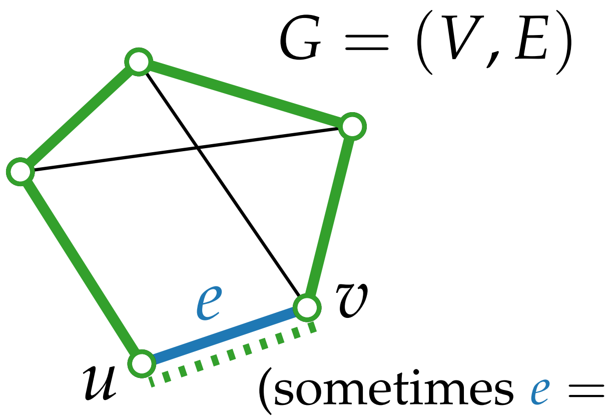

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Sequence of $\ell + 1$ distinct adjacent vertices (and ℓ connecting edges), starting with u and ending with v:
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simple cycle: u-u-path

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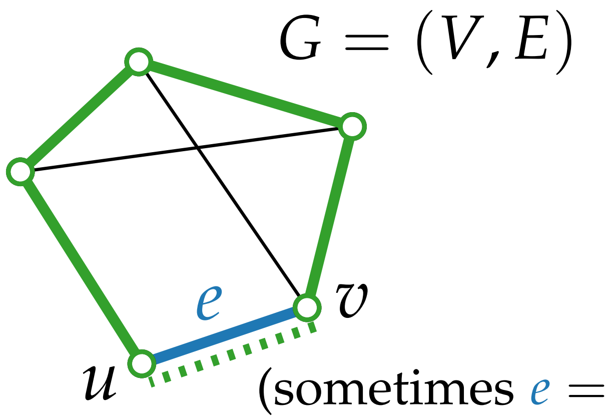

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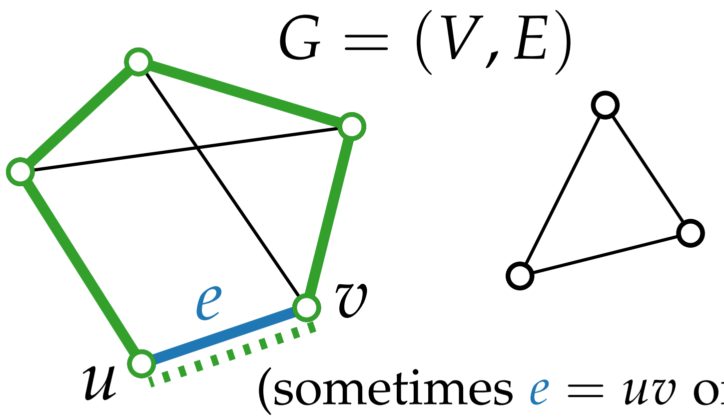

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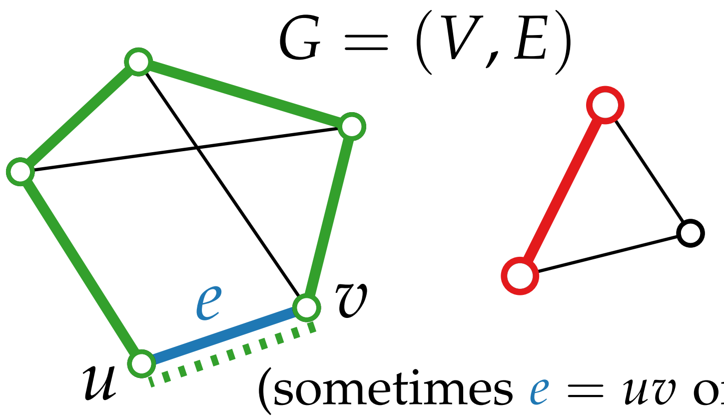

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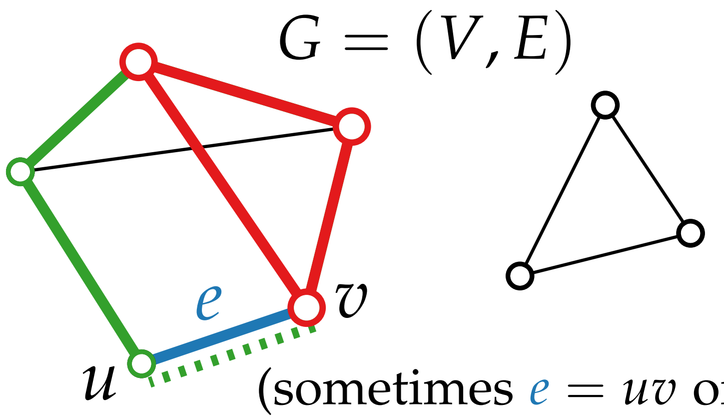

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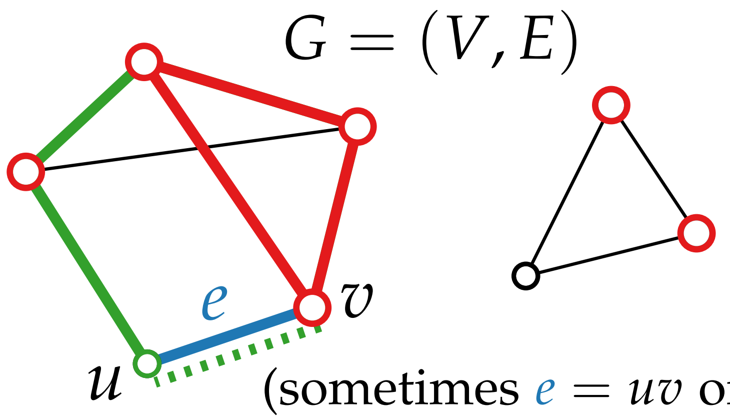

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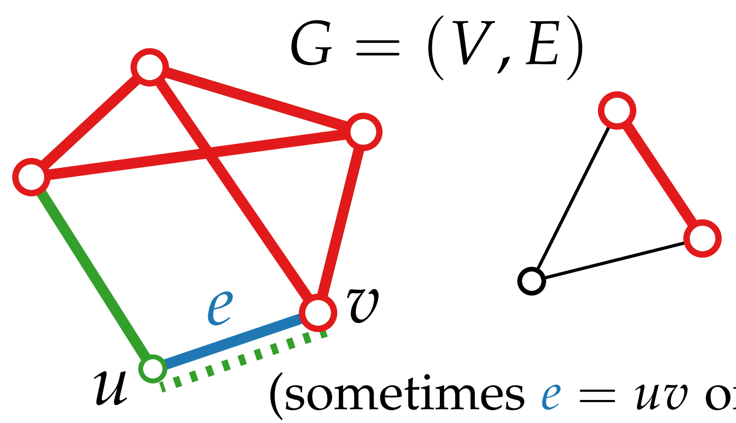

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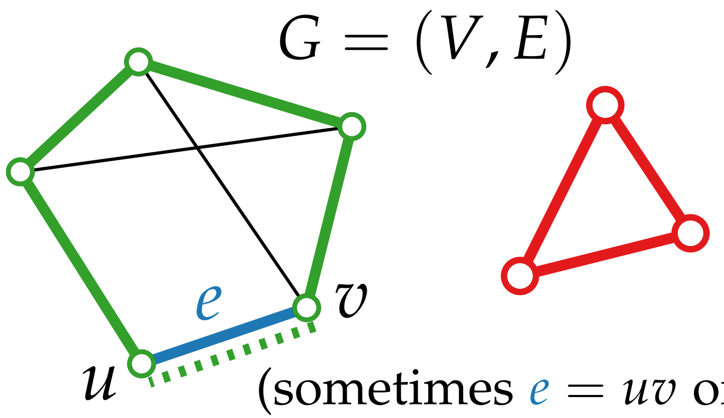

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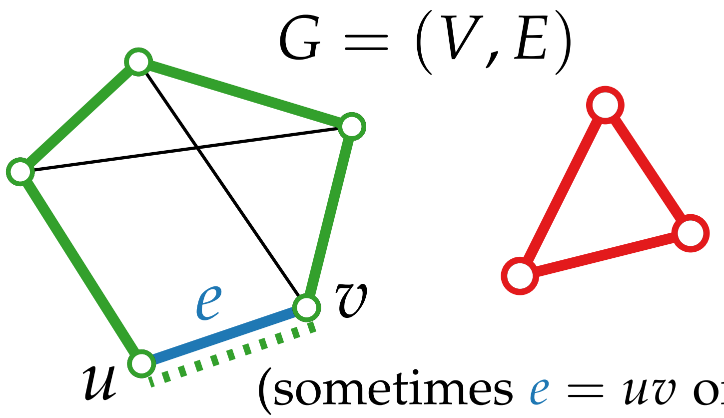

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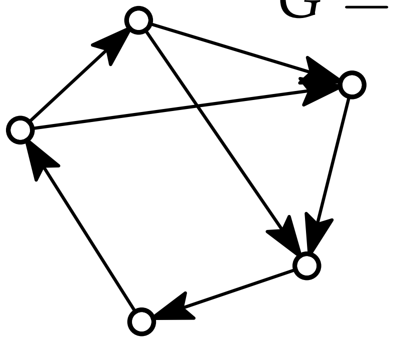
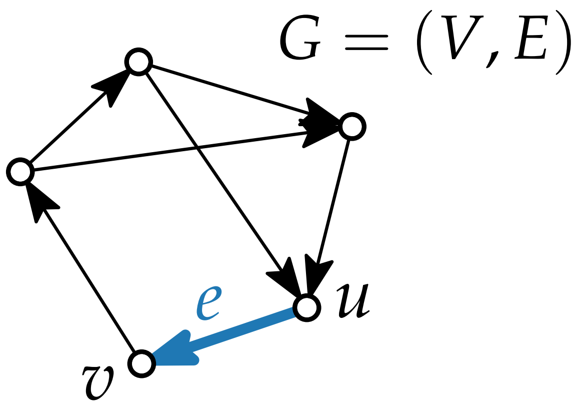


$G = (V, E)$

Directed Graphs



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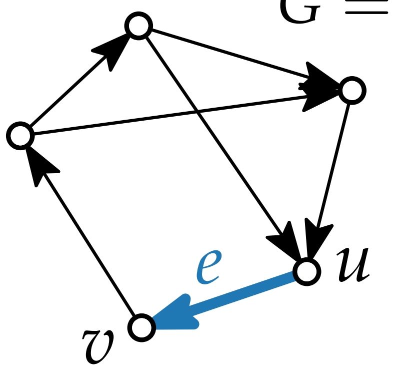
Directed Graphs



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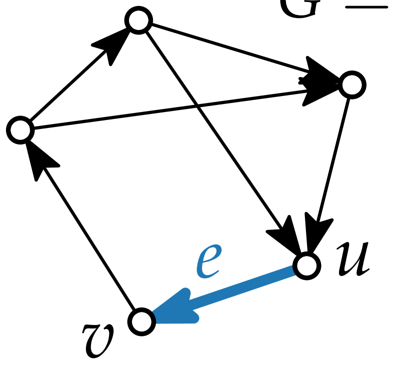

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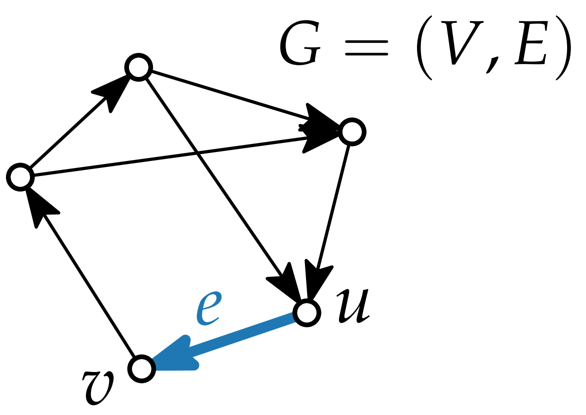

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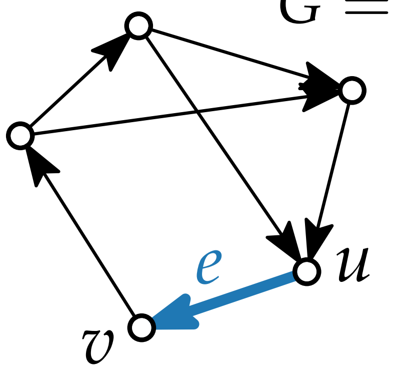

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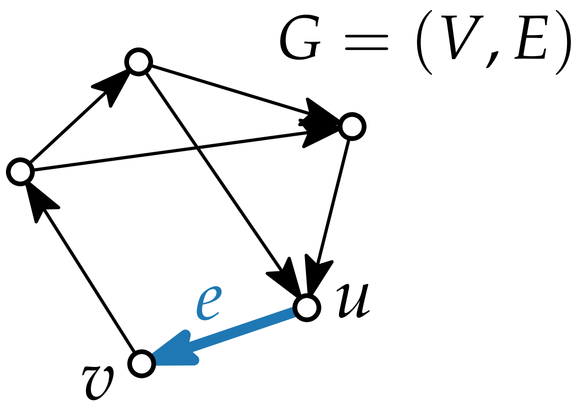

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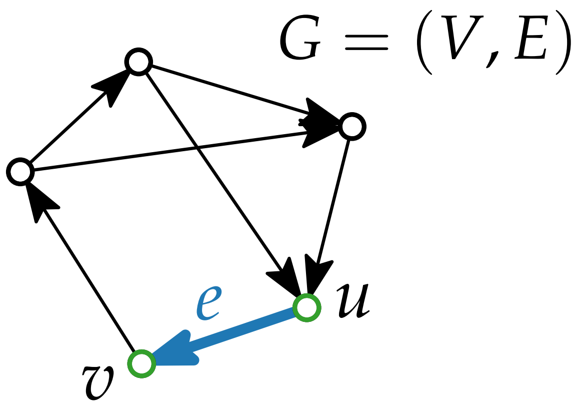

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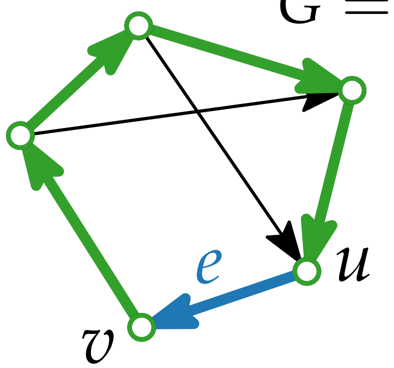

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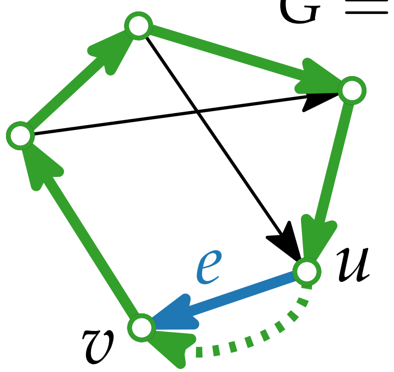

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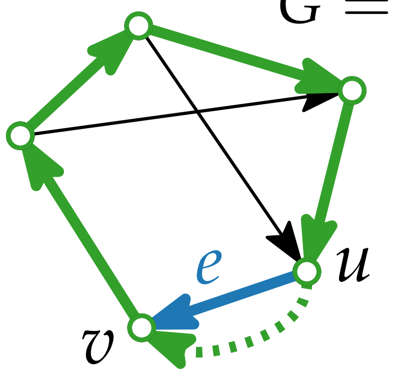

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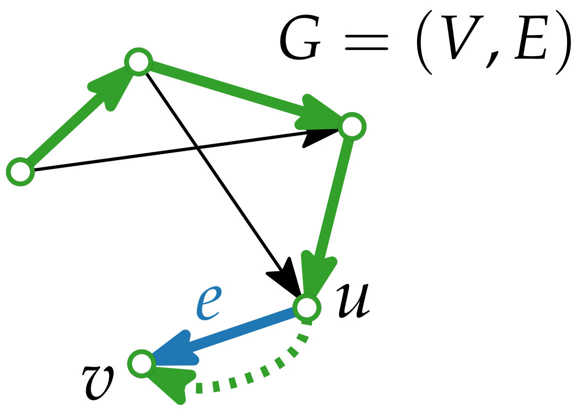
Directed Graphs



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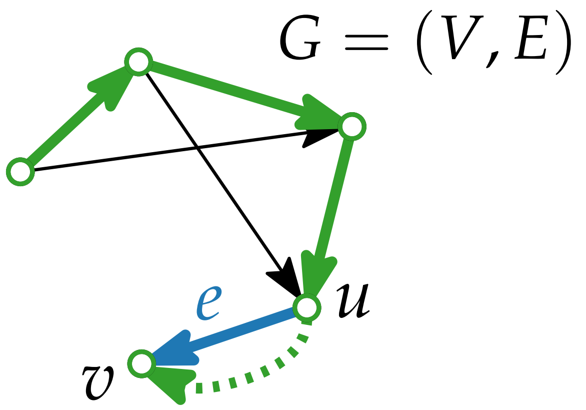

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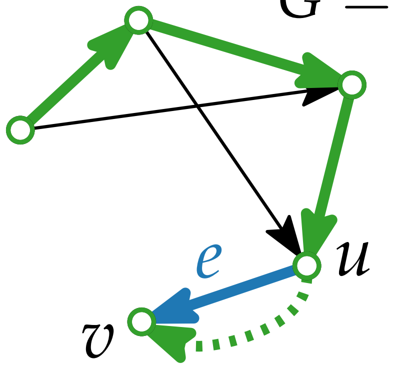

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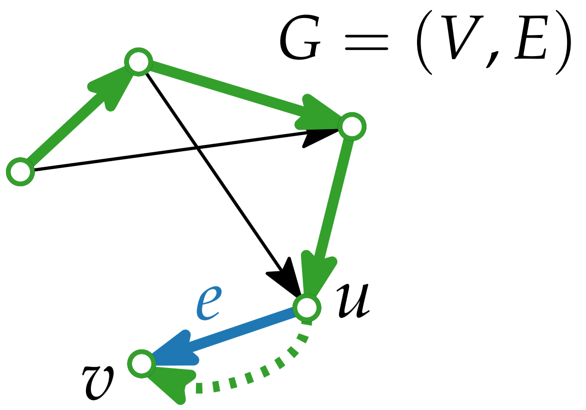

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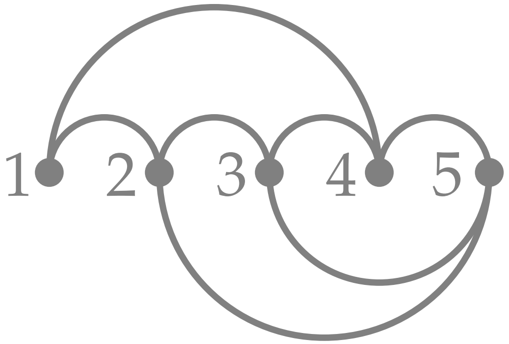
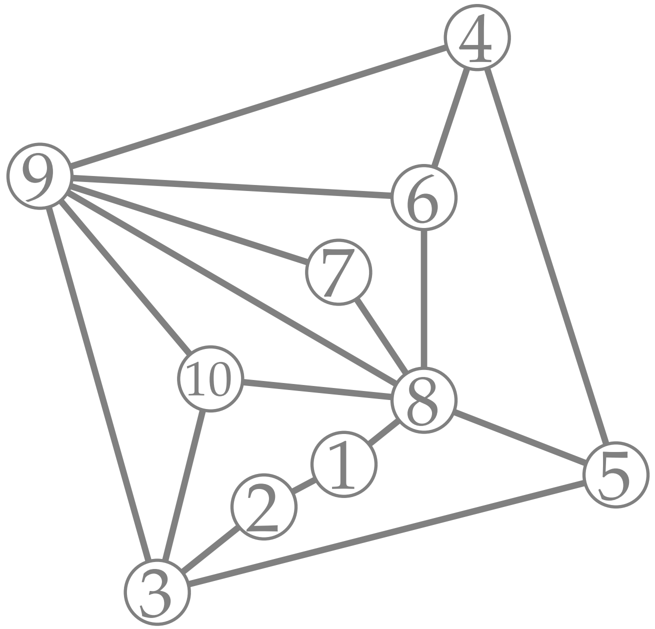
~~connected component~~ **connected component**

Universität Trier

Visualization of Graphs



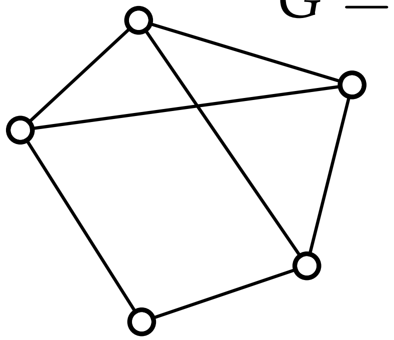
Lecture 1:
The Graph Visualization Problem



Part IV:
Planarity

Philipp Kindermann
Summer Semester 2021

Planar Graphs



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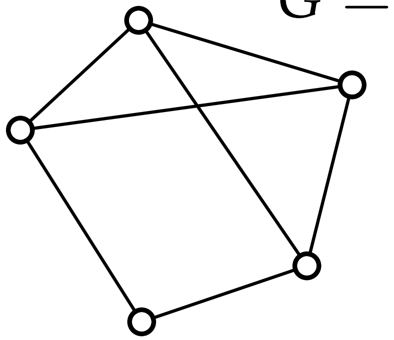
Planar Graphs



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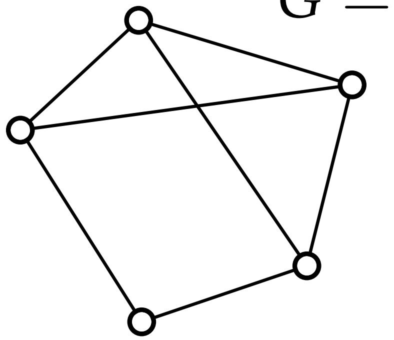
G is **planar**:
it can be drawn in such a way
that no edges cross each other.

Planar Graphs



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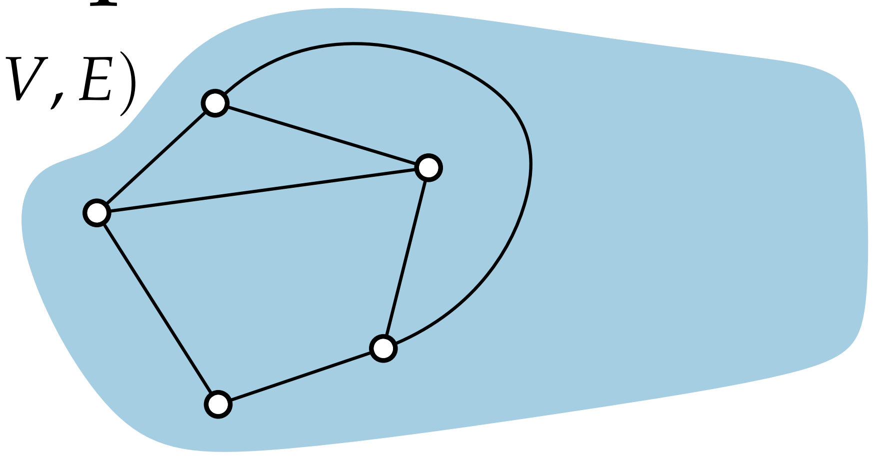

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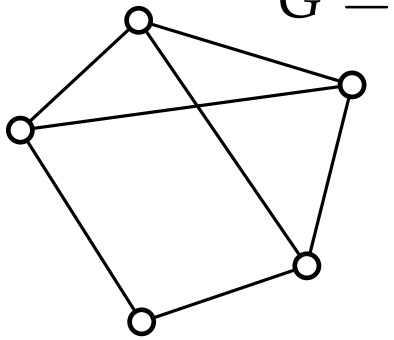

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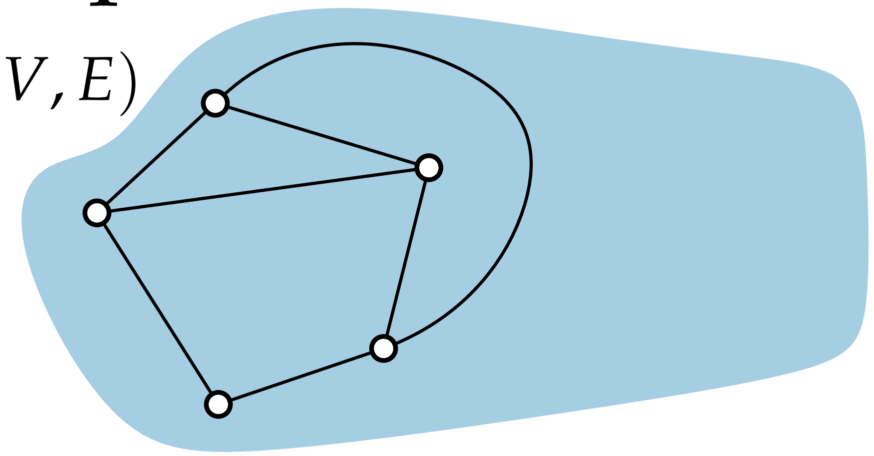

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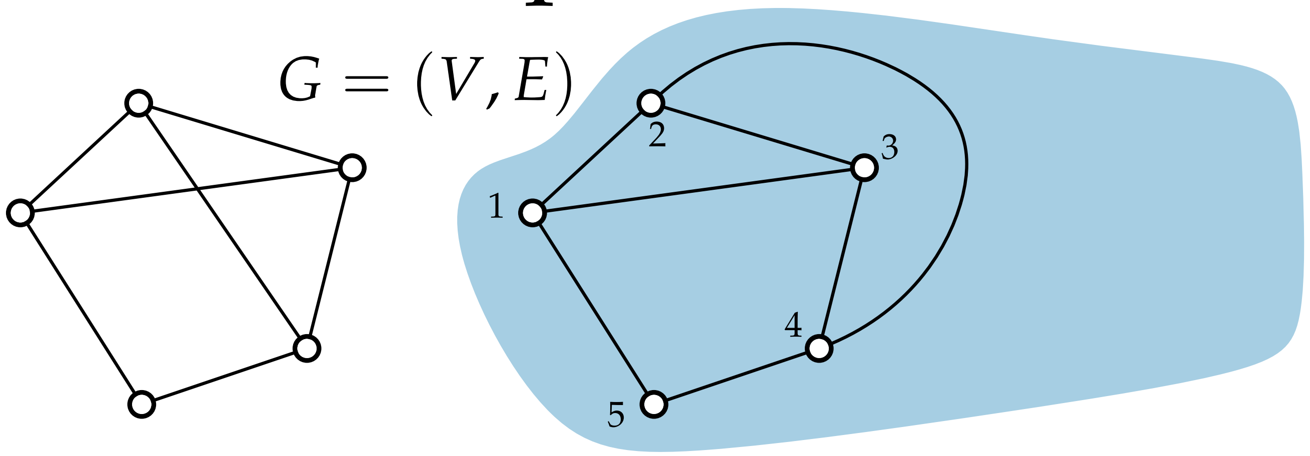

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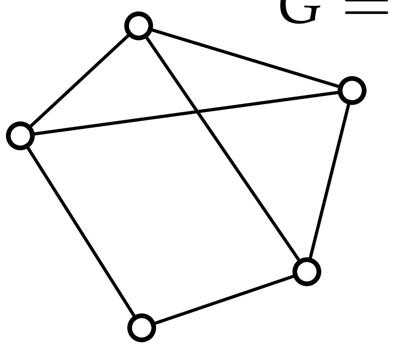

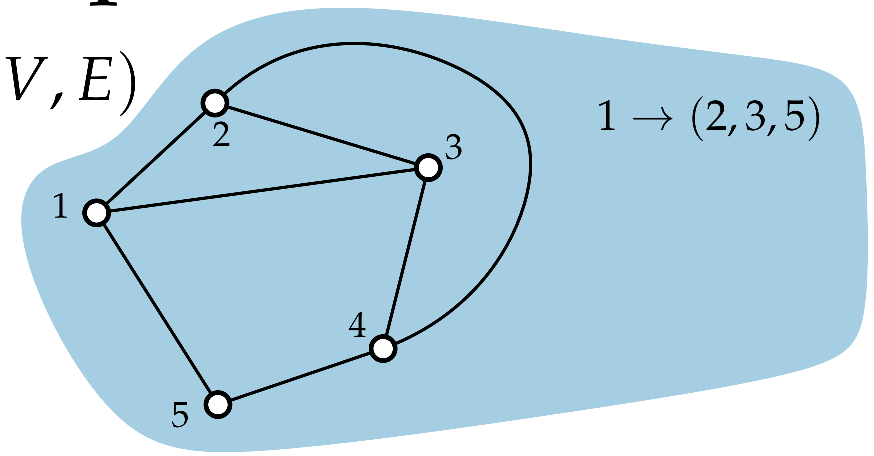

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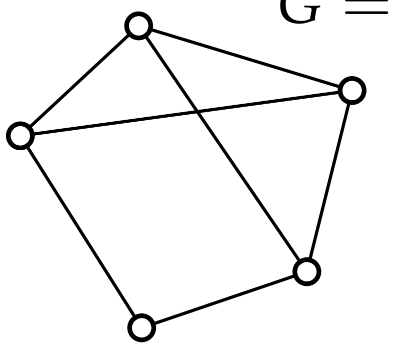

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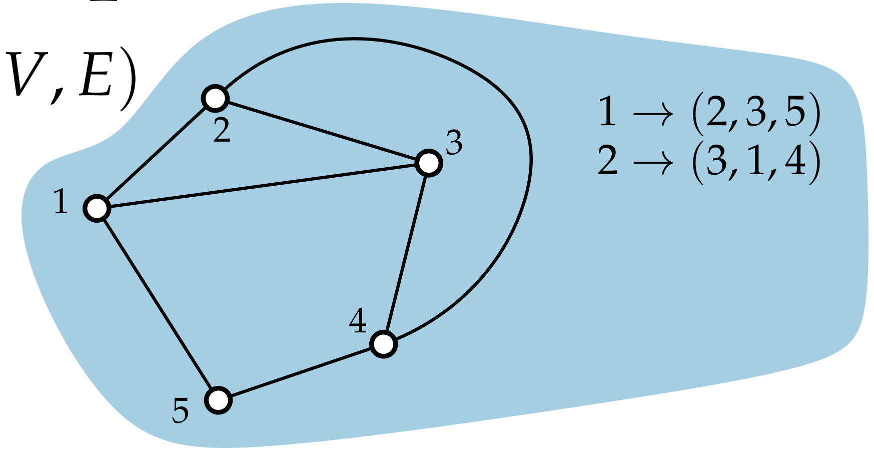

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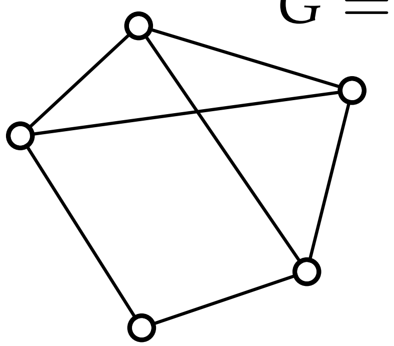

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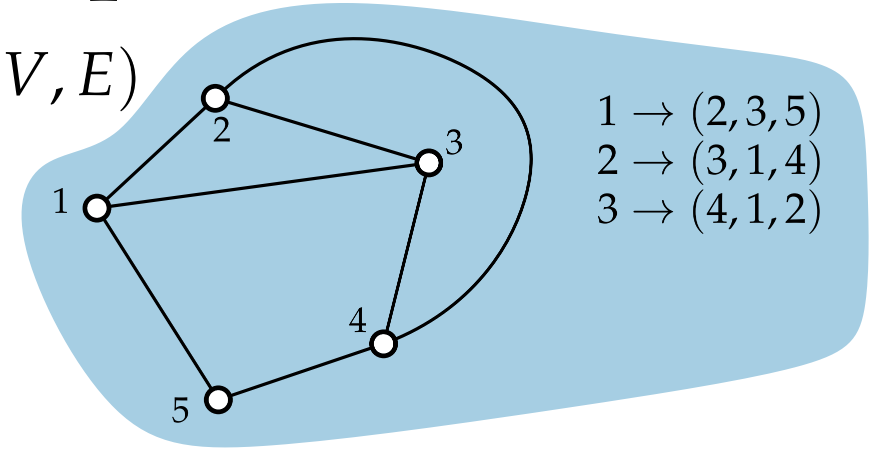

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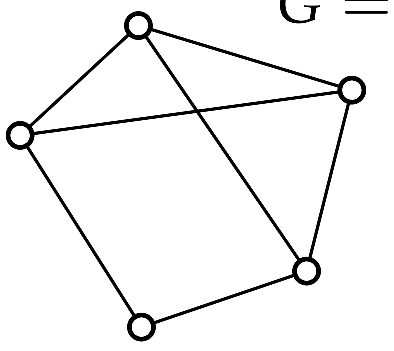

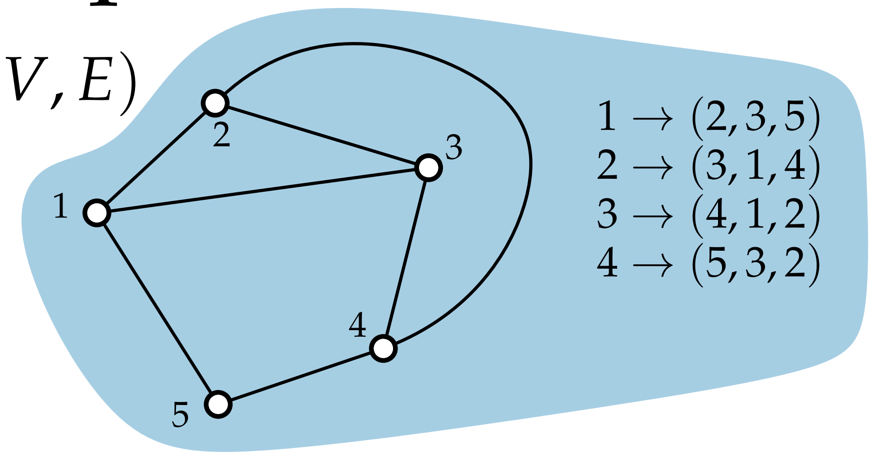

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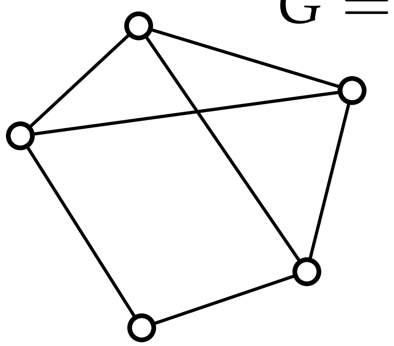

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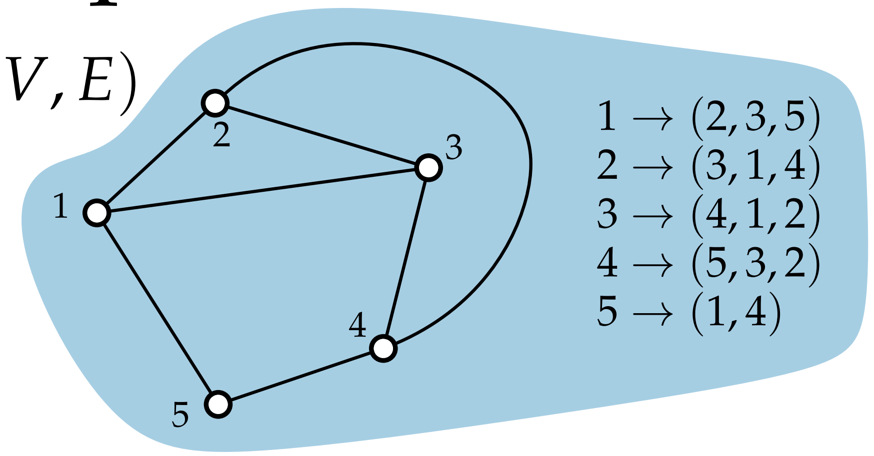

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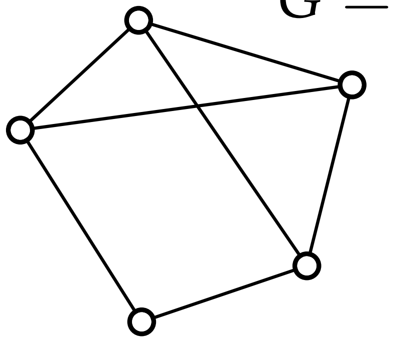

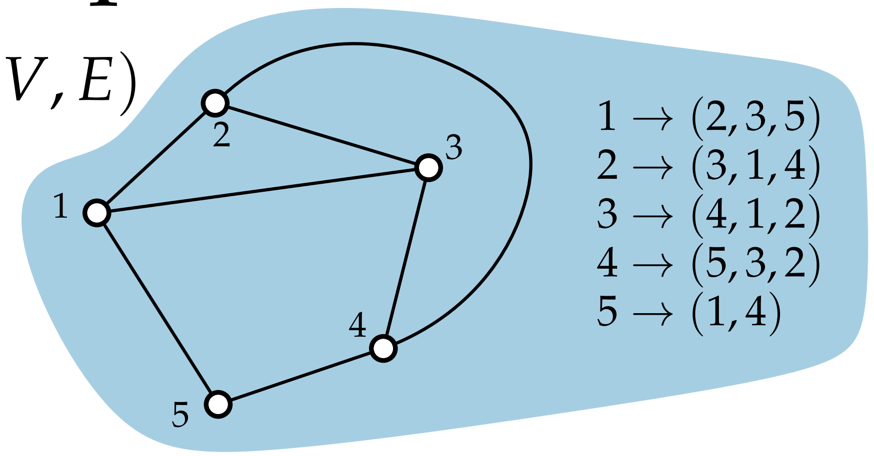

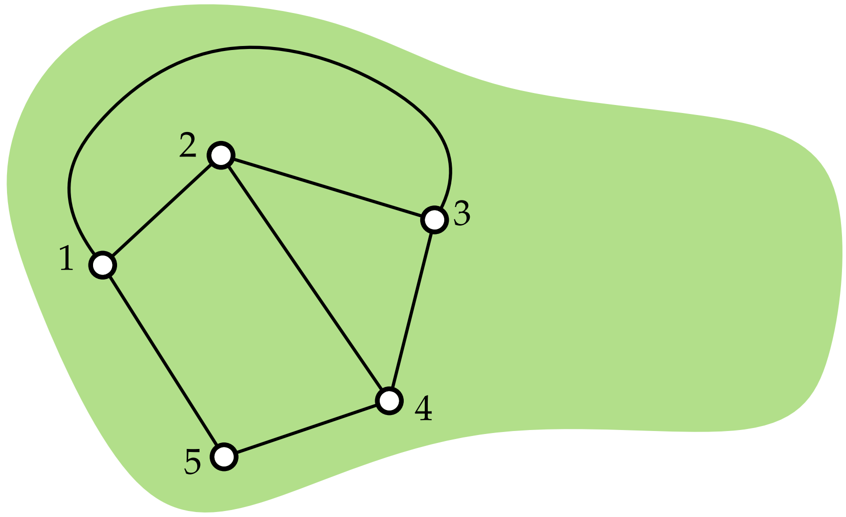

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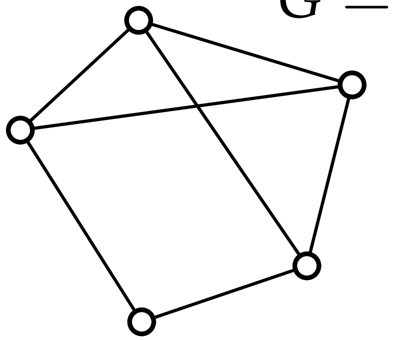
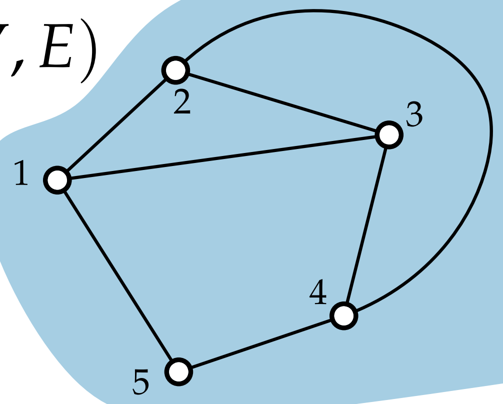
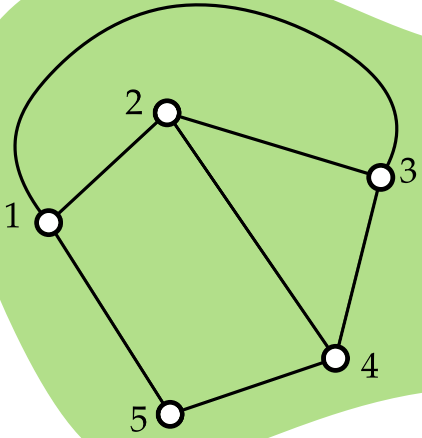




$1 \to (2,3,5)$
$2 \to (3,1,4)$
$3 \to (4,1,2)$
$4 \to (5,3,2)$
$5 \to (1,4)$



$1 \to (2,5,3)$
$2 \to (3,4,1)$
$3 \to (4,2,1)$
$4 \to (5,2,3)$
$5 \to (1,4)$

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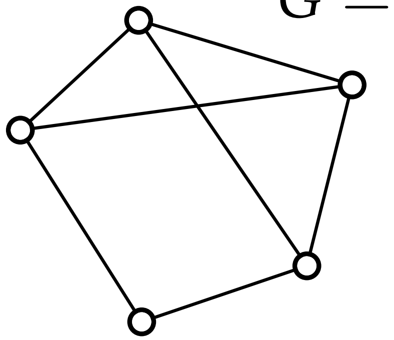

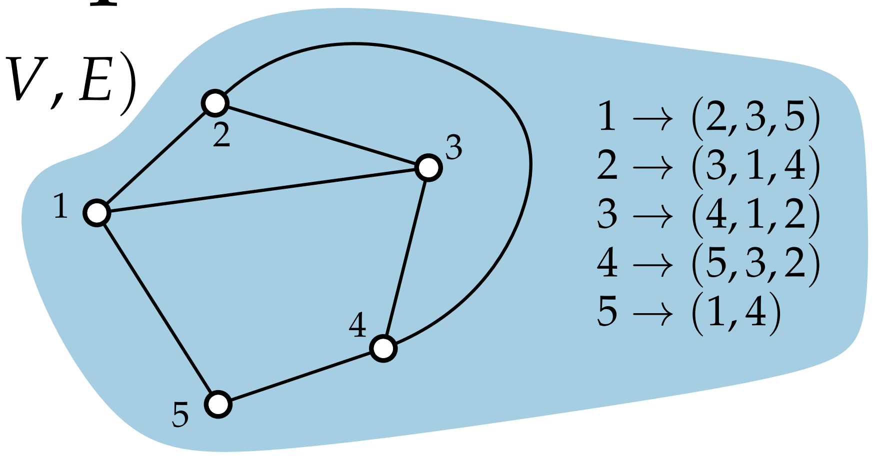

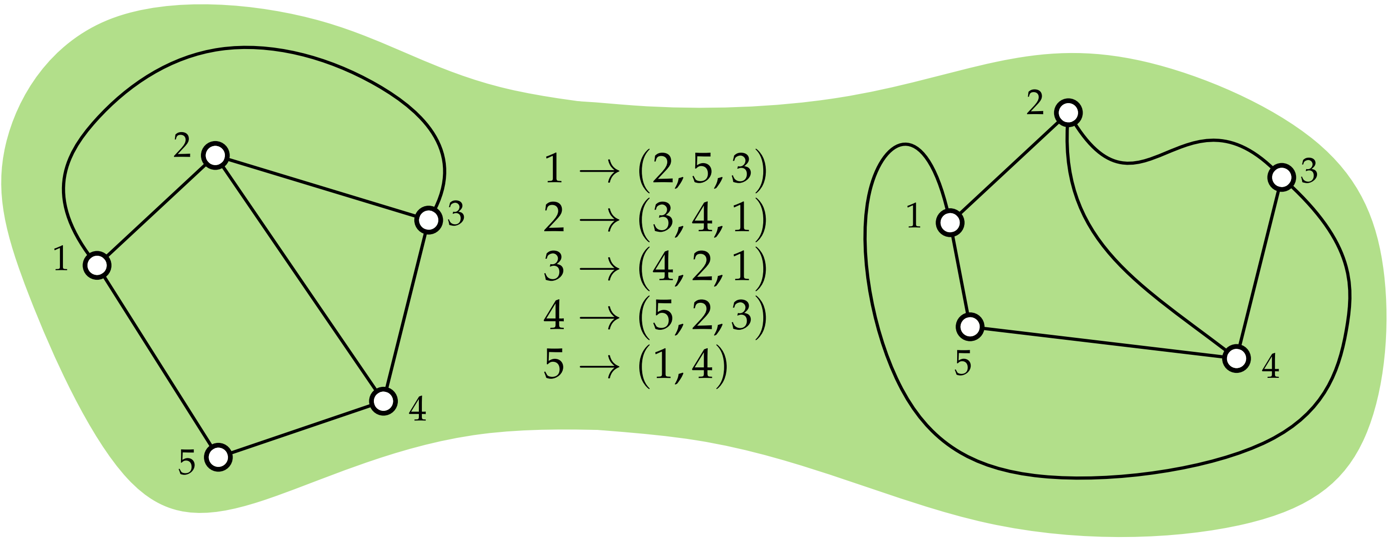

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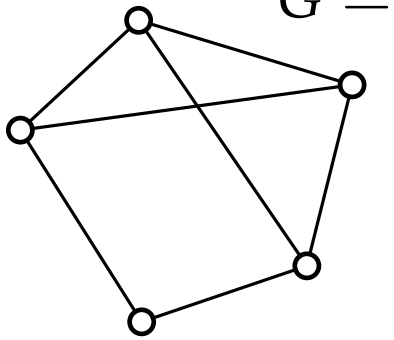

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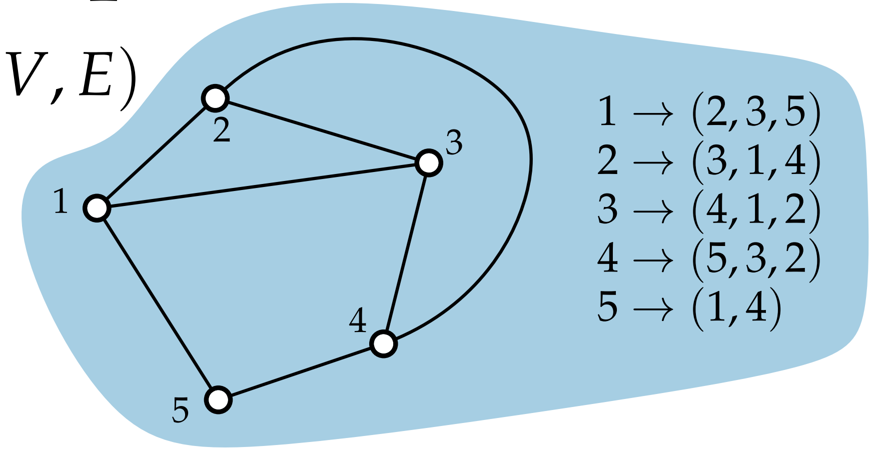

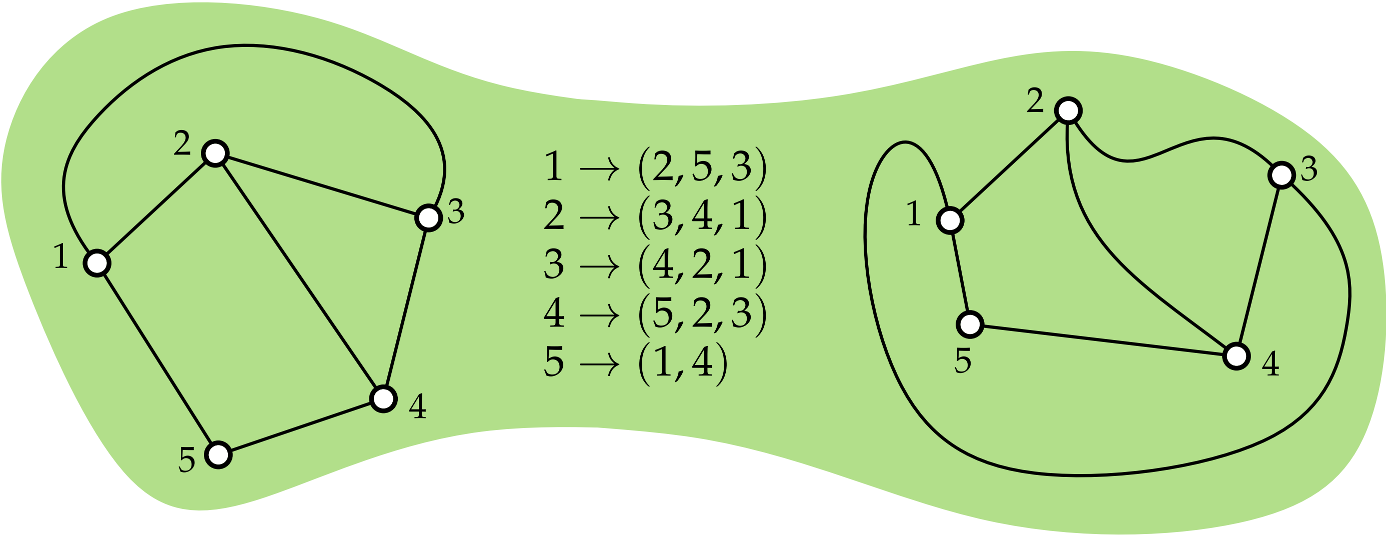

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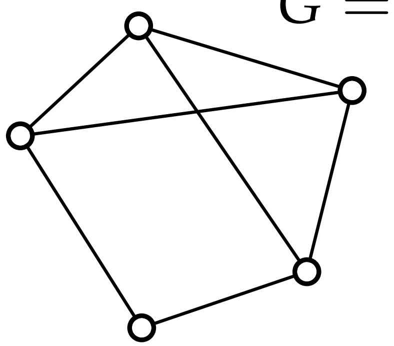

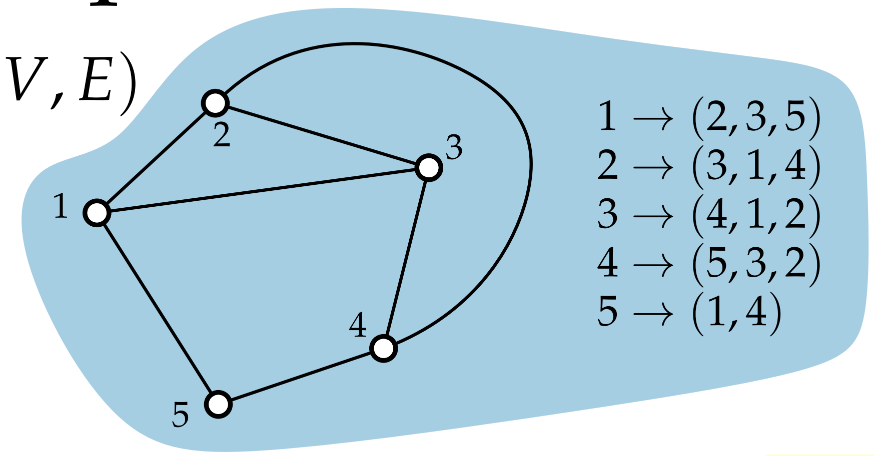

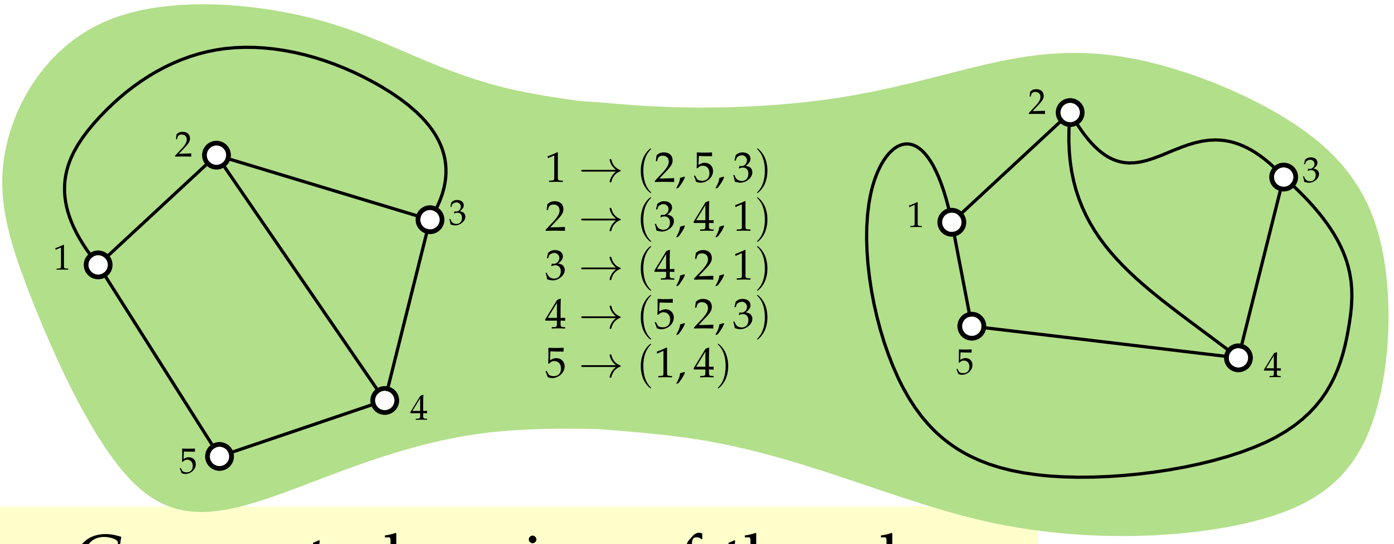

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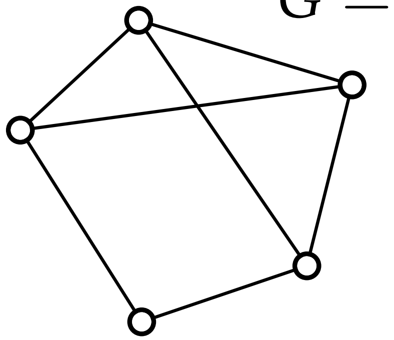

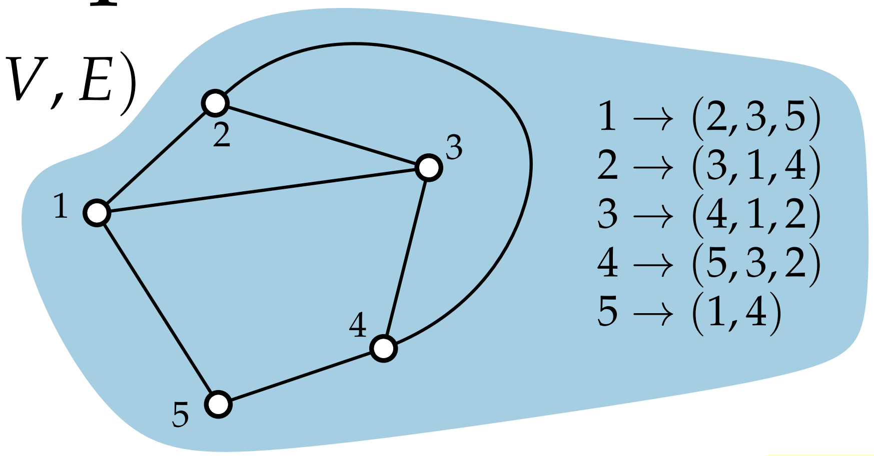

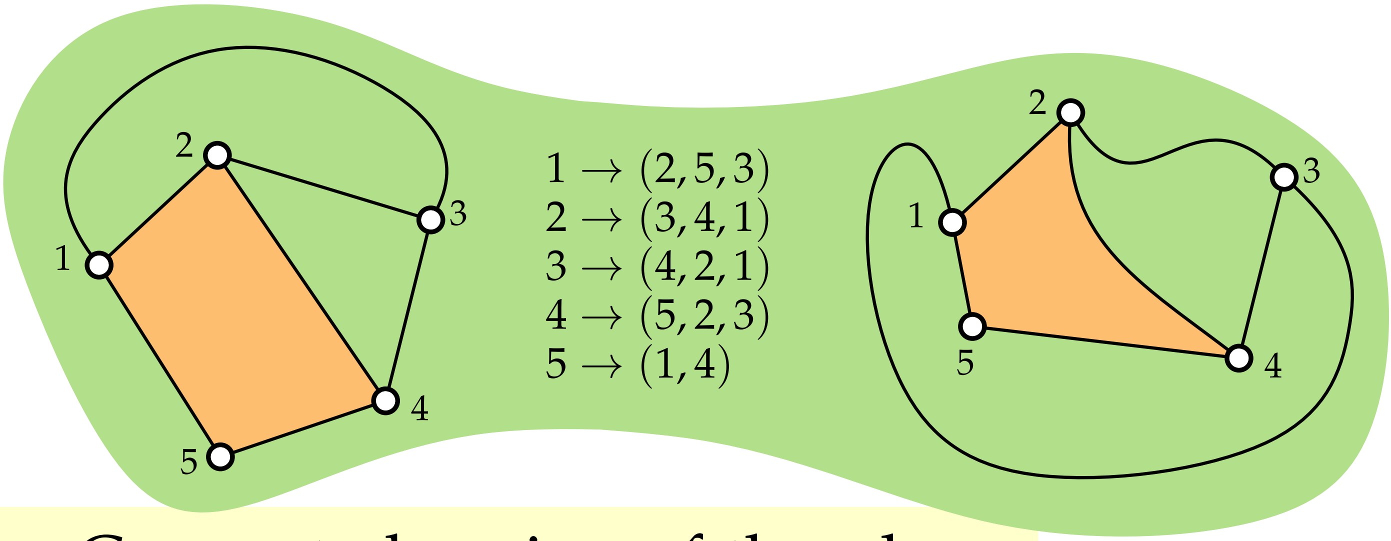

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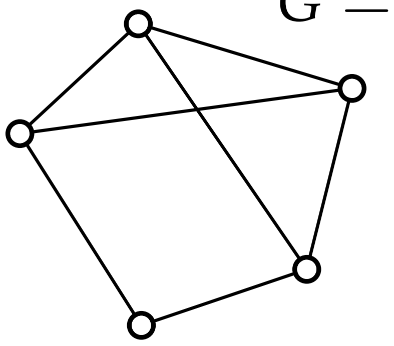

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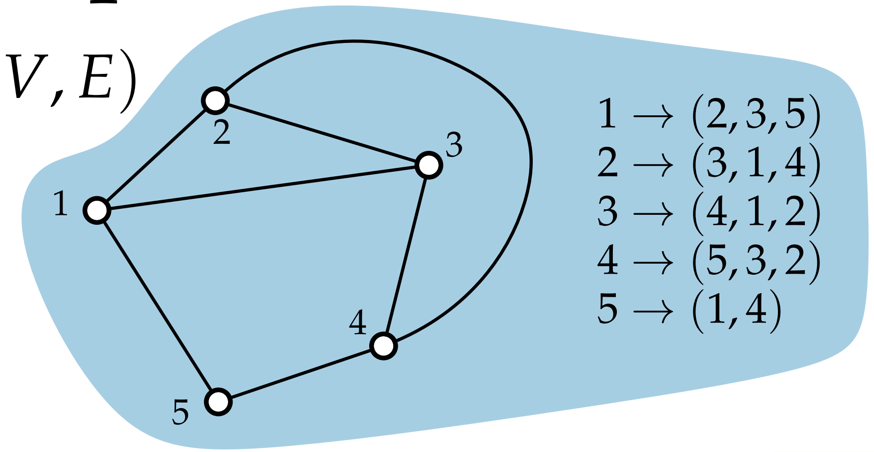

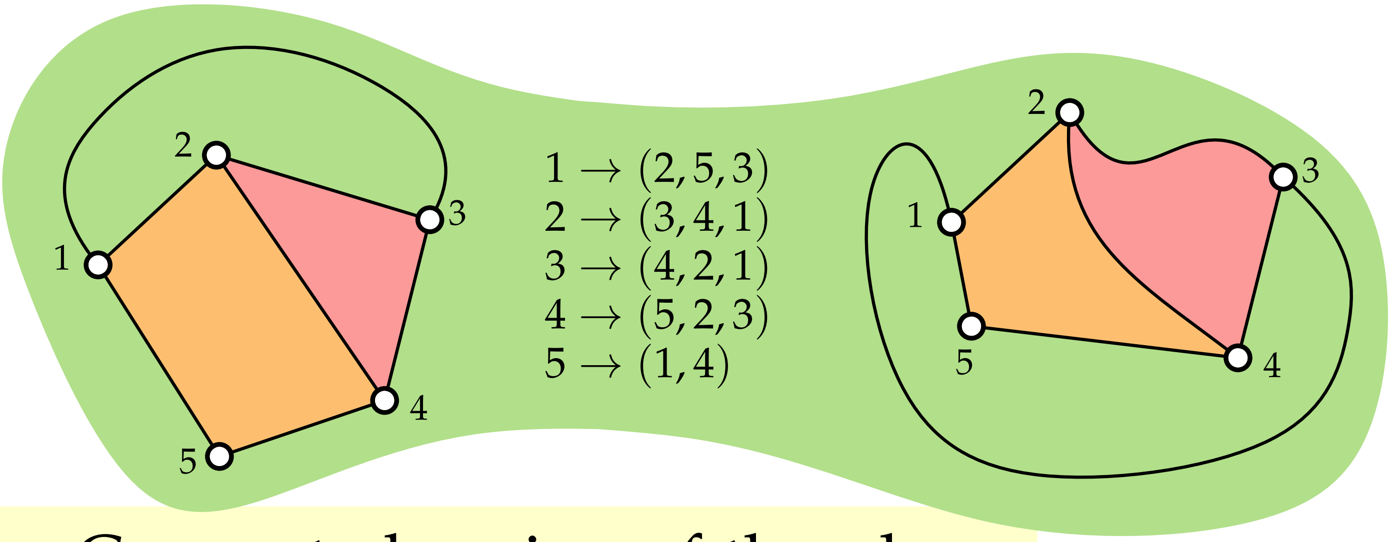

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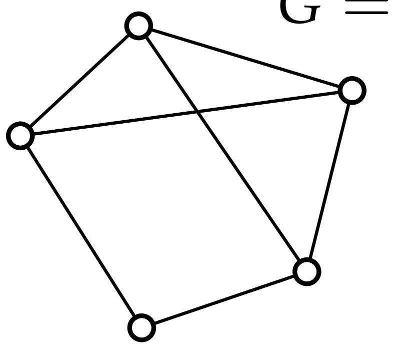

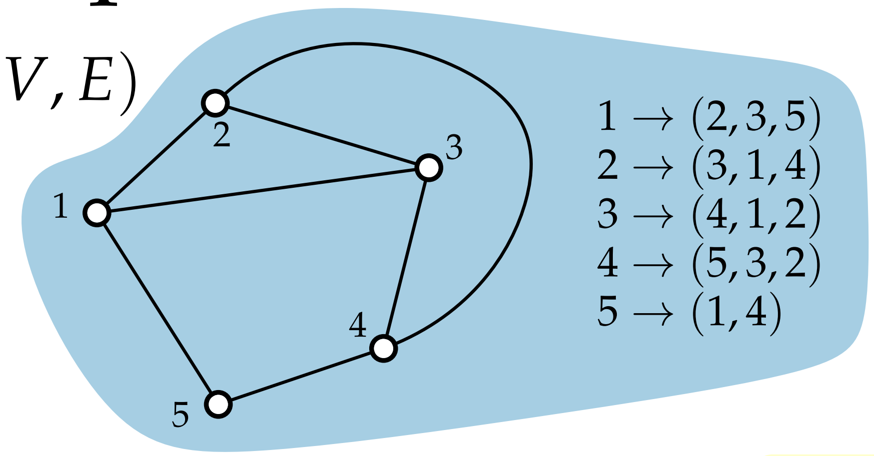

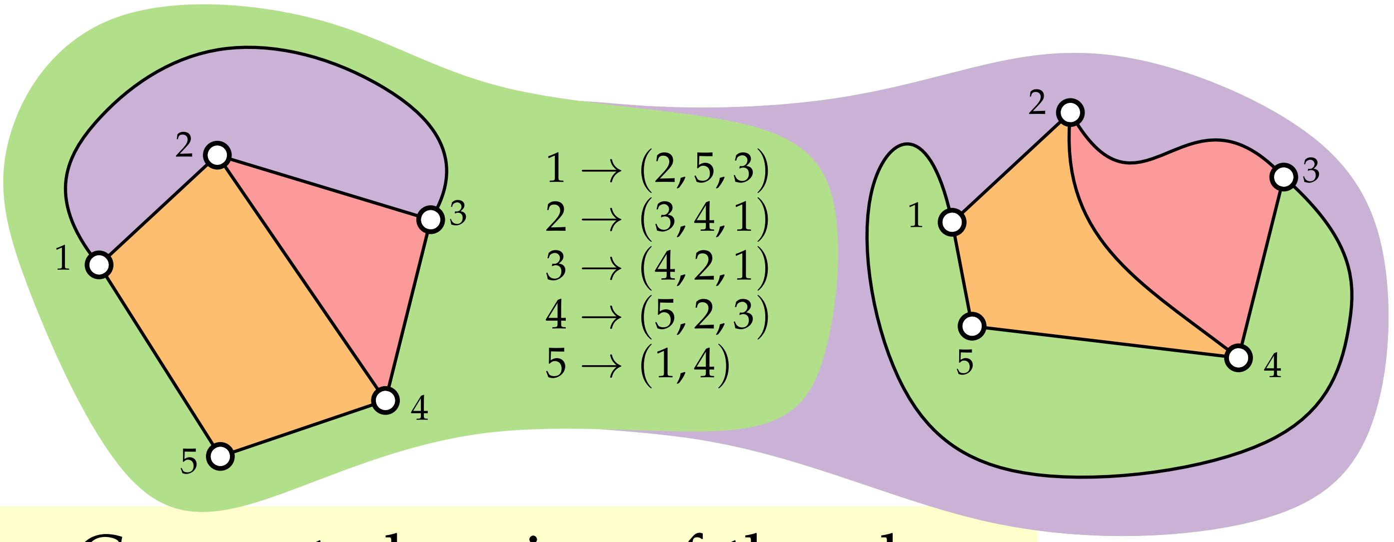

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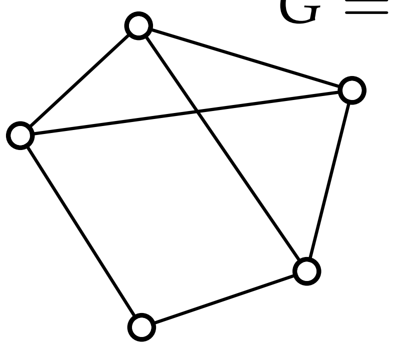

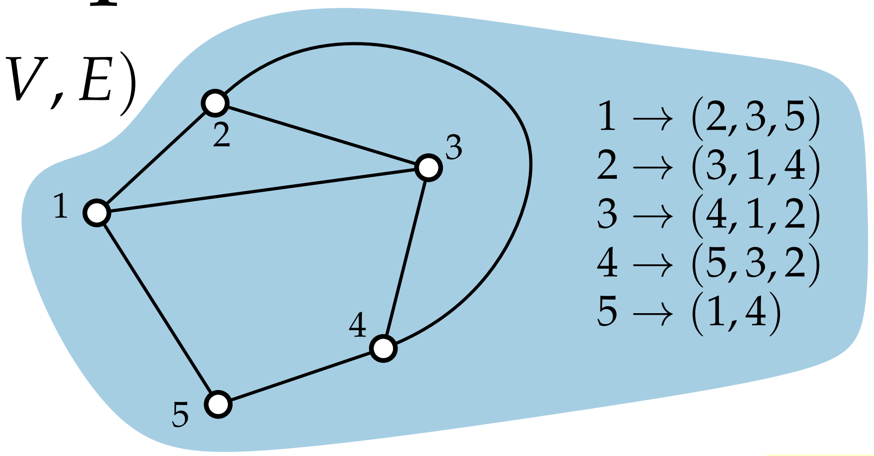

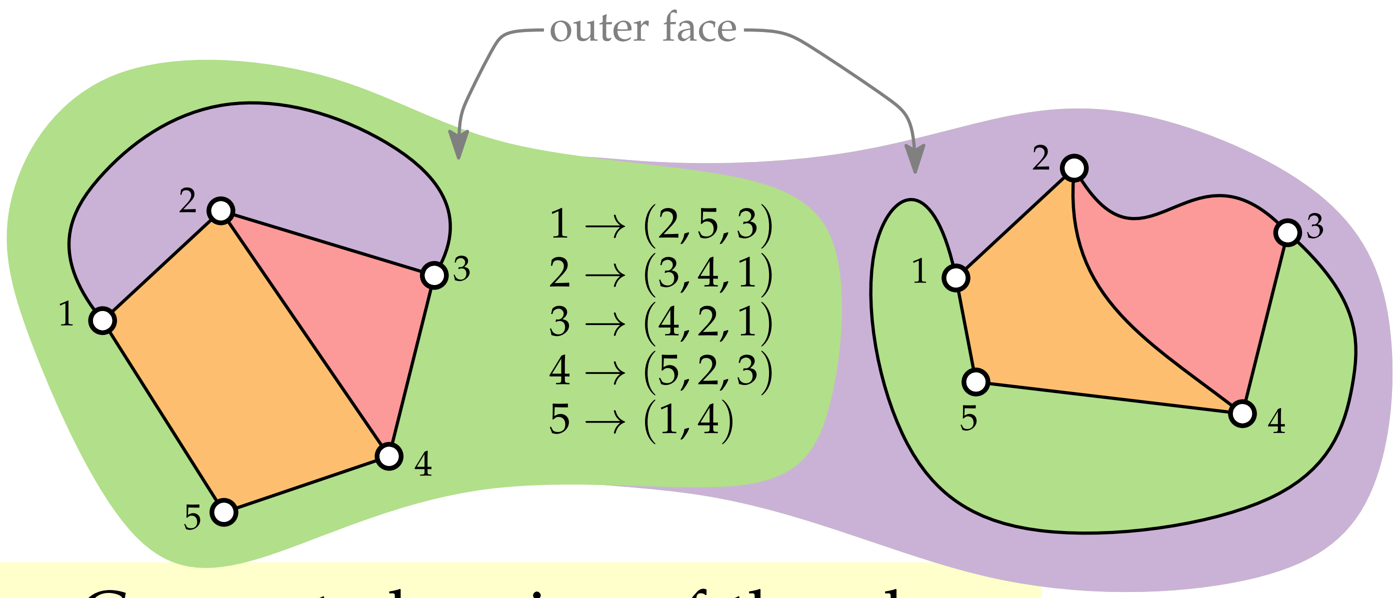

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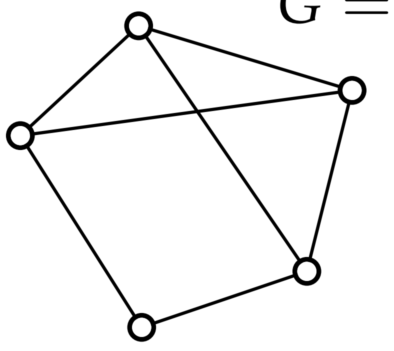

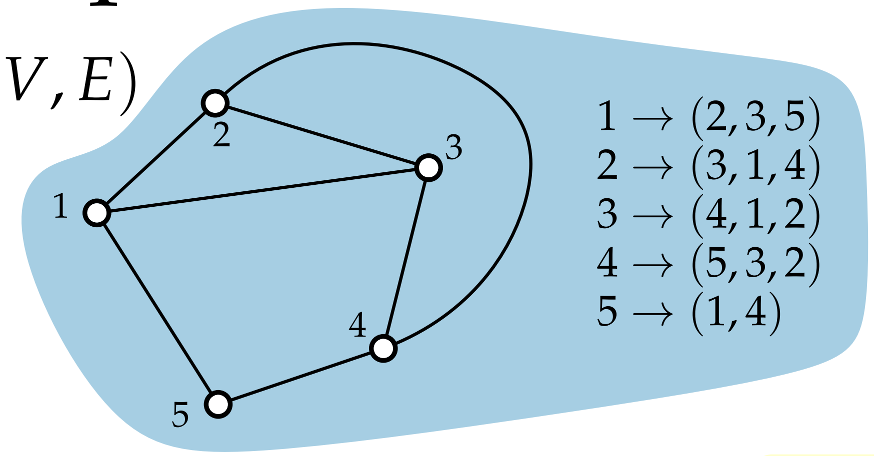

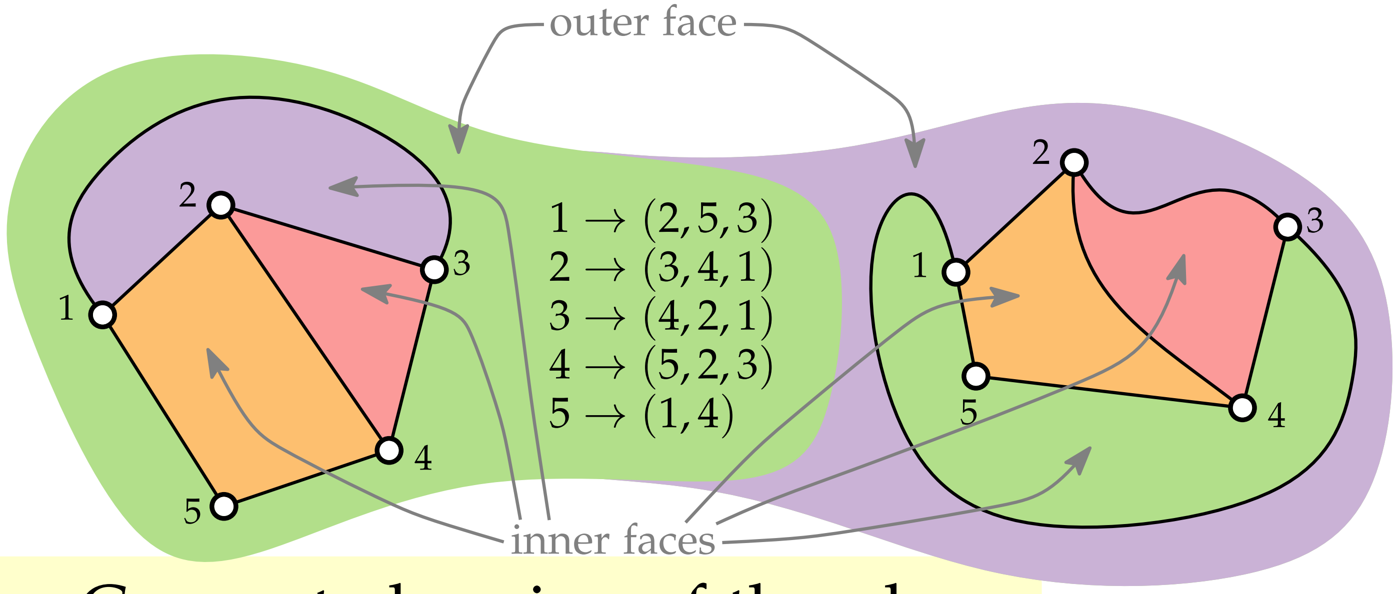

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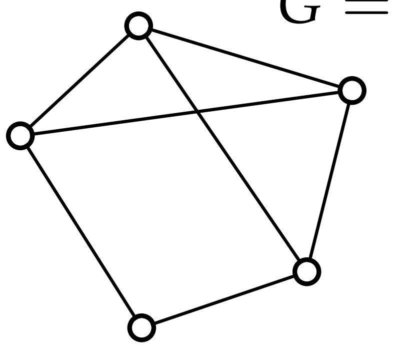
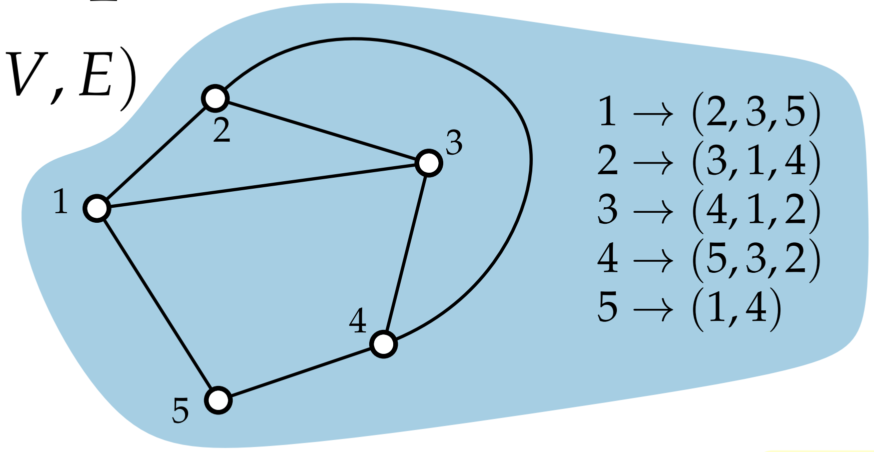
Euler's polyhedra formula.

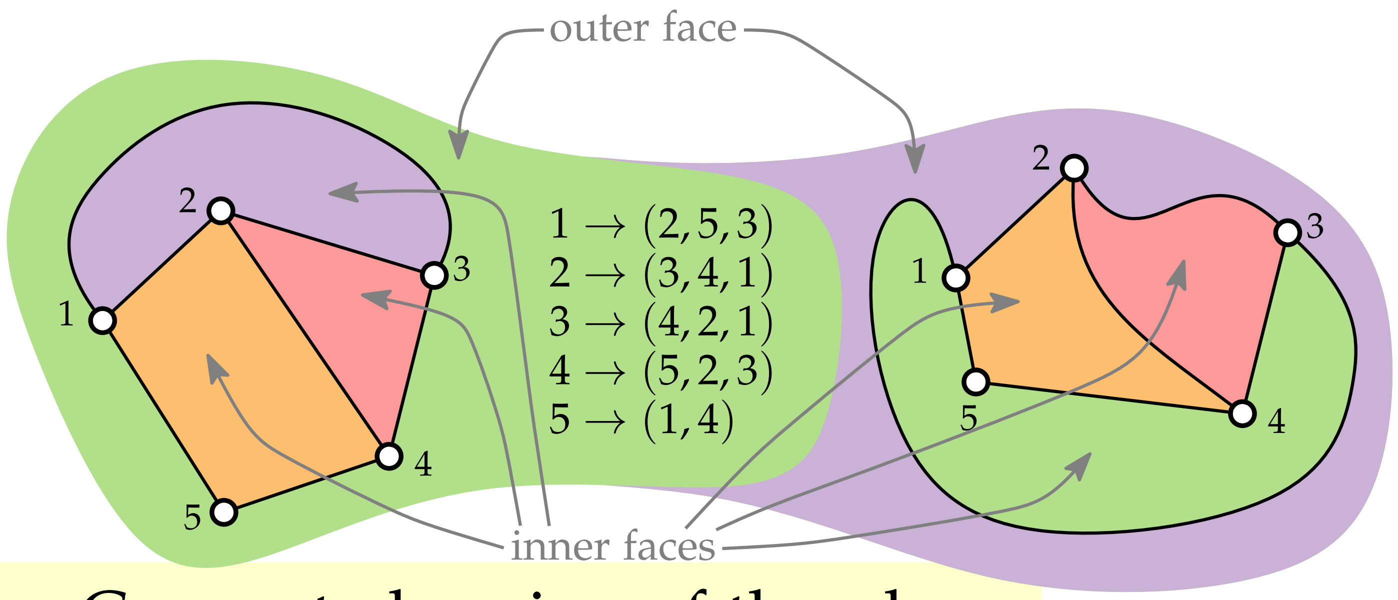
#faces - #edges + #vertices = #conn.comp. + 1

$$f - m + n = c + 1$$

Planar Graphs







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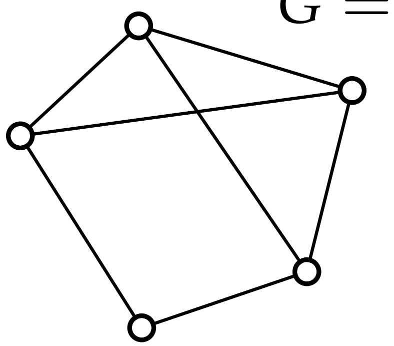
Proof.

Planar Graphs



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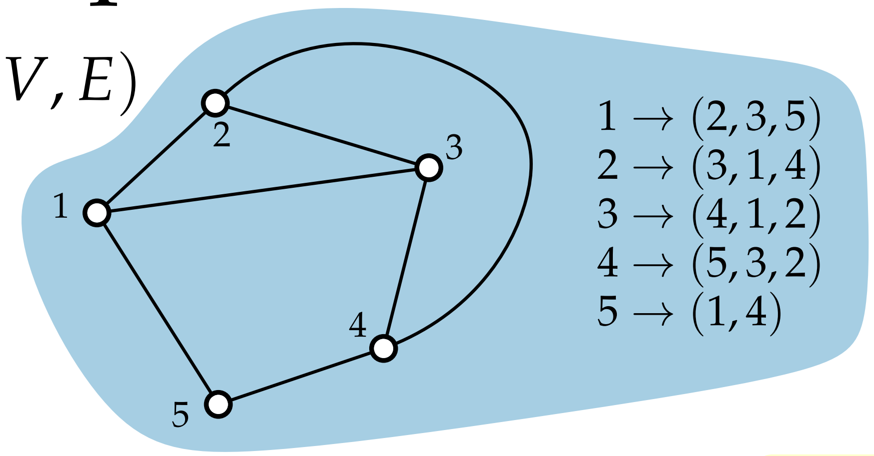

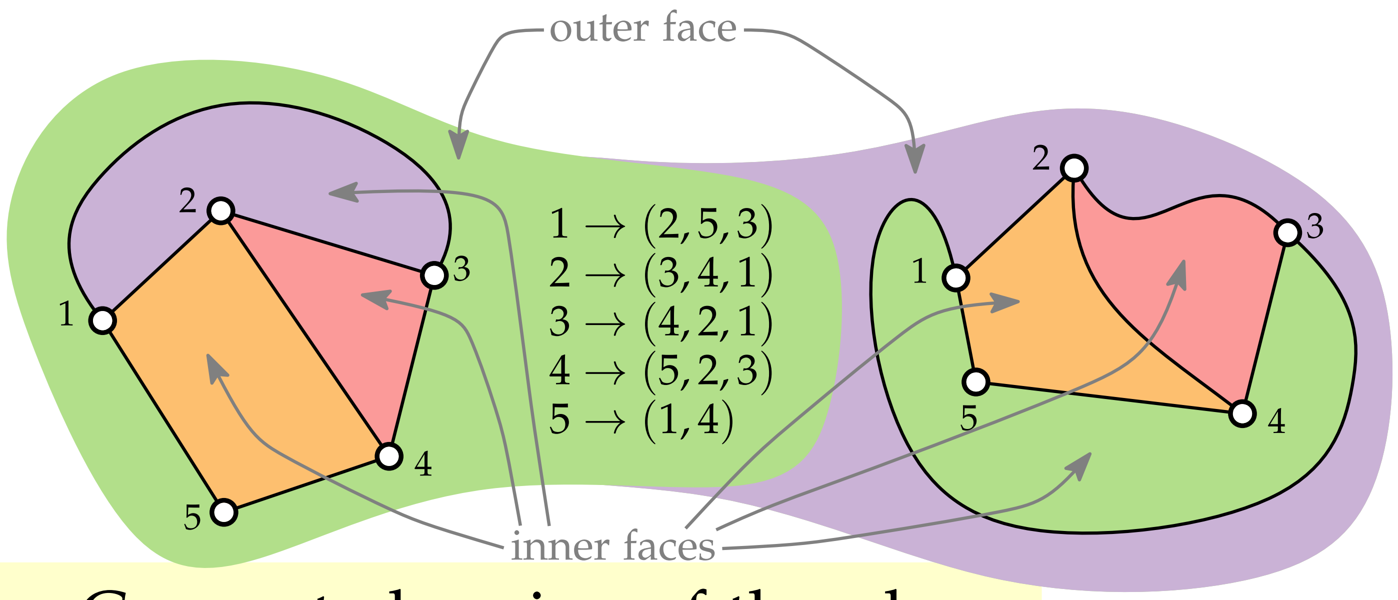

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Proof. By induction on m:

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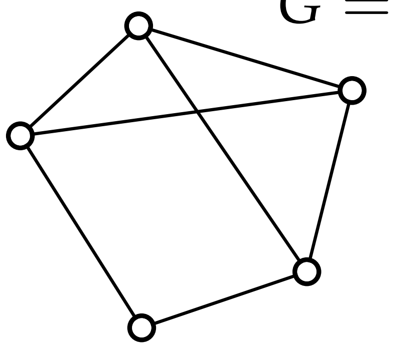

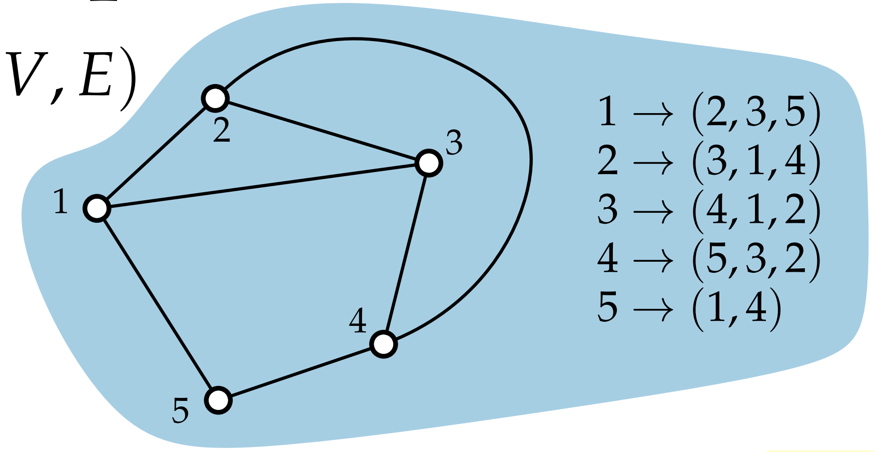

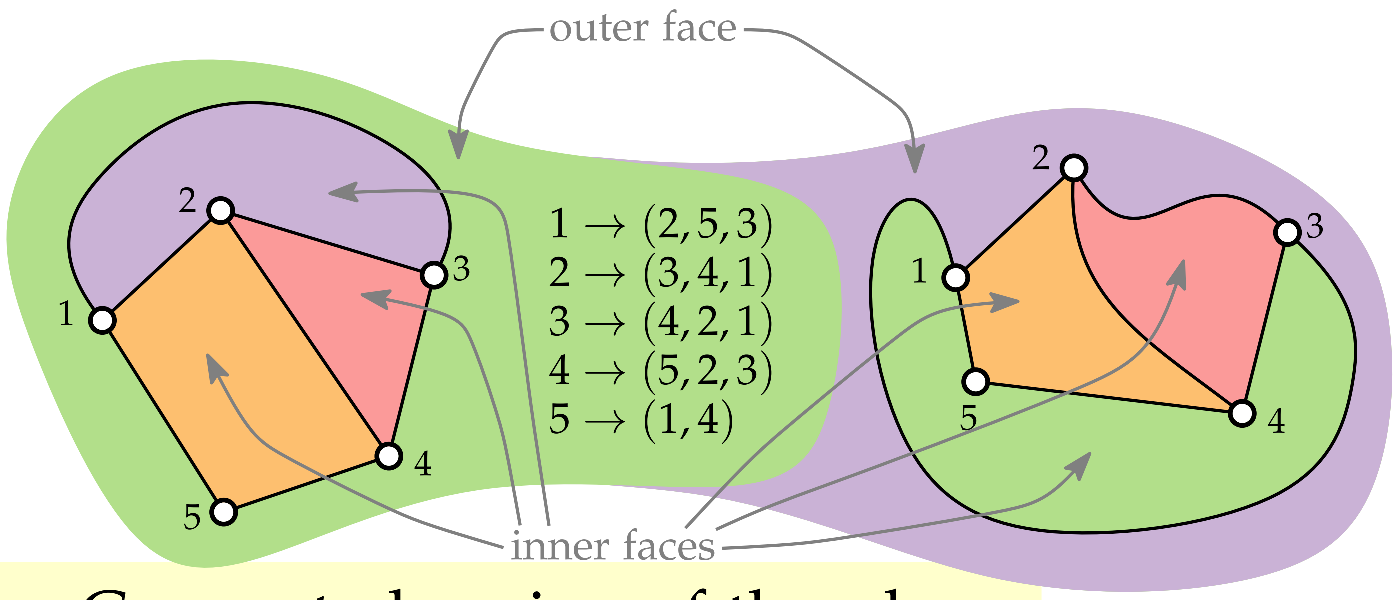

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$m = 0 \Rightarrow$

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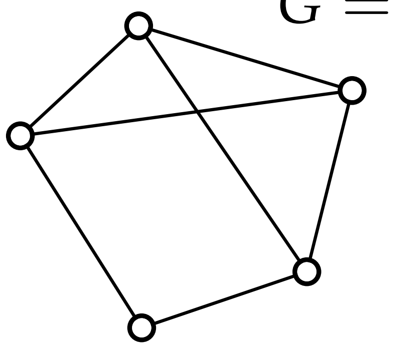

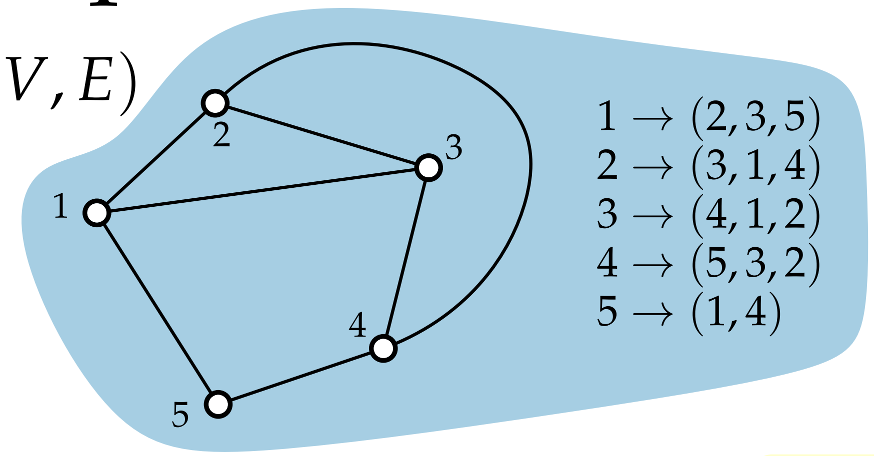

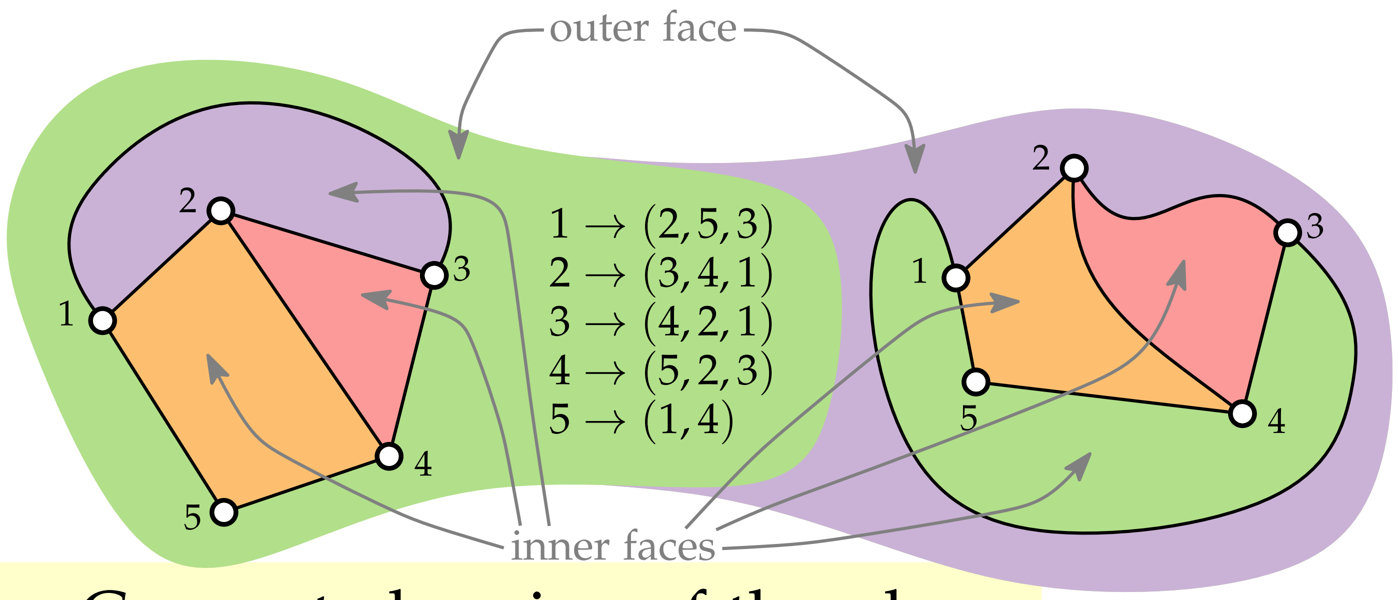

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Planar Graphs

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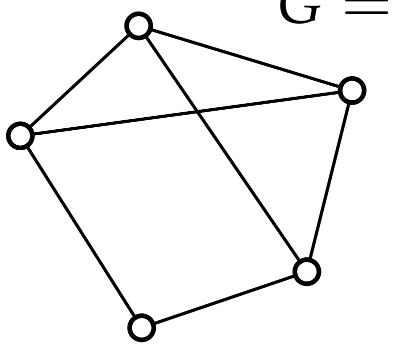

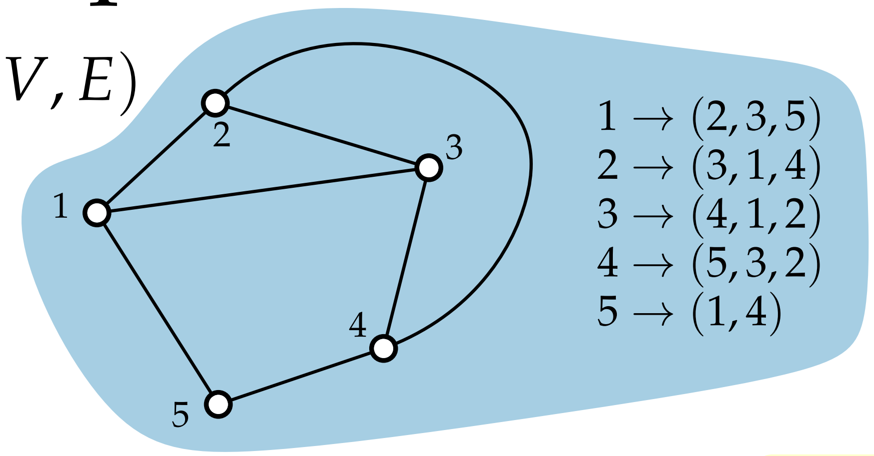

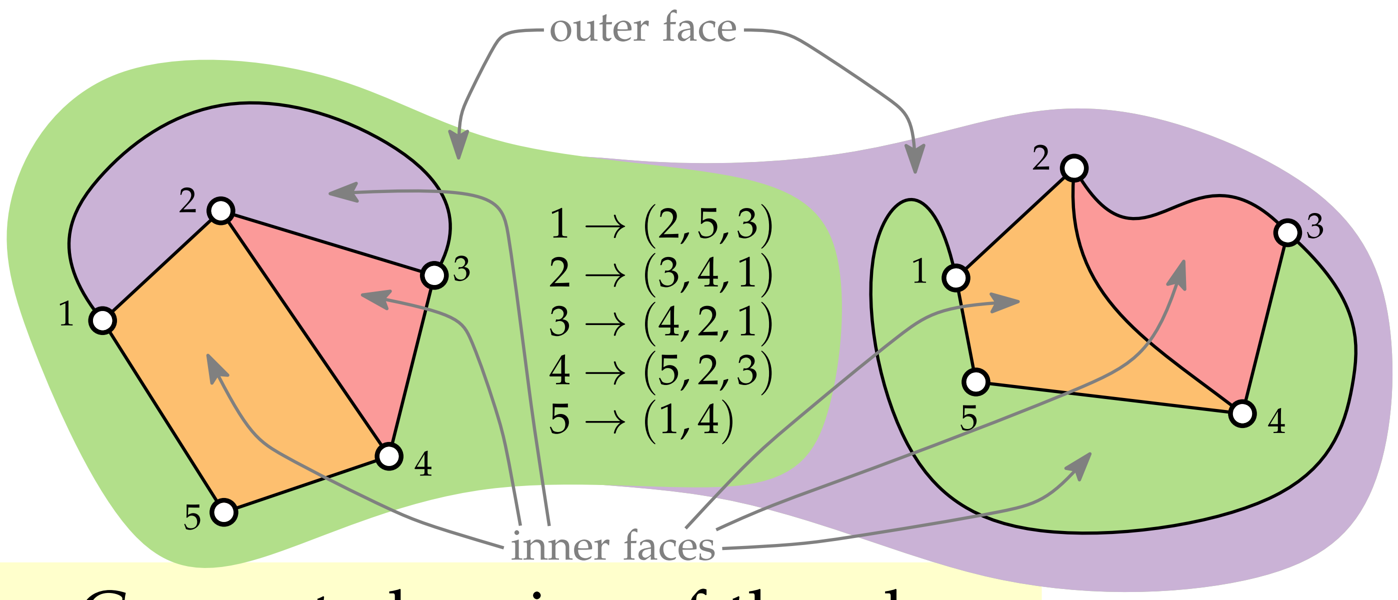

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Proof. By induction on m:
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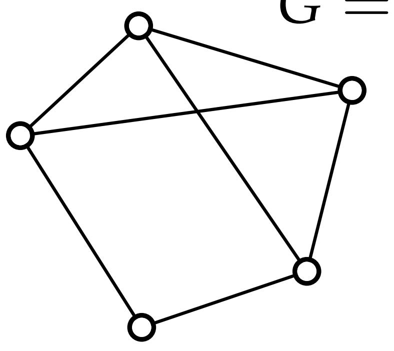
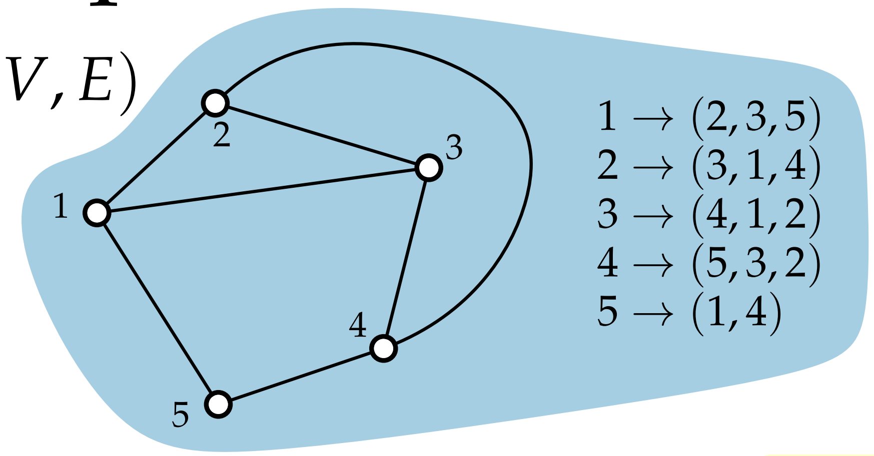
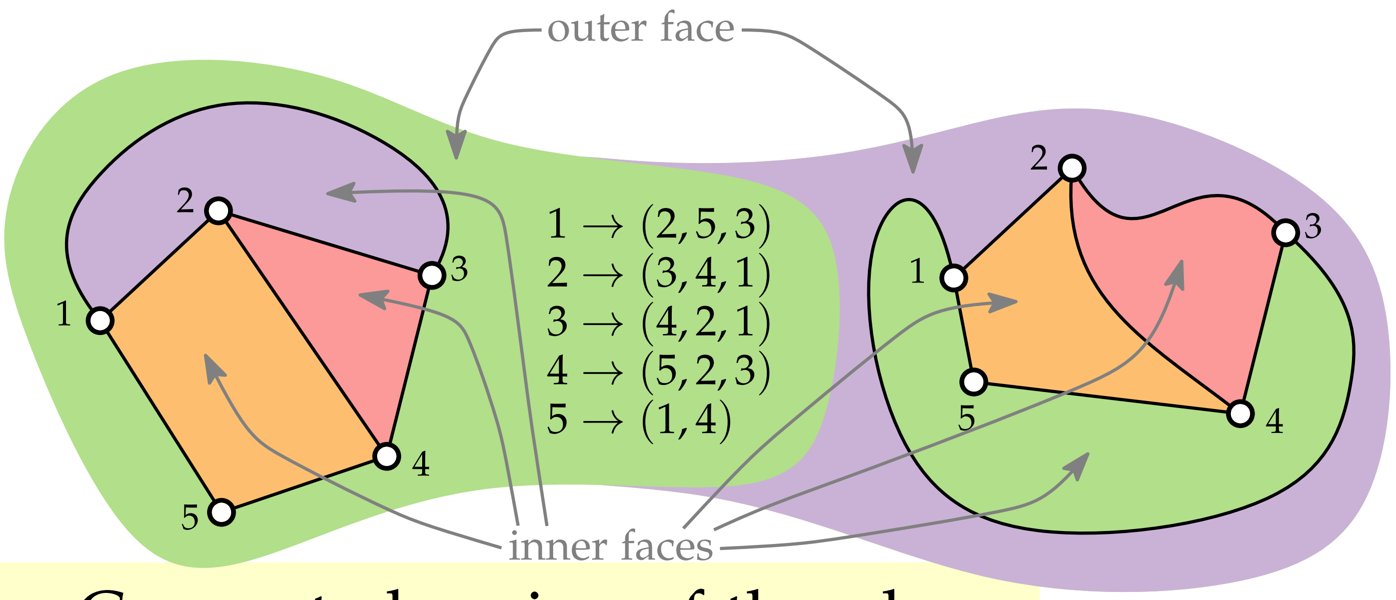
Planar Graphs

$G = (V, E)$







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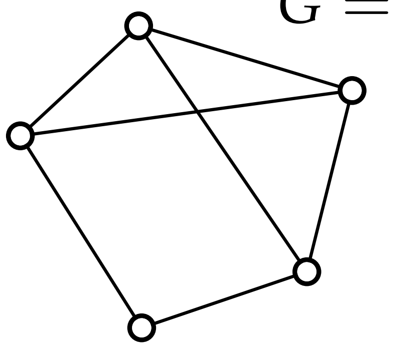

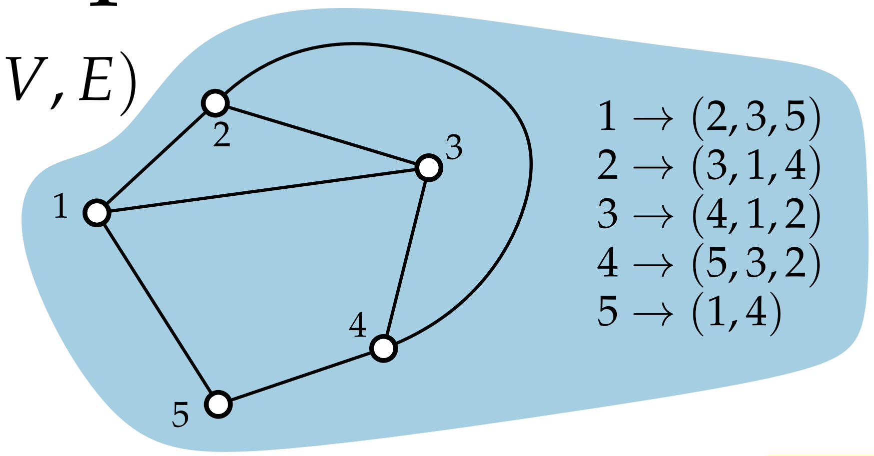

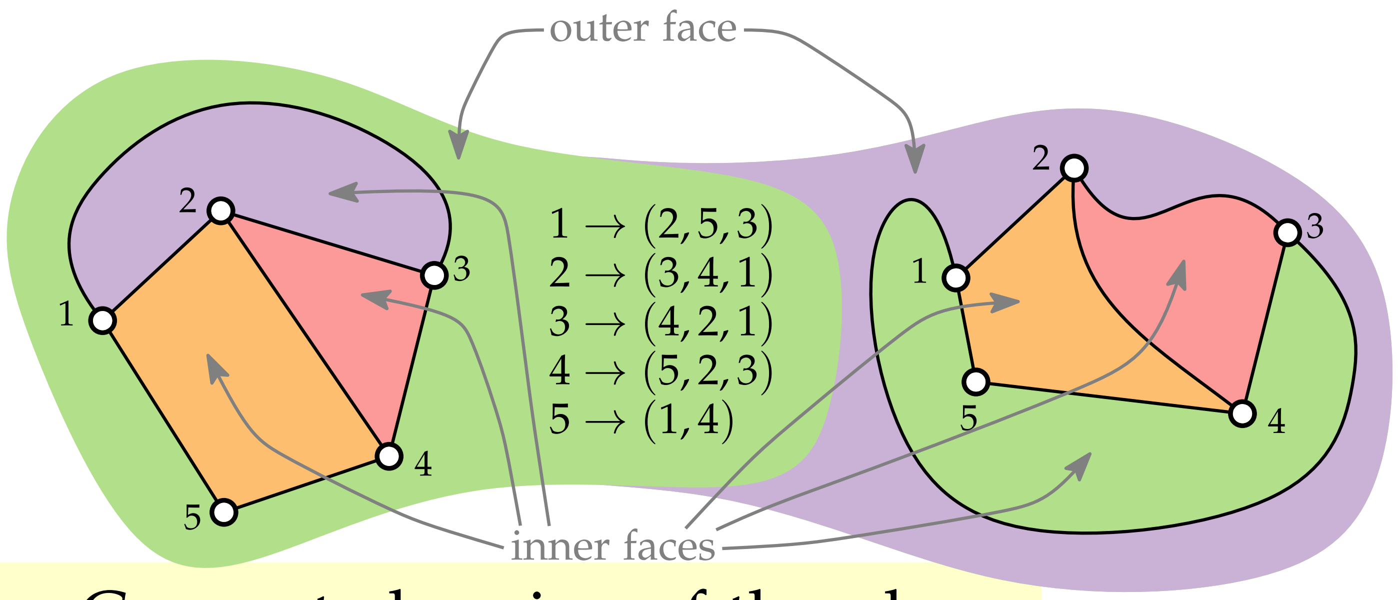

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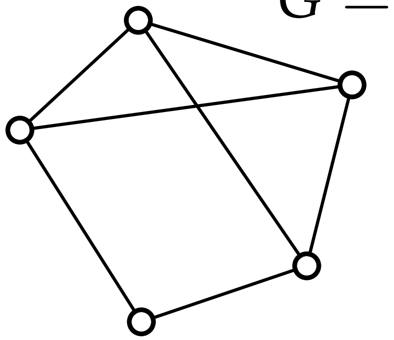

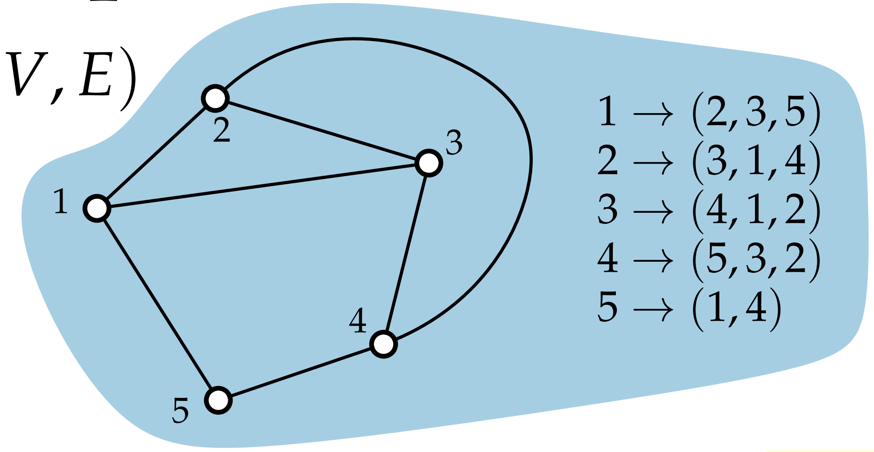

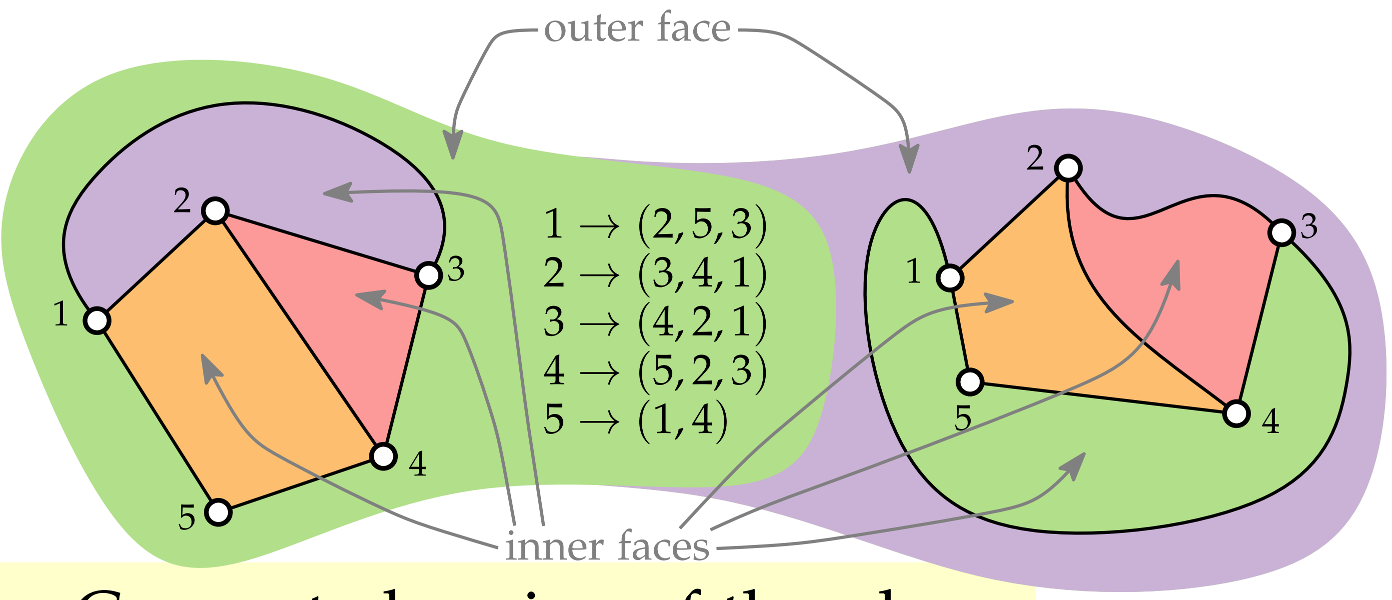

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$m > 1 \Rightarrow$

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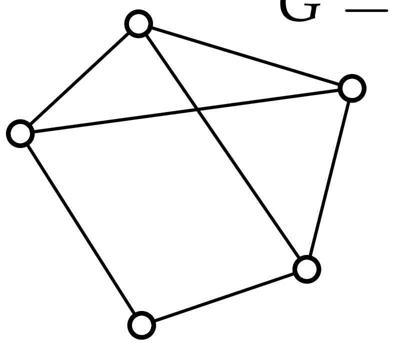

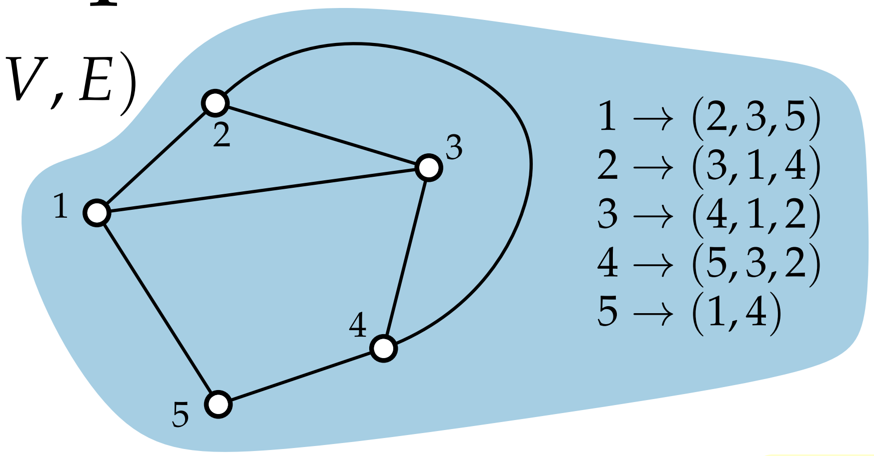

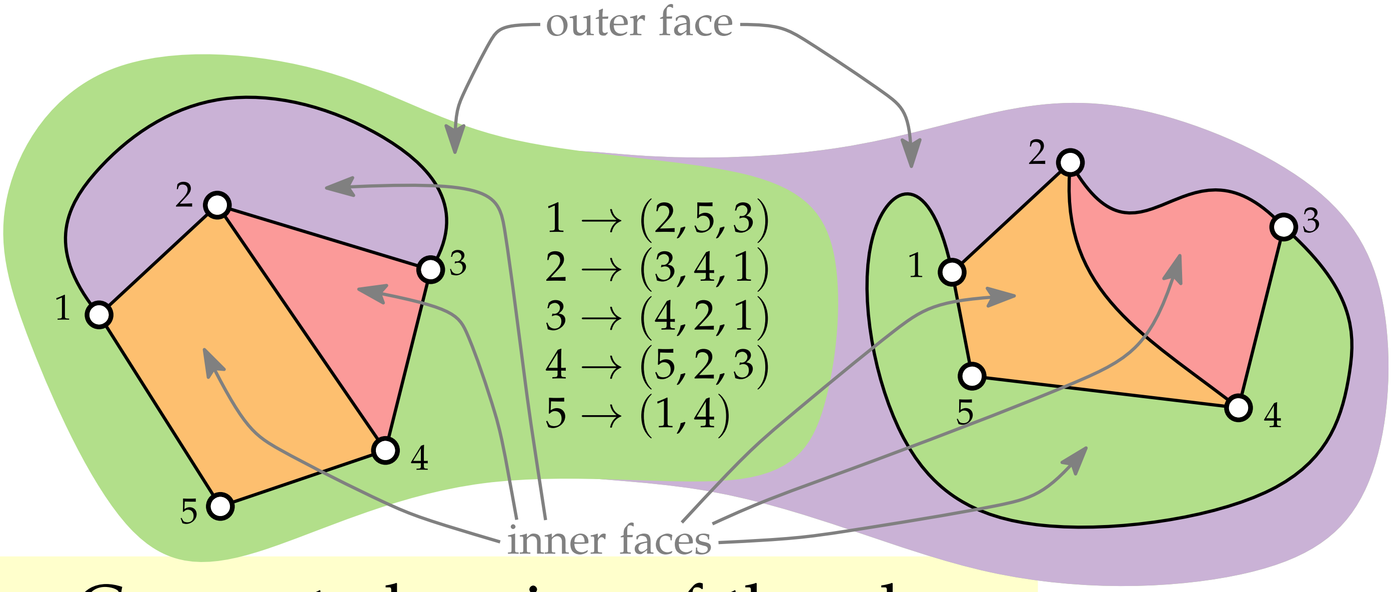

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$m > 1 \Rightarrow$ remove 1 edge e

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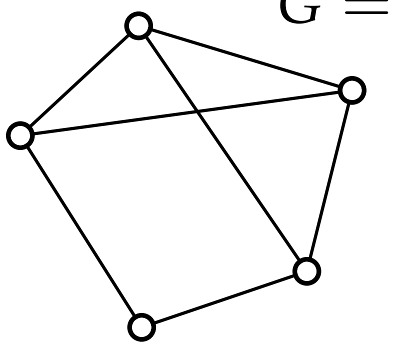

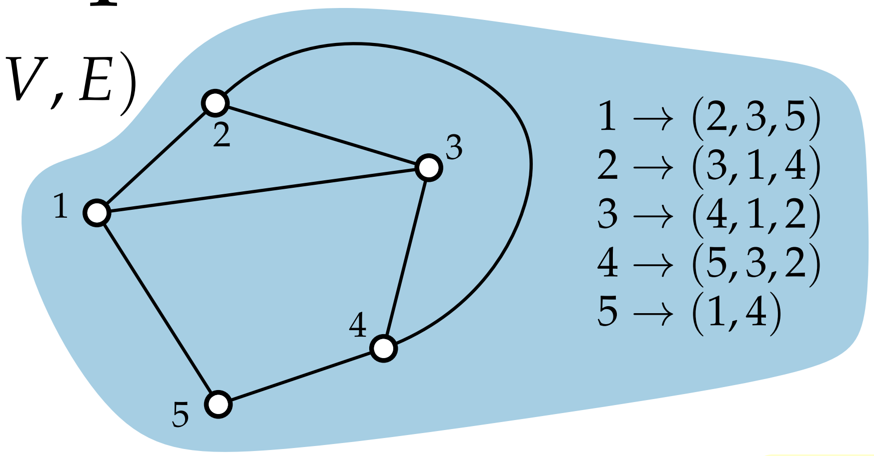

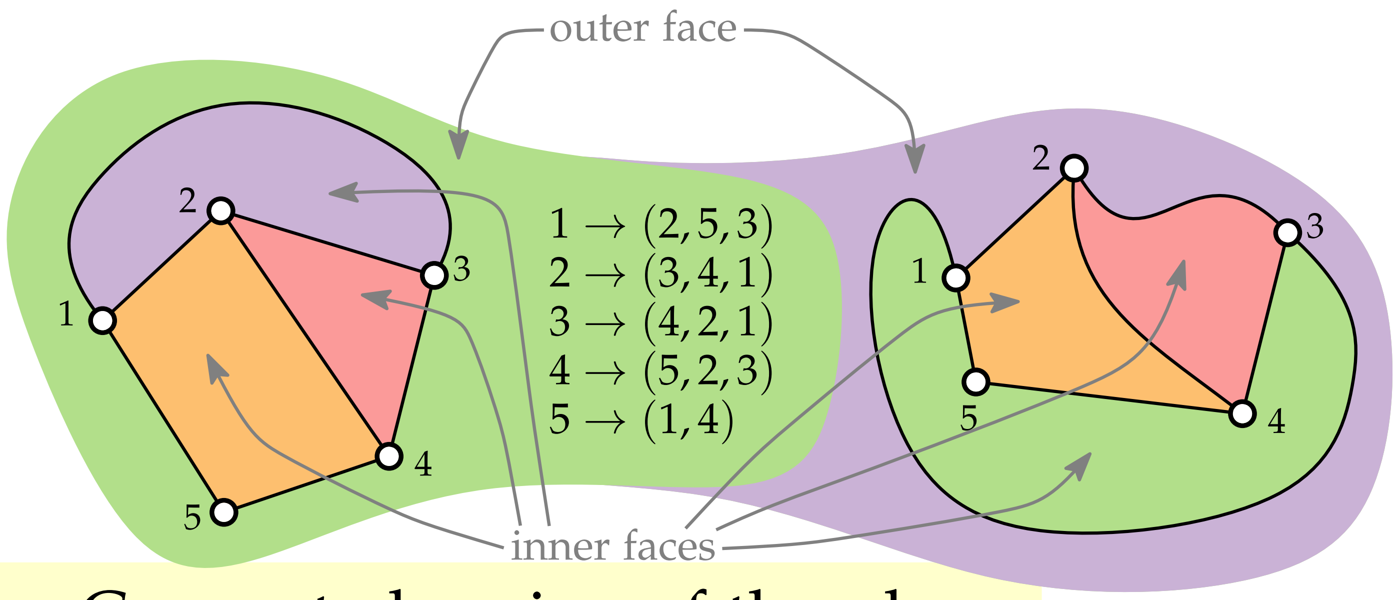

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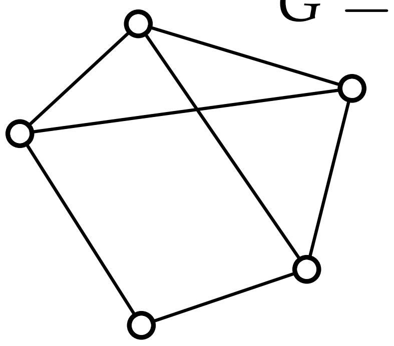

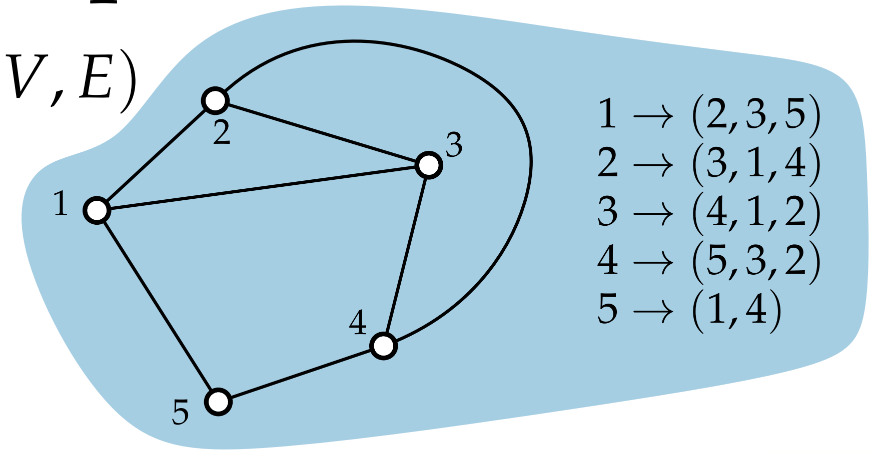

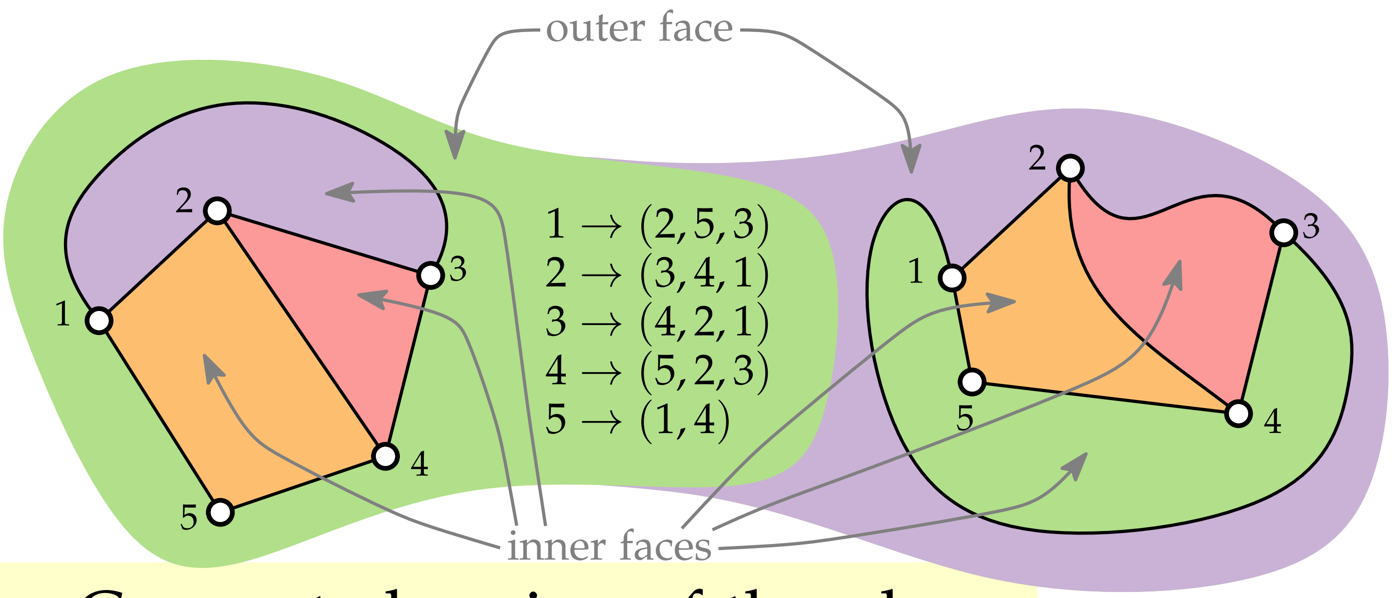

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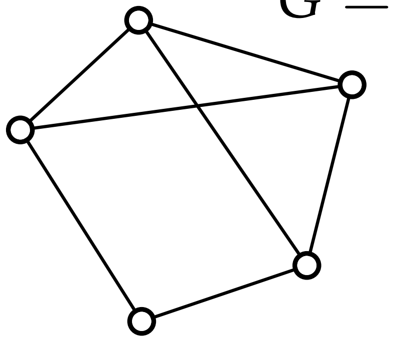

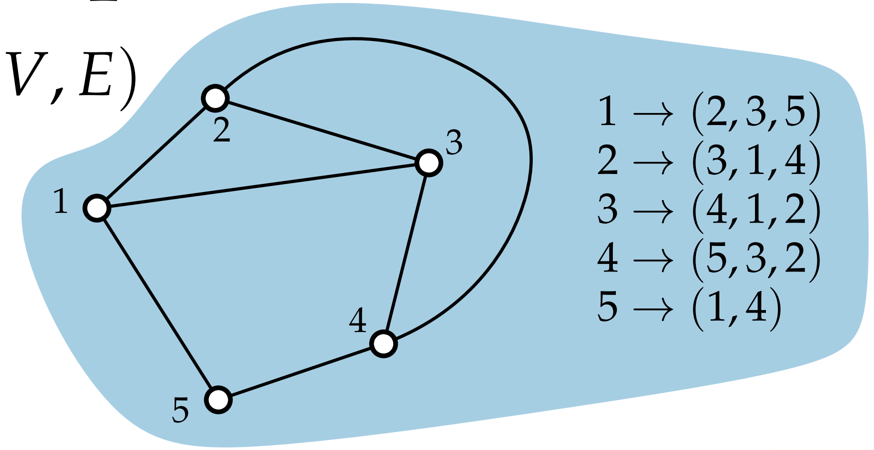

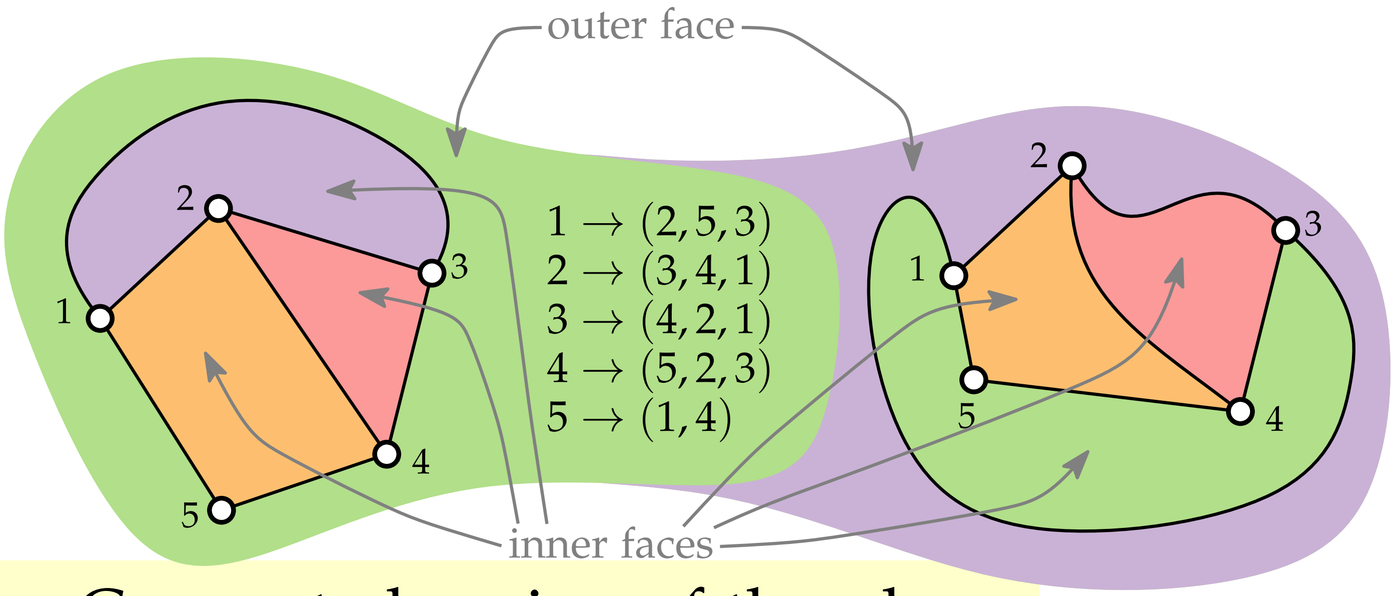

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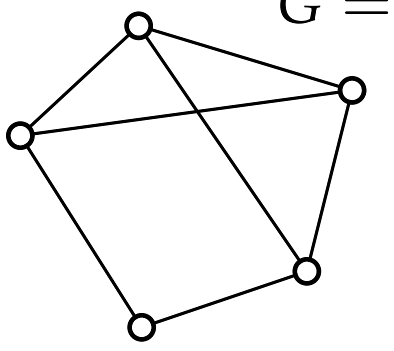

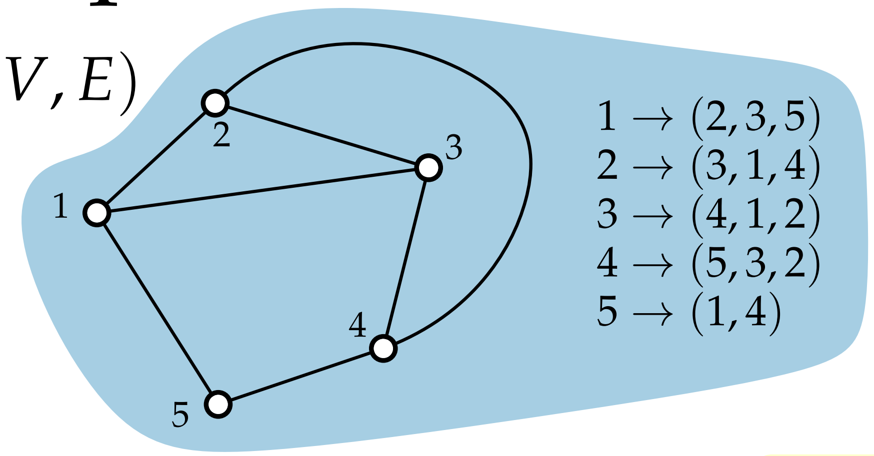

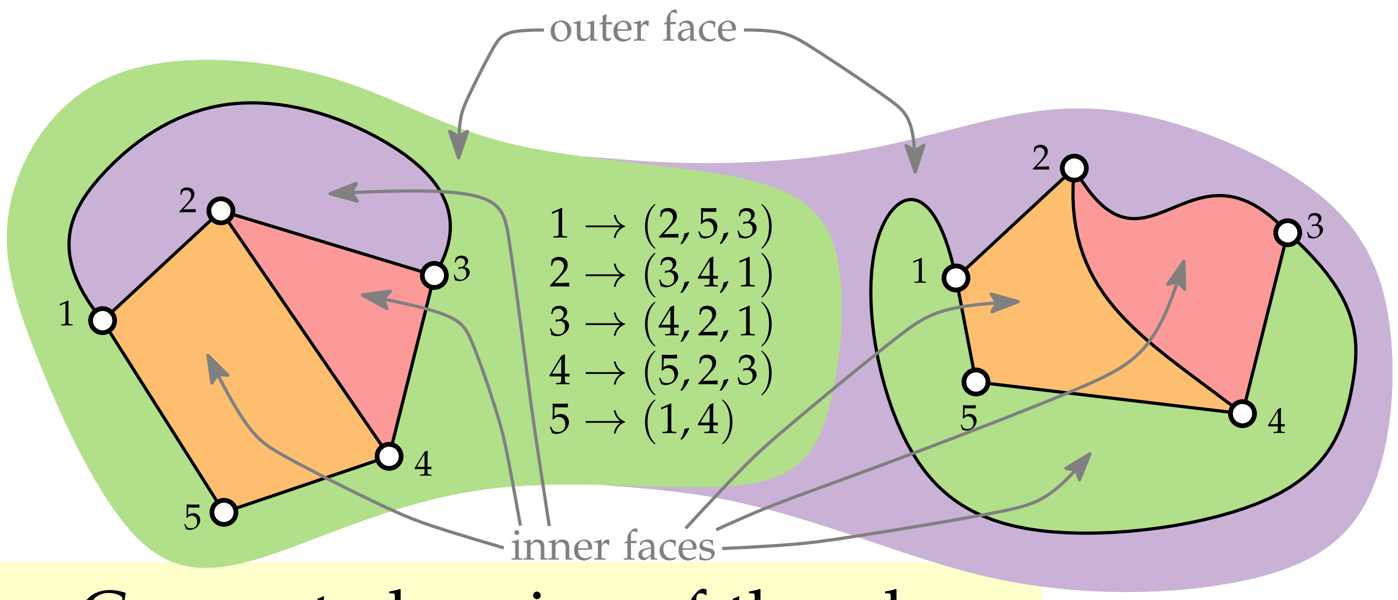

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Properties of Planar Graphs

Euler's polyhedra formula.

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Properties of Planar Graphs

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Theorem. G simple planar graph with $n \geq 3$.

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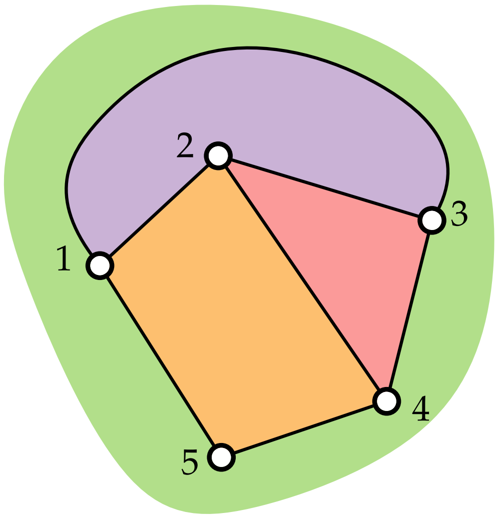

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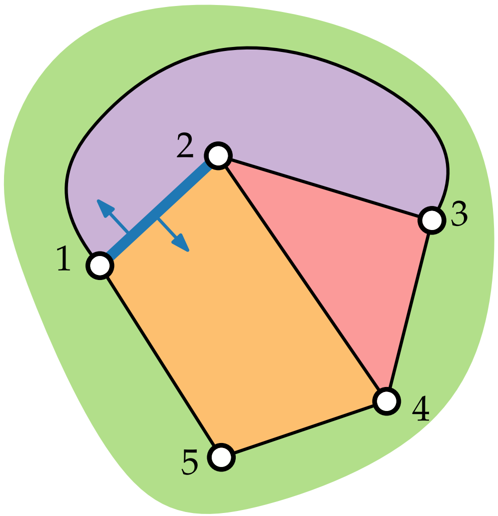

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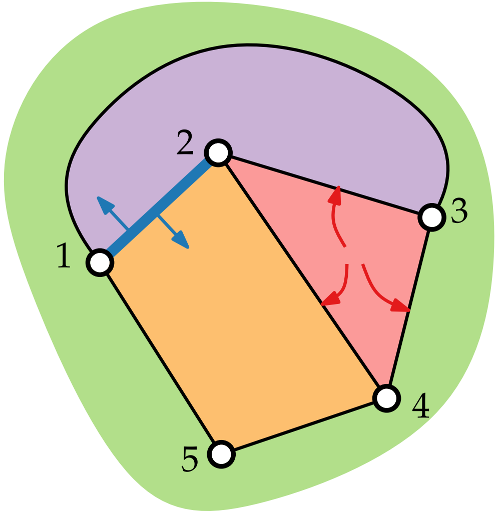

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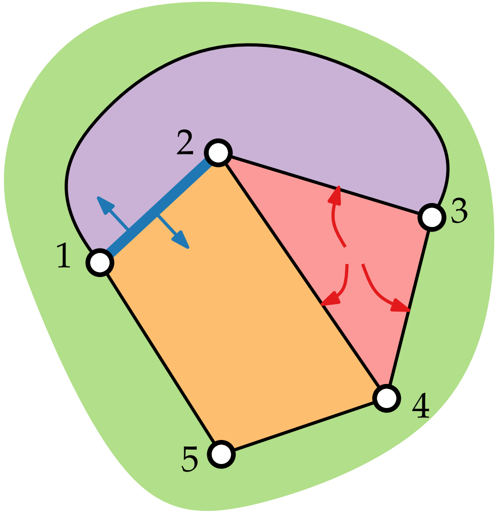

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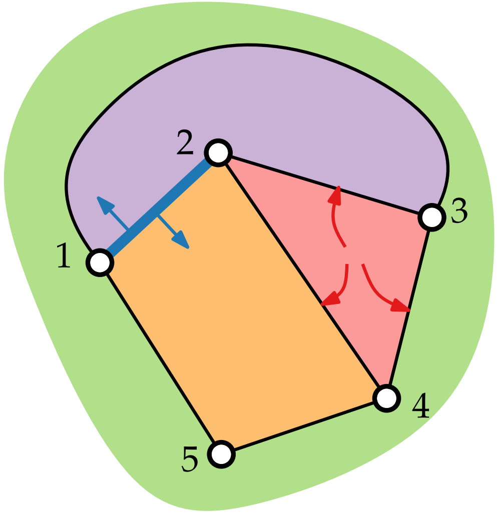

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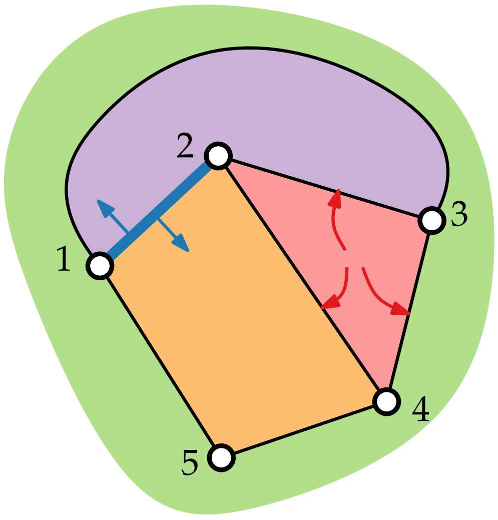

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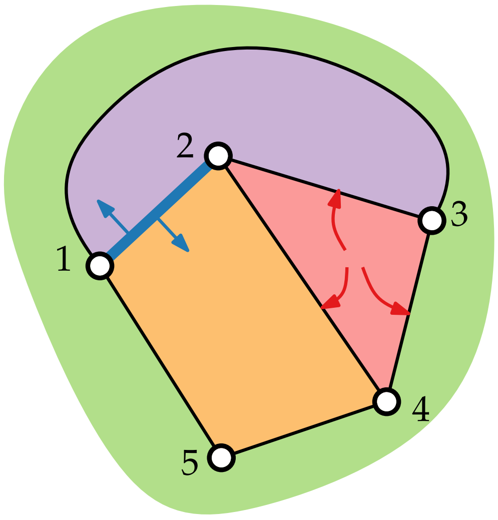

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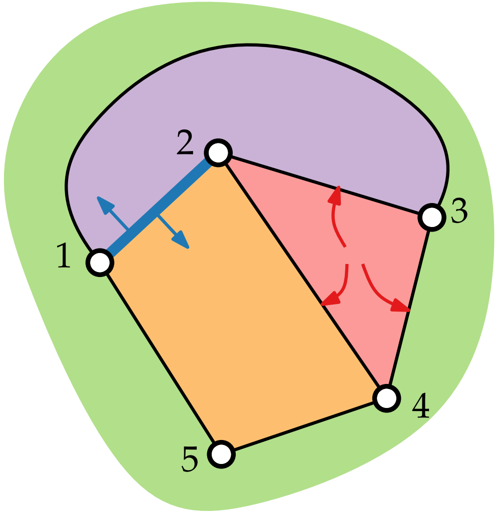

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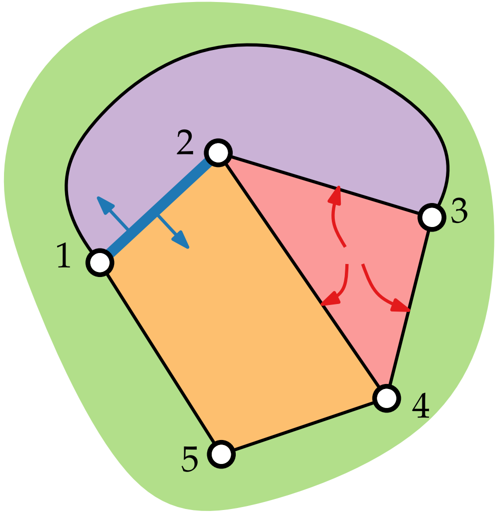

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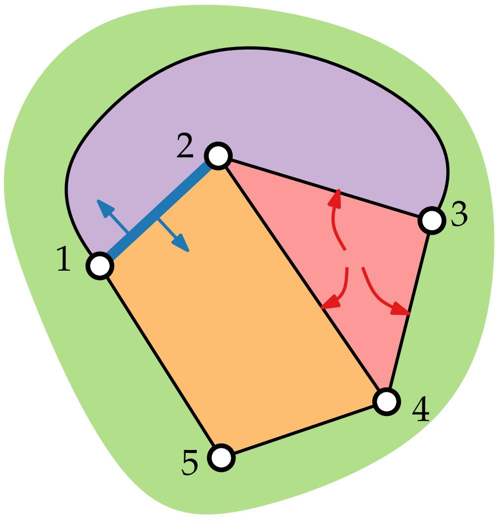

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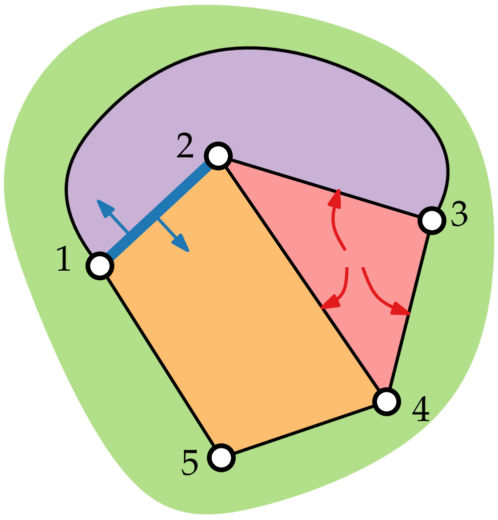

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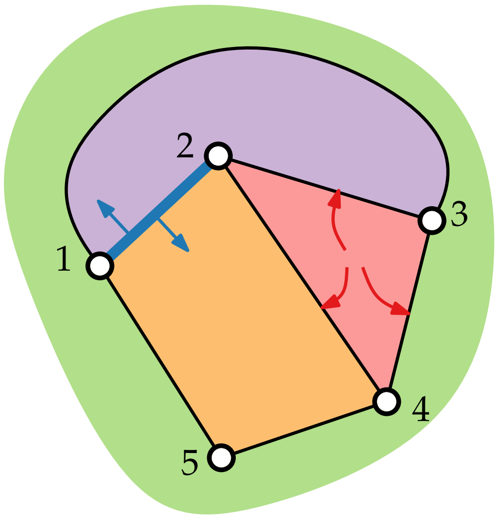

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2. $3f \leq 2m \leq 6n - 12$

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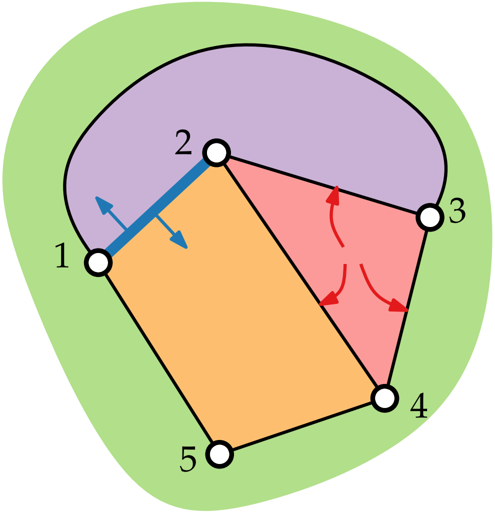

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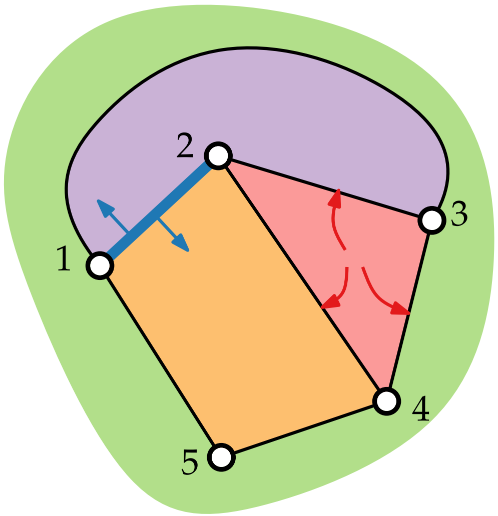

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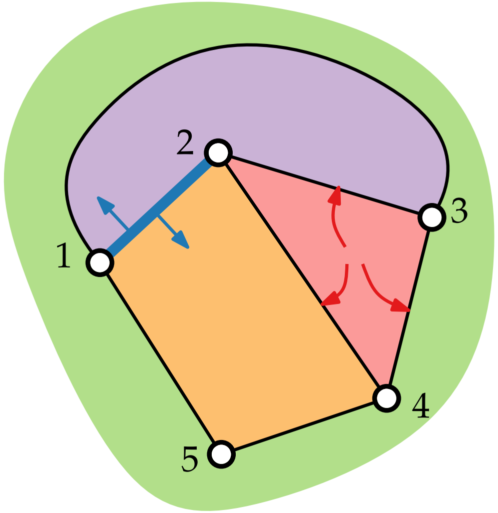

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$\Rightarrow 6 \leq 3c + 3 \leq 3f - 3m + 3n \leq 2m - 3m + 3n = 3n - m$
$\Rightarrow m \leq 3n - 6$

2. $3f \leq 2m \leq 6n - 12 \Rightarrow f \leq 2n - 4$

3. $\sum_{v \in V} \deg(v)$

Properties of Planar Graphs

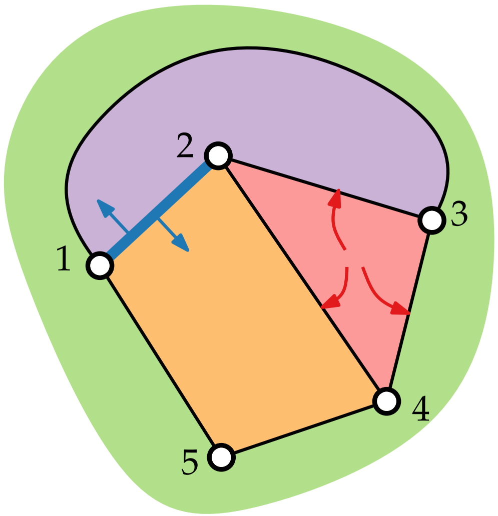
Euler's polyhedra formula.

#faces - #edges + #vertices = #conn.comp. + 1

f - m + n = c + 1

Theorem. G simple planar graph with $n \geq 3$.

1. $m \leq 3n - 6$ 2. $f \leq 2n - 4$
3. There is a vertex of degree at most five



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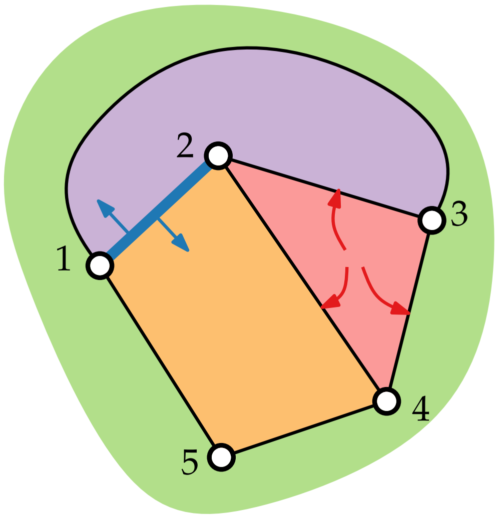

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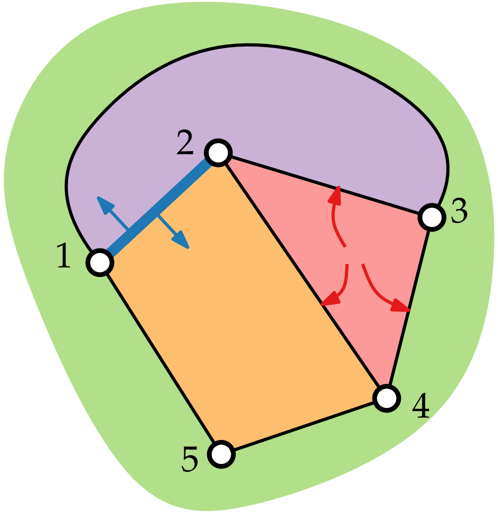

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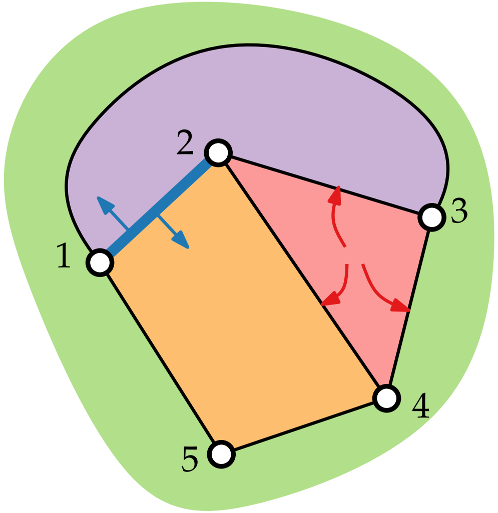

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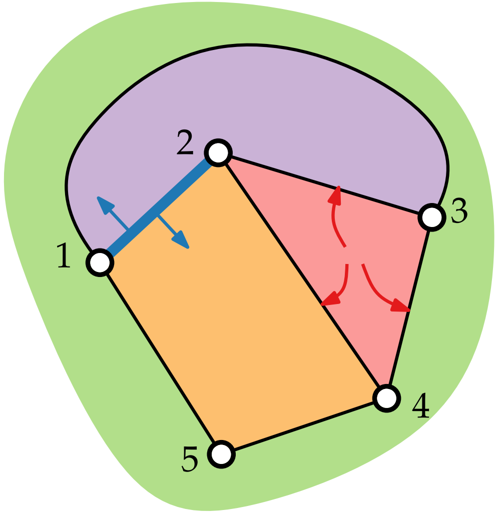

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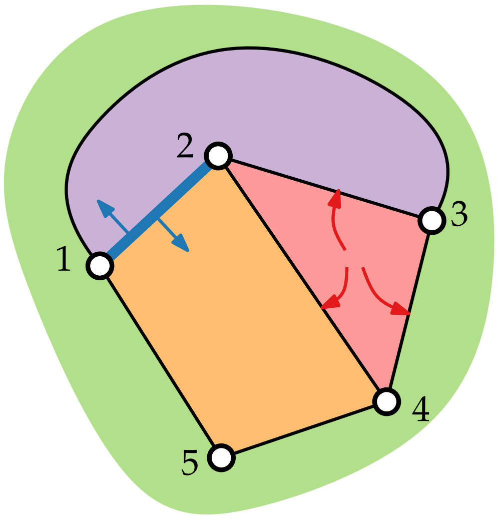

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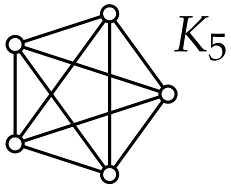
Complete graphs



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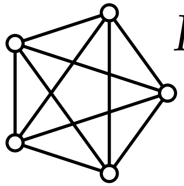

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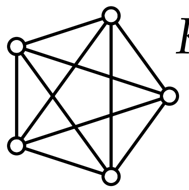
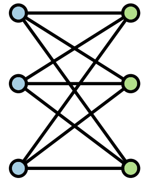
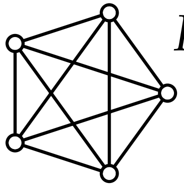


K_5



$K_{3,3}$

Complete graphs



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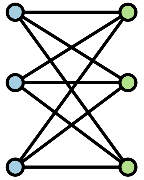

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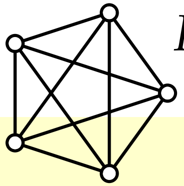

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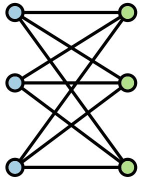
V_1 V_2



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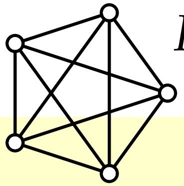
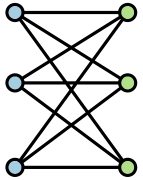
Complete graphs





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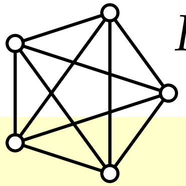

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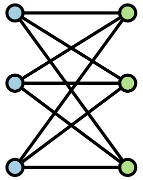

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K_5:

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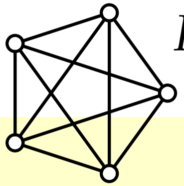
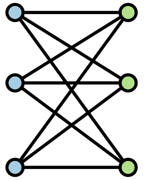

K_5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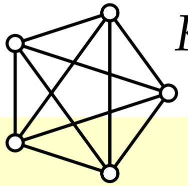

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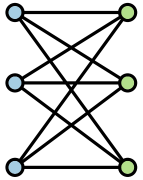

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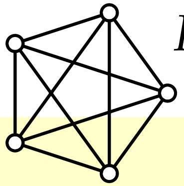

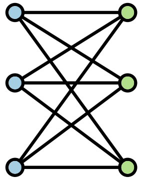

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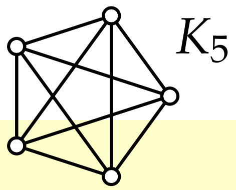
Complete graphs



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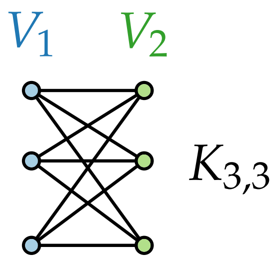

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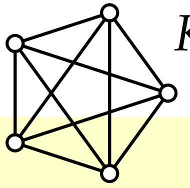

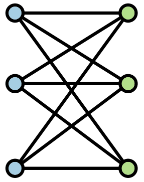

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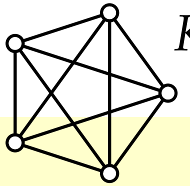

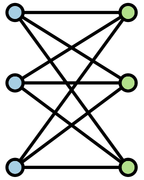

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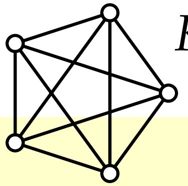

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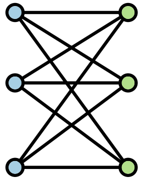

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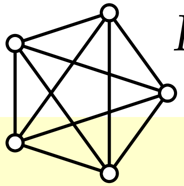

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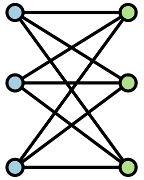

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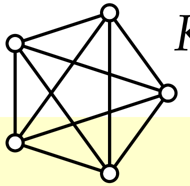

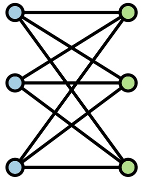

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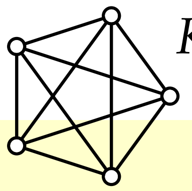
Complete graphs



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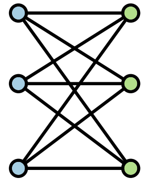

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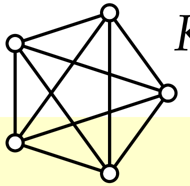
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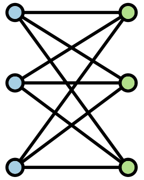

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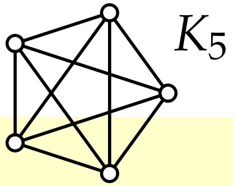

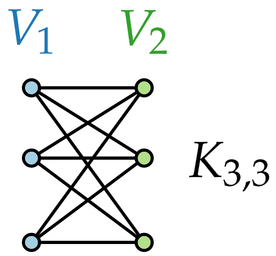

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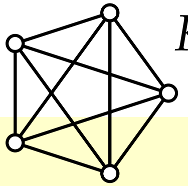

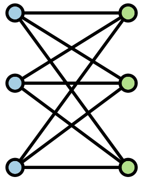

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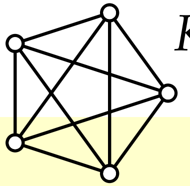

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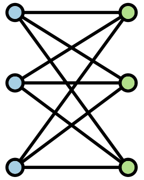

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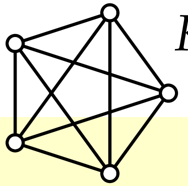
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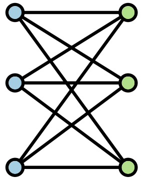
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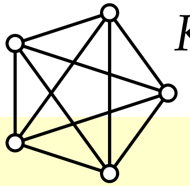

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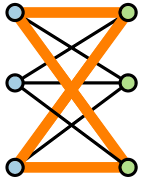

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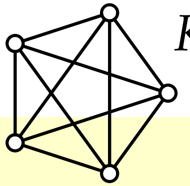
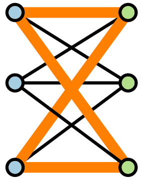
Complete graphs





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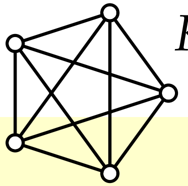
Every face incident to ≥ 4 edges (in hypothetical planar drawing)

Complete graphs



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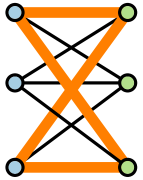

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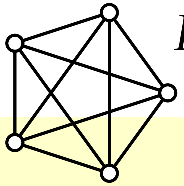

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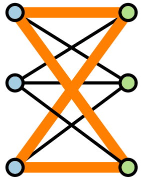

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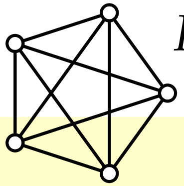

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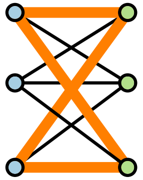

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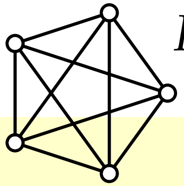

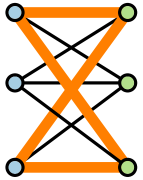

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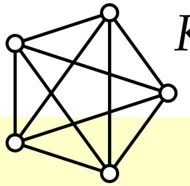

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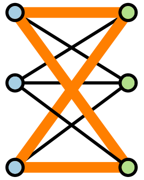

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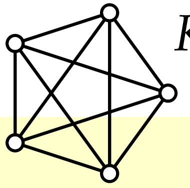

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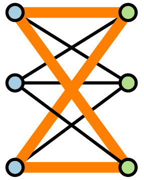

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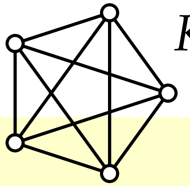
Complete graphs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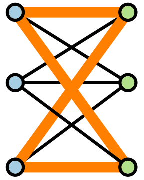
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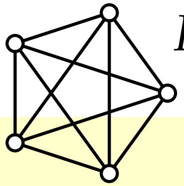

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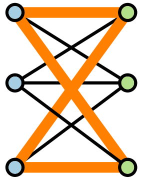

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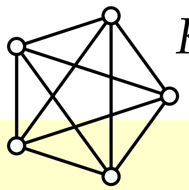
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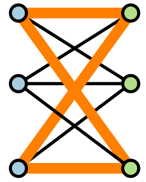
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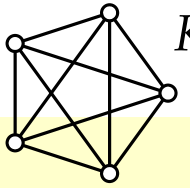

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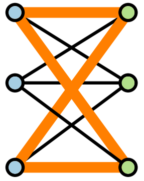

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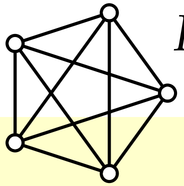

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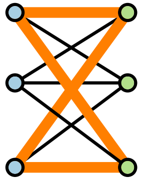

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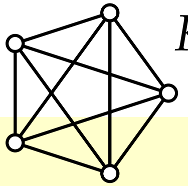

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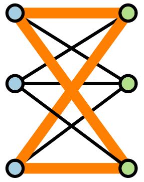

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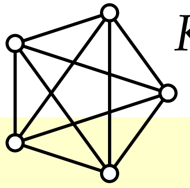

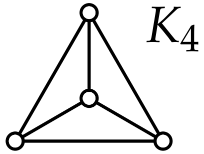

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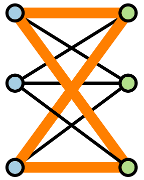

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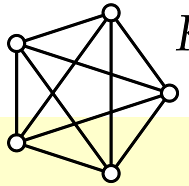

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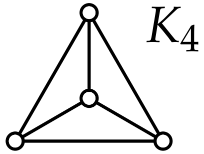

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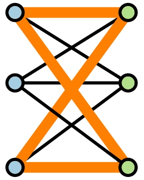

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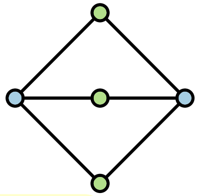

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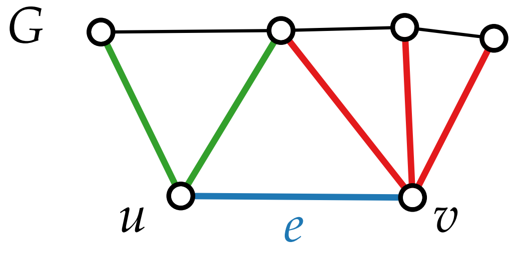

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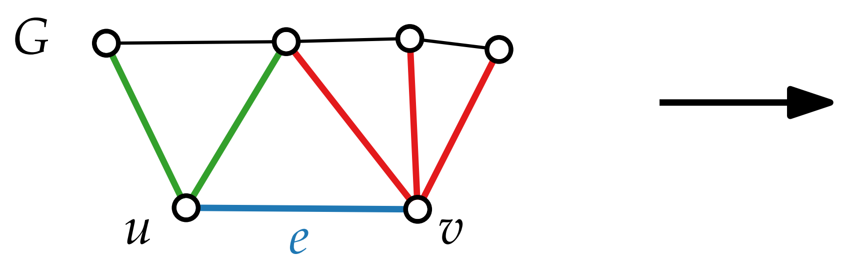

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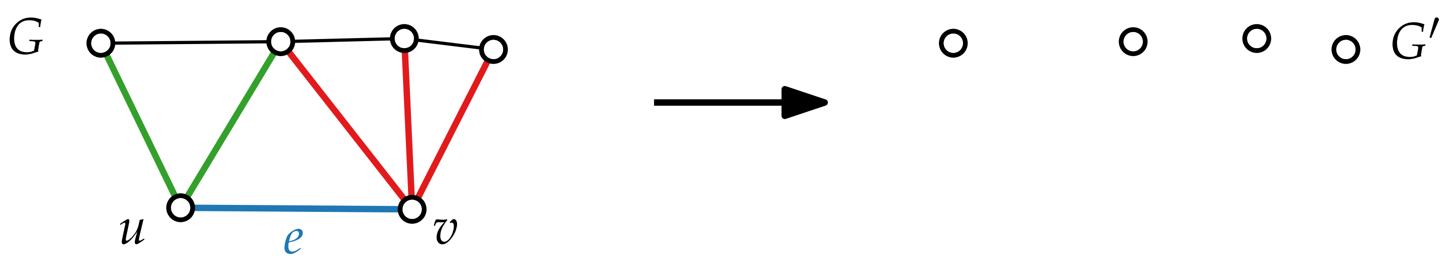

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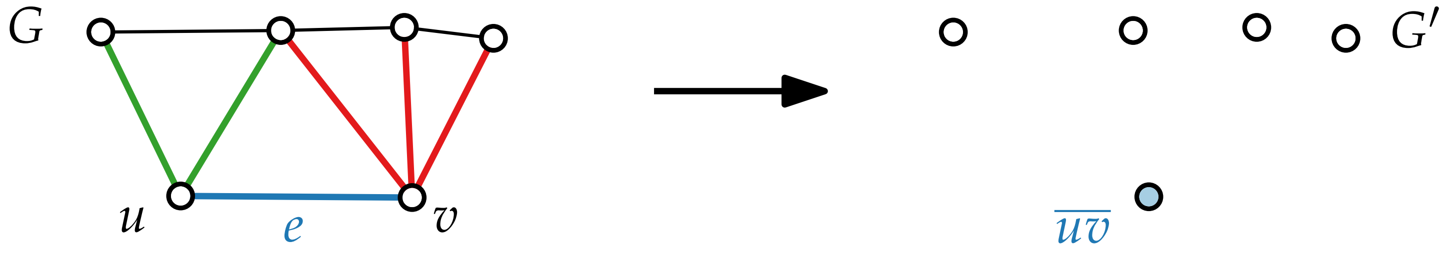

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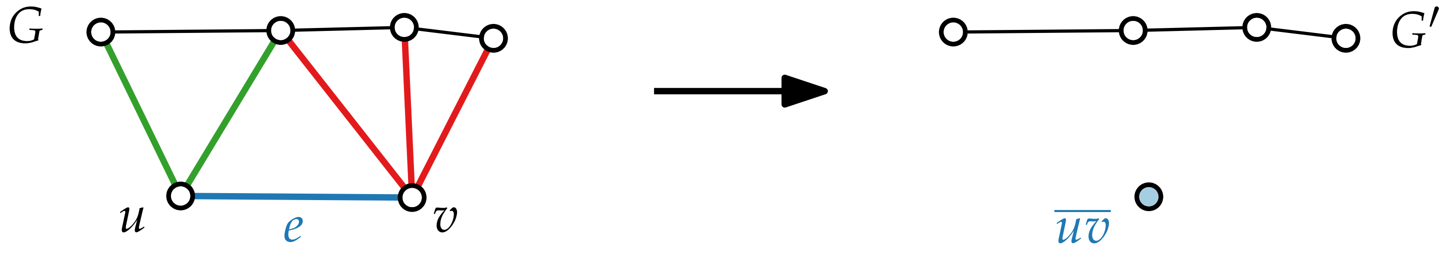

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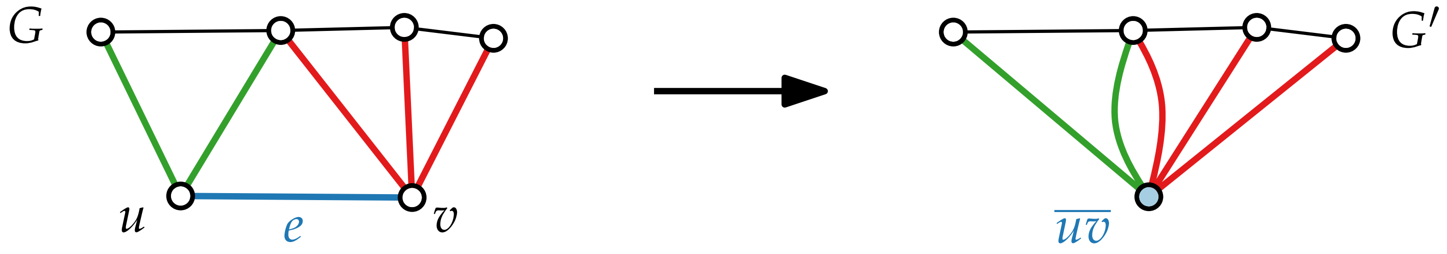

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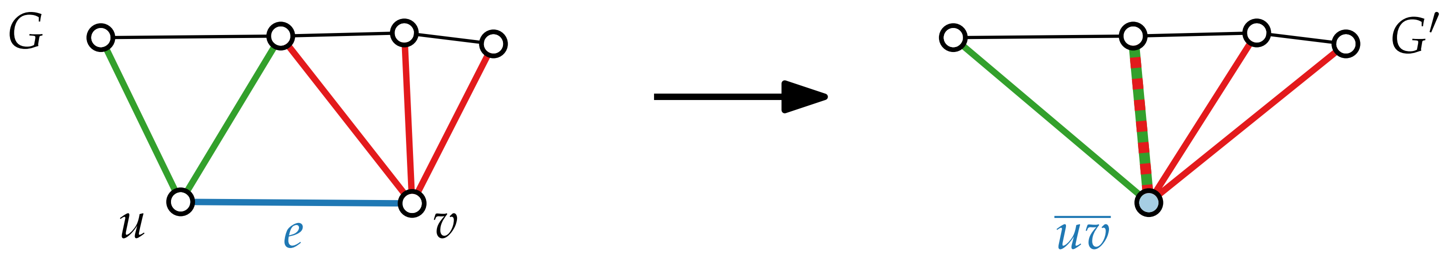

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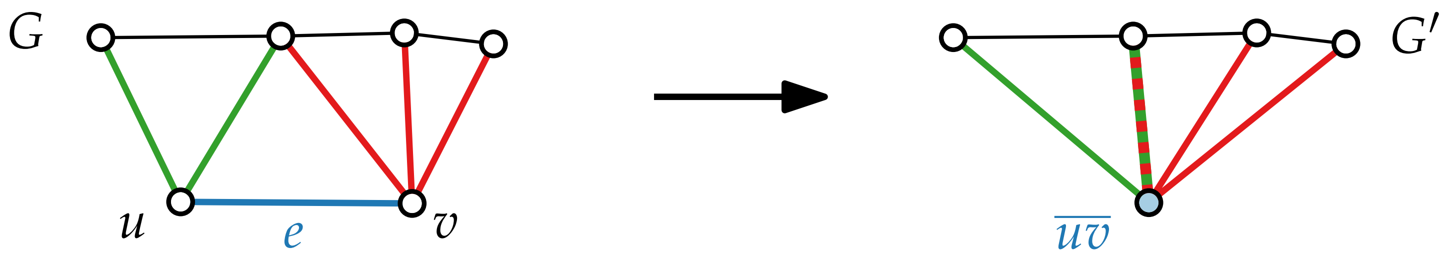

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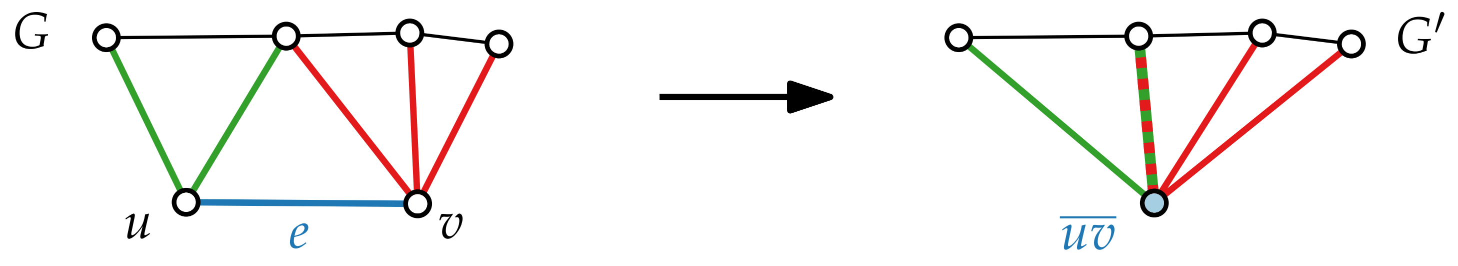

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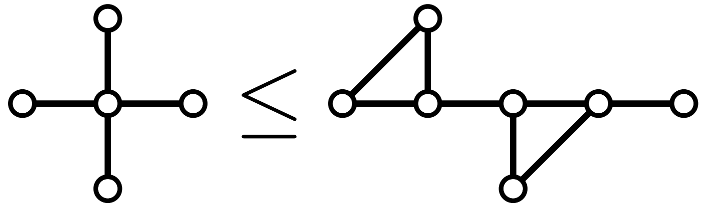

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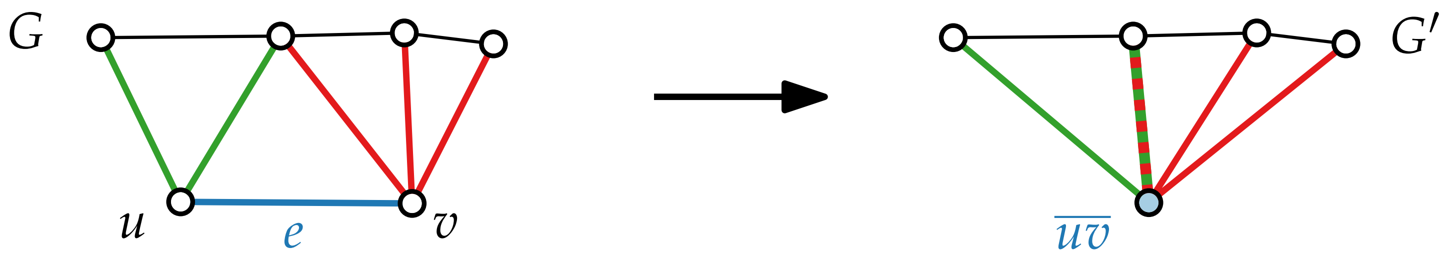

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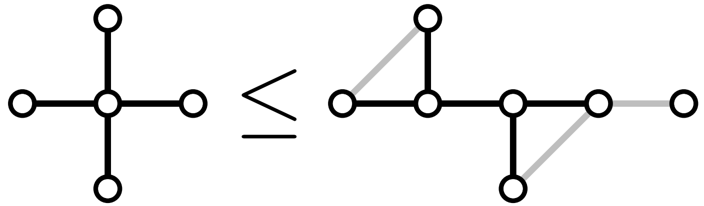

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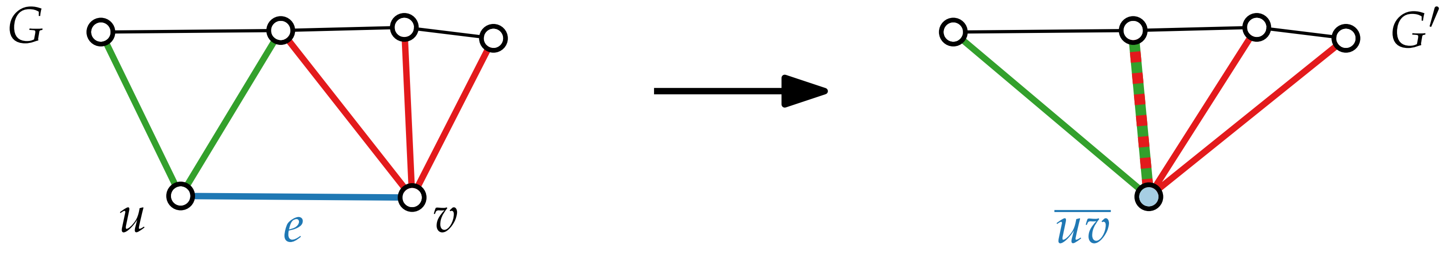
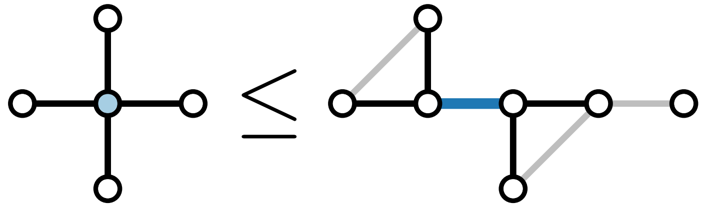


A graph H is a **minor** of G (write $H \leq G$) if it is obtained by a set of contractions from a subgraph of G.



Contractions and Minors

G simple graph and $e = uv \in E$

Contracting e gives the graph $G' = (V', E')$
$V' = V \setminus \{u, v\} \cup \overline{uv}$
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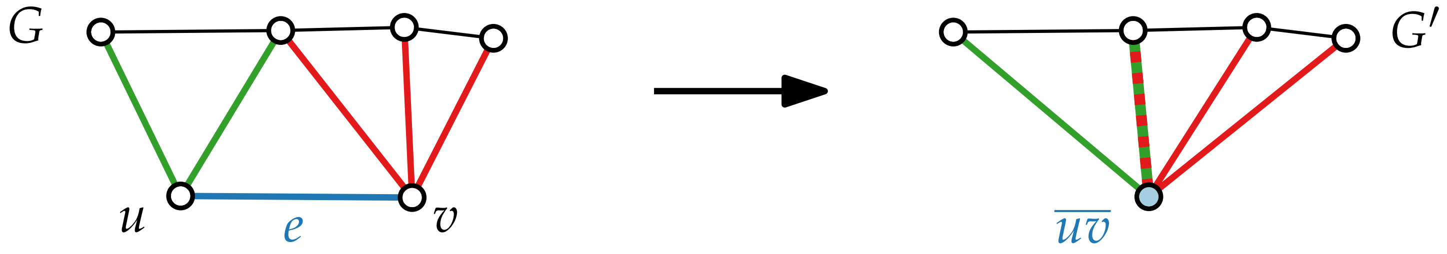

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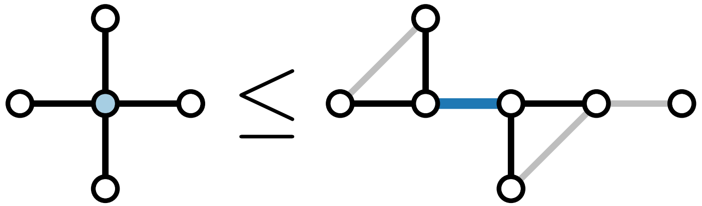

Observation.
G planar, $H \leq G \Rightarrow H$ planar

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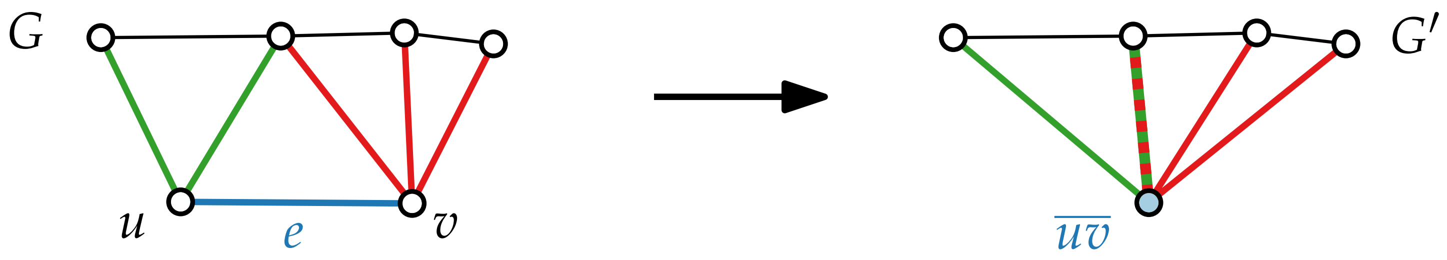

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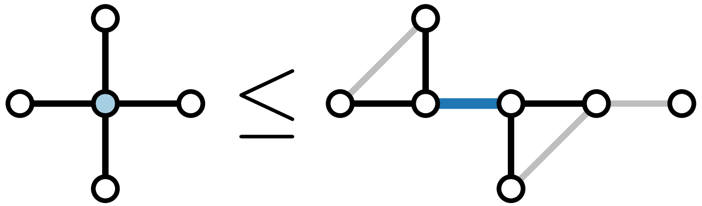

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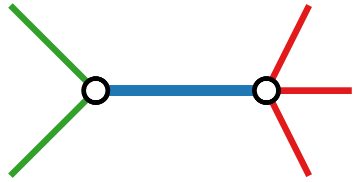

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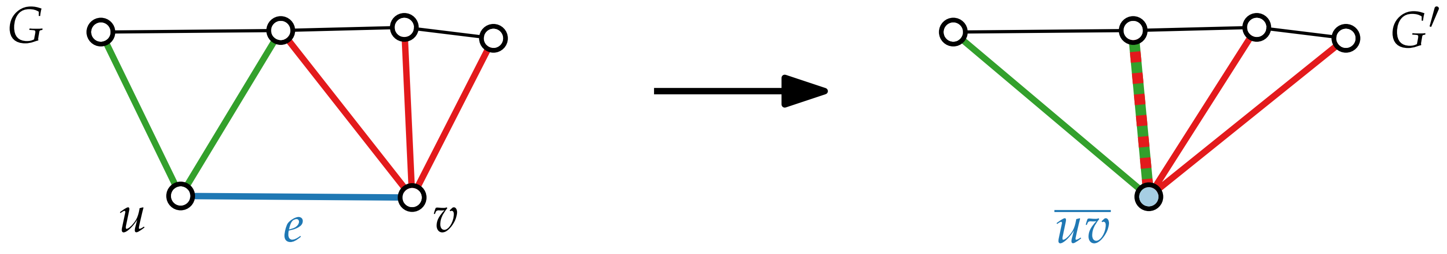

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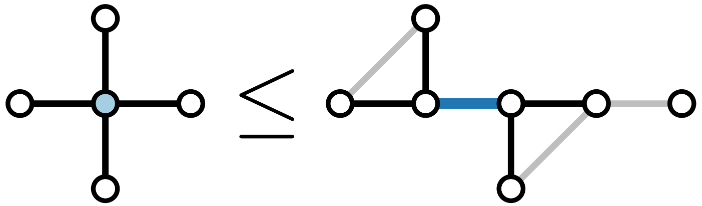

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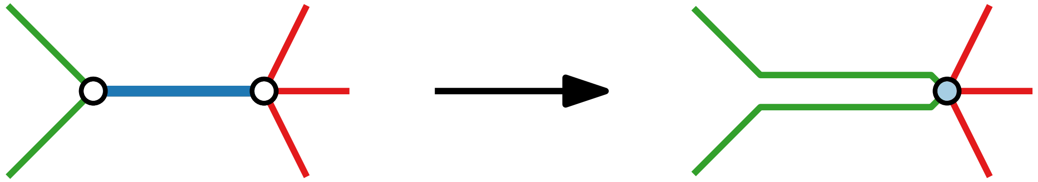

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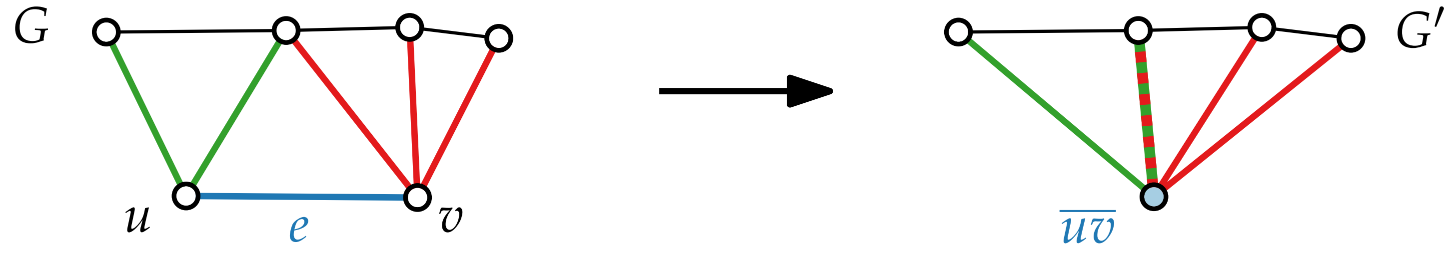

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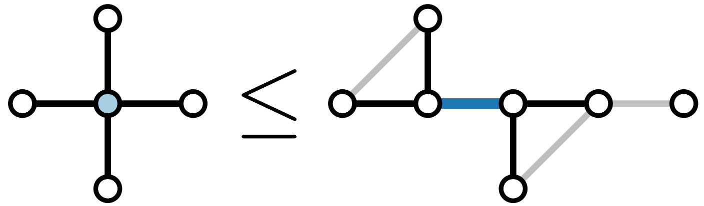

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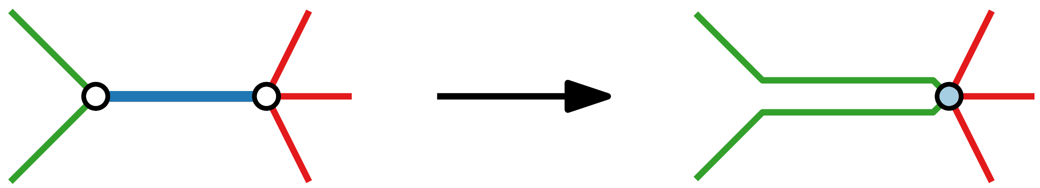
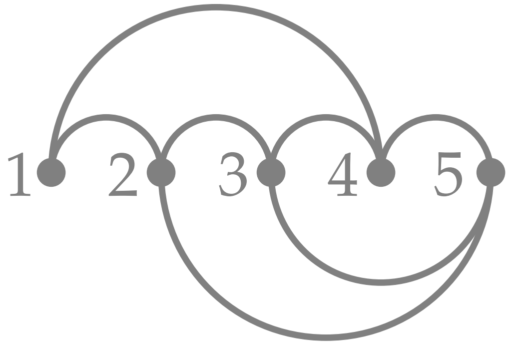


Theorem. [Kuratowski 1930]
G planar $\Leftrightarrow$
neither K_5 nor $K_{3,3}$ minor of G

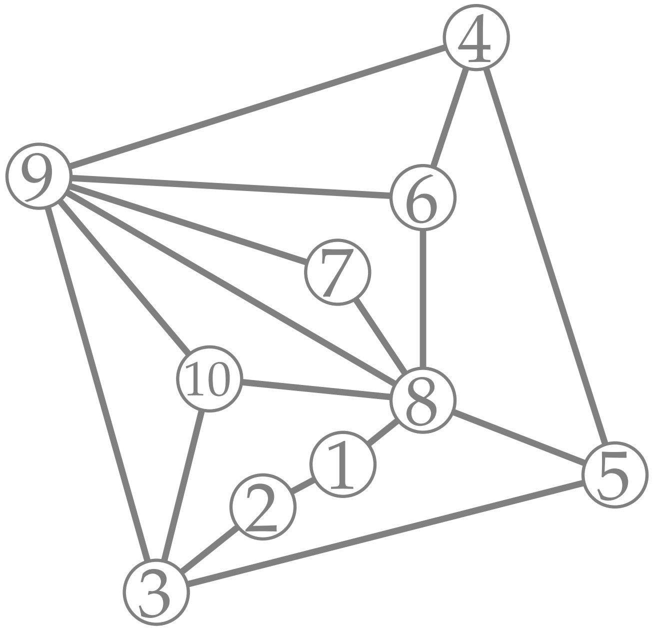
Kazimierz Kuratowski
Warschau 1896–1980 Warschau

Universität Trier

Visualization of Graphs



Lecture 1:
The Graph Visualization Problem

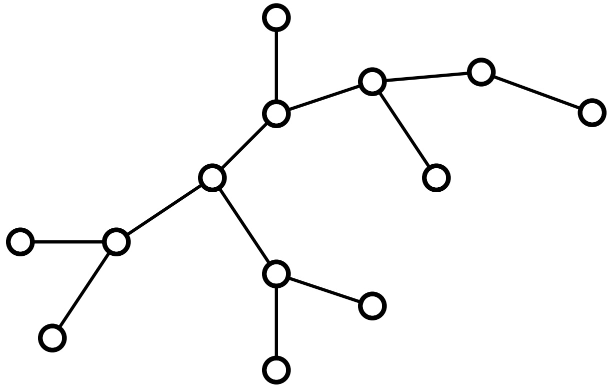


Part V:
Binary Search Trees

Philipp Kindermann
Summer Semester 2021

(Rooted) Trees

G is a **tree** if the following equivalent conditions hold:



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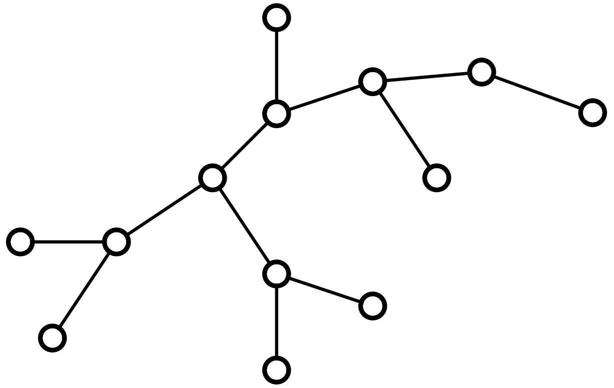

(Rooted) Trees

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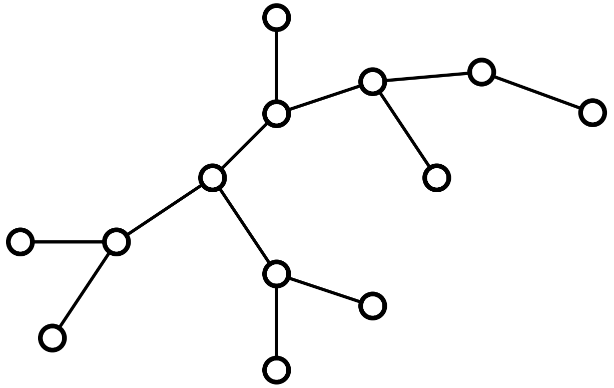

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G is a **tree** if the following equivalent conditions hold:

1. there is exactly one v-w-path between any $v, w \in V$
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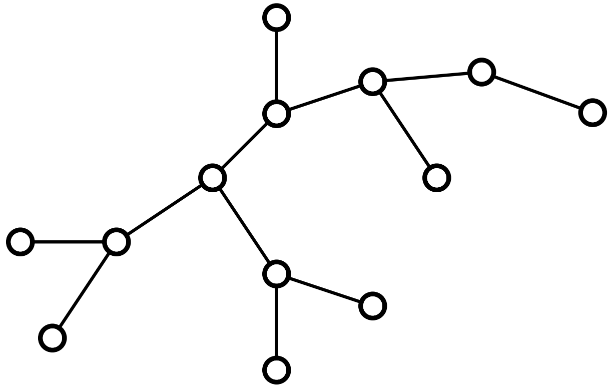
3. G cycle-free and $m = n - 1$



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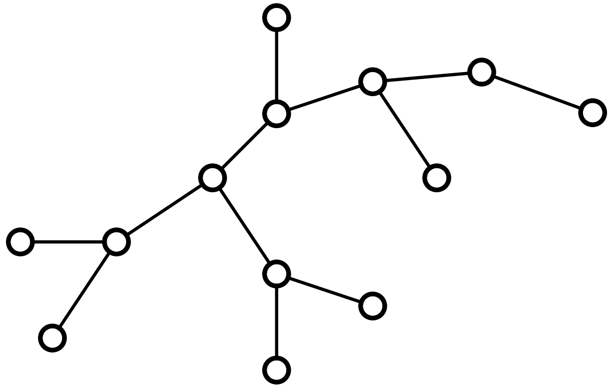

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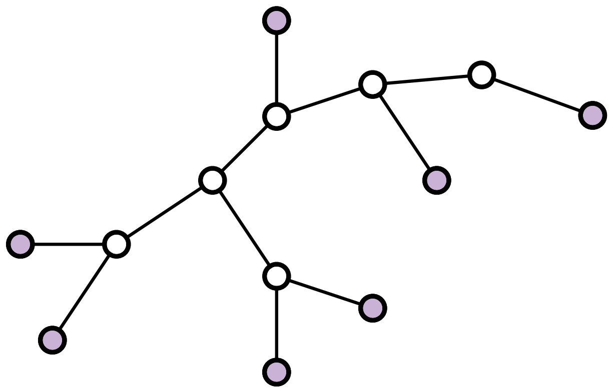

Leaf: Vertex of degree 1

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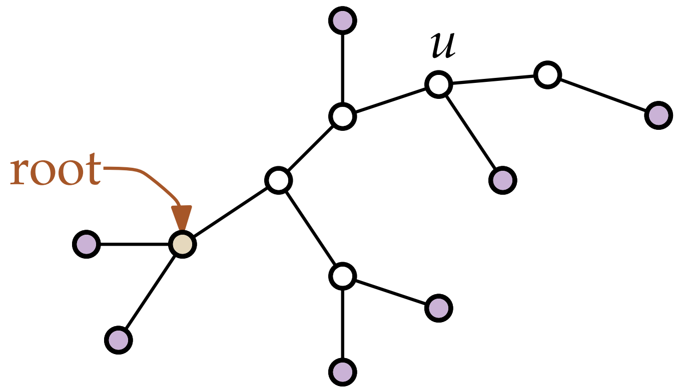

Leaf: Vertex of degree 1

Rooted tree: tree with designated **root**

Parent: Neighbor on path to root

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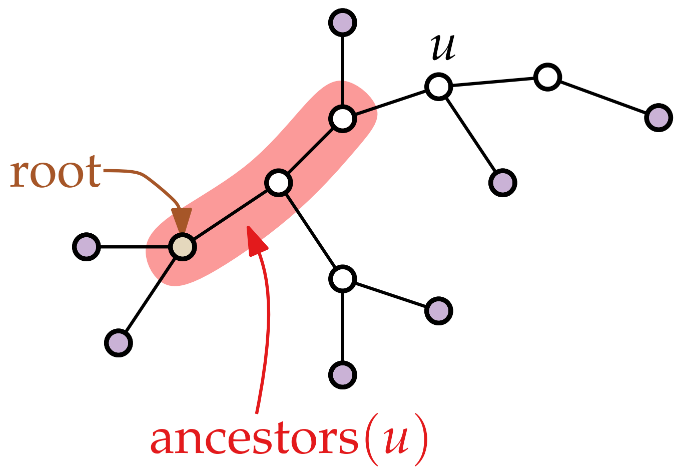

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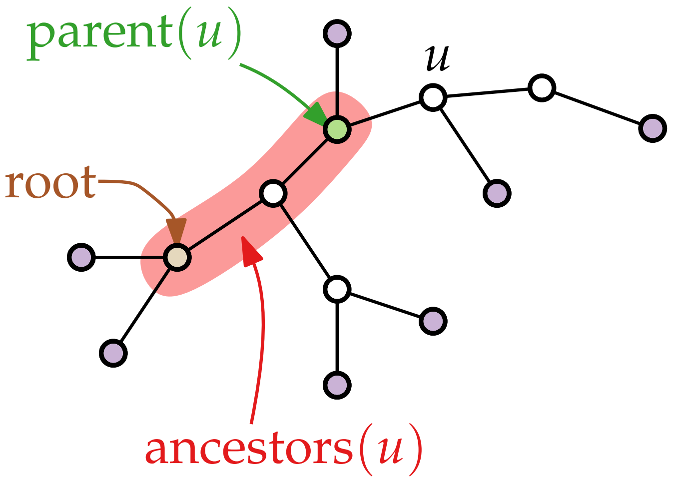

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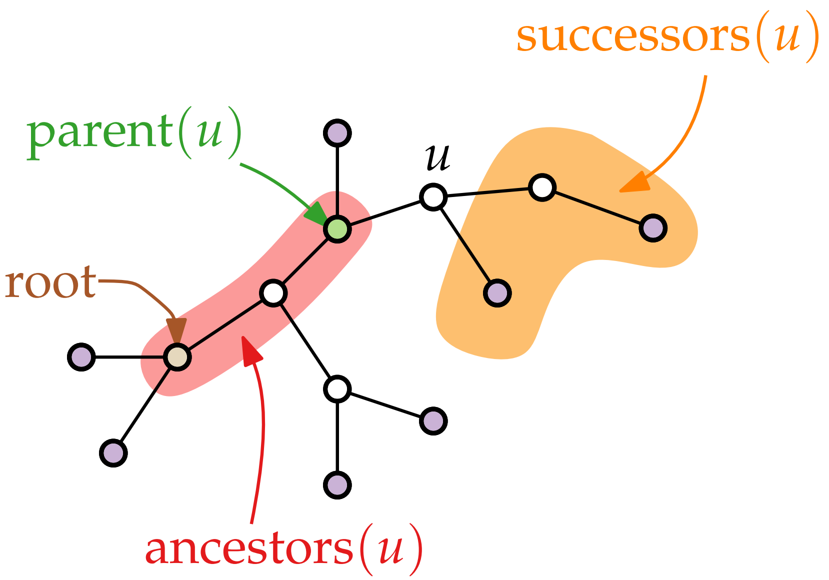

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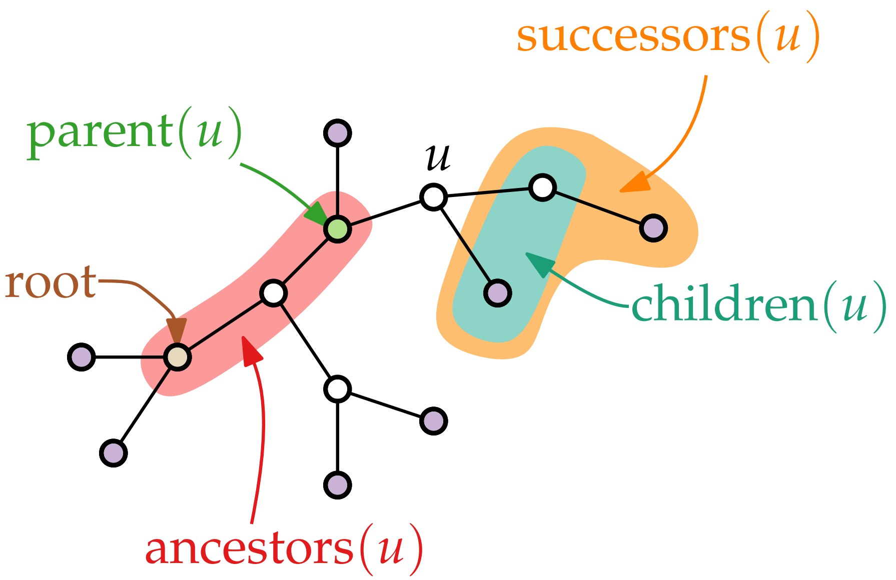

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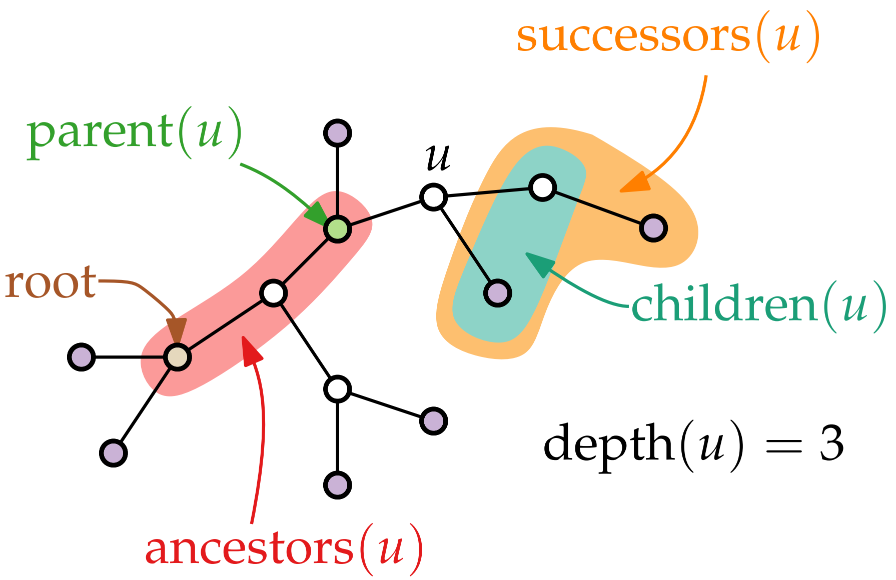

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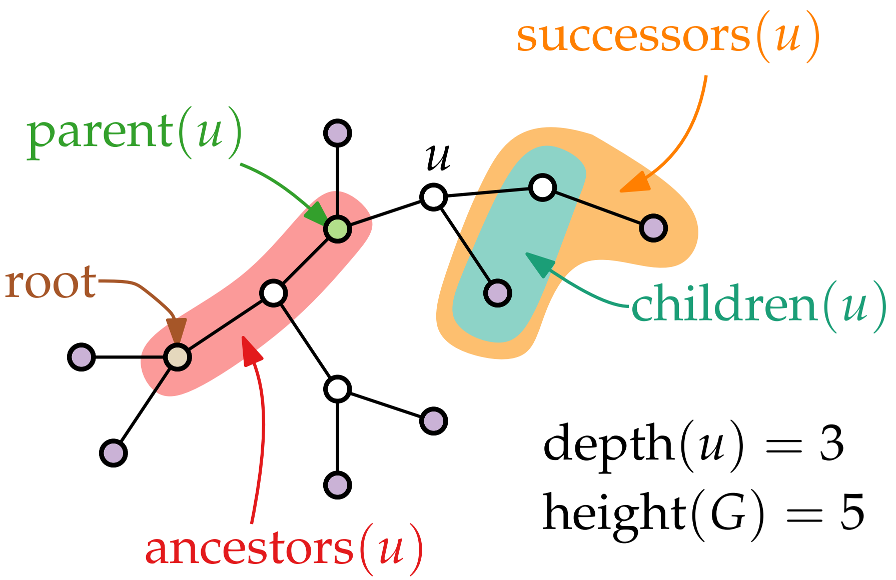

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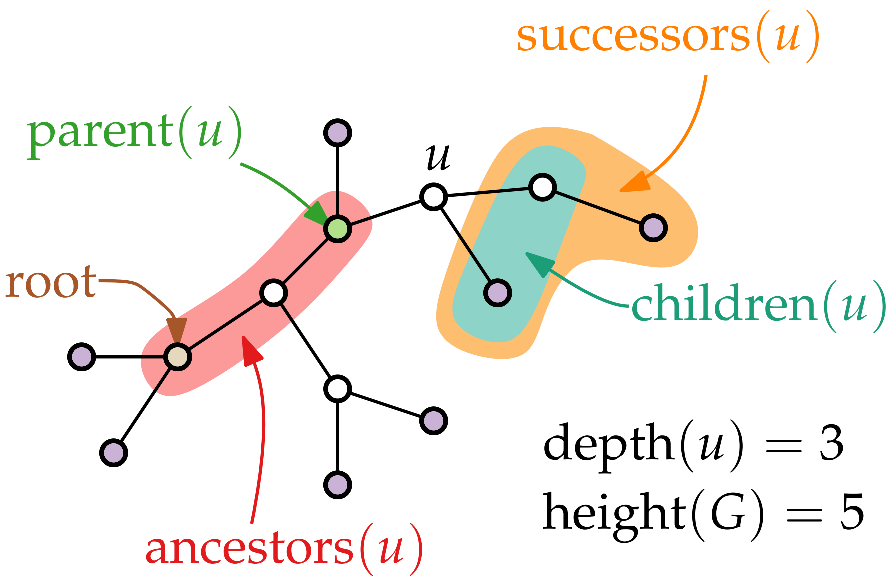

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Binary Tree: At most two children per vertex (left / right child)

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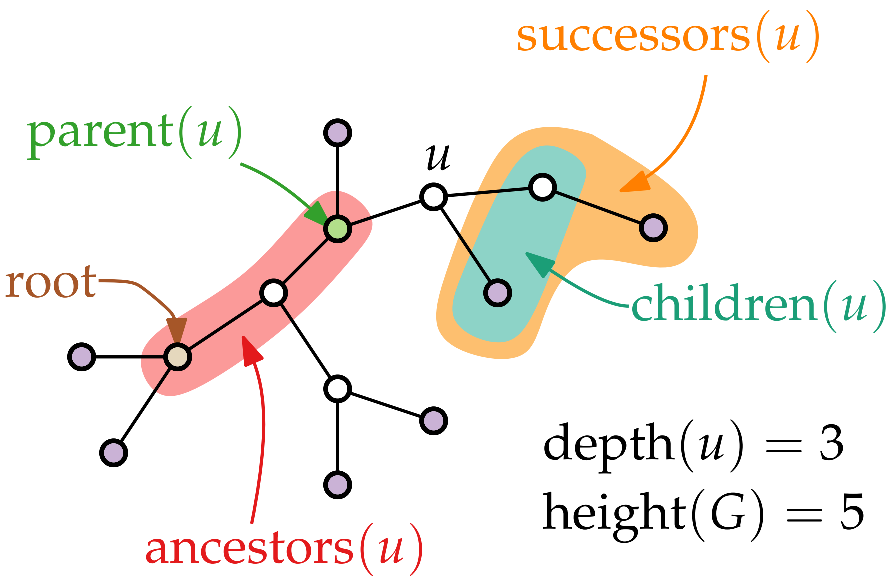

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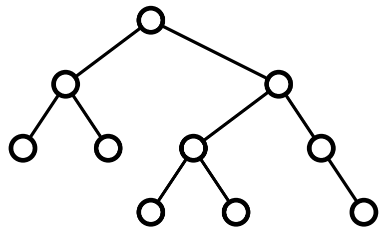

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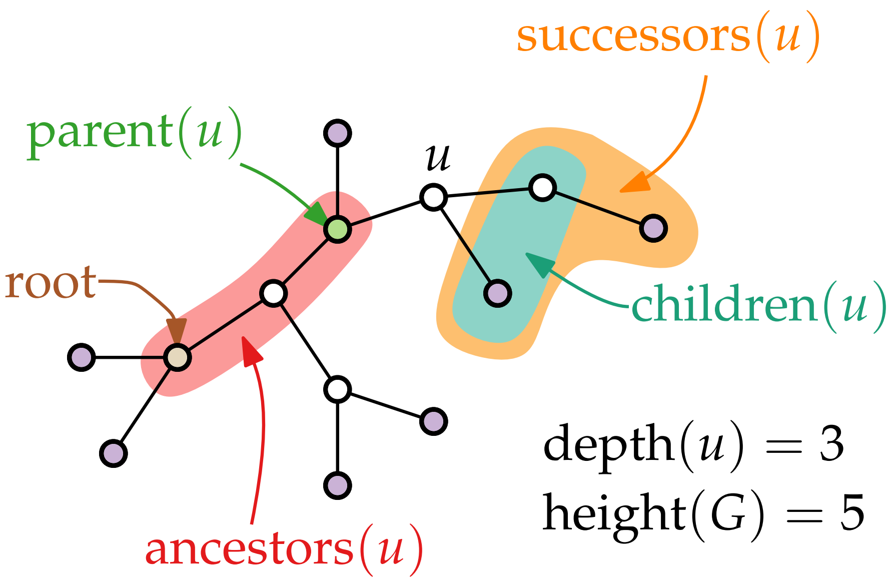

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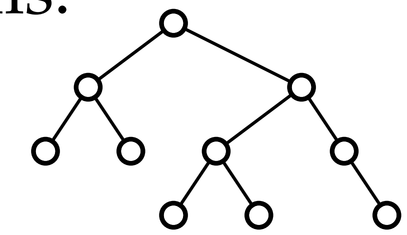
3 traversals:



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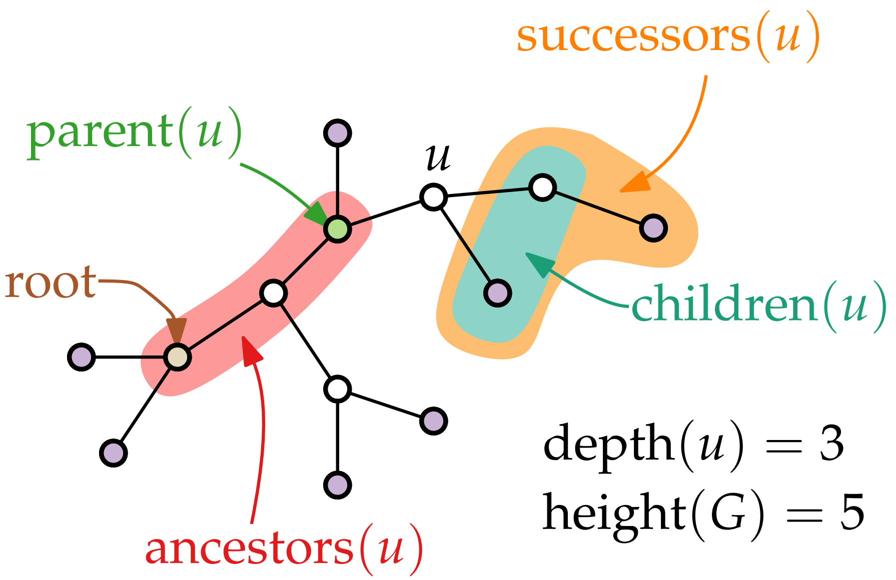

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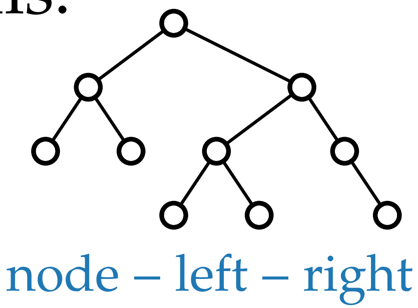
3 traversals:

preorder



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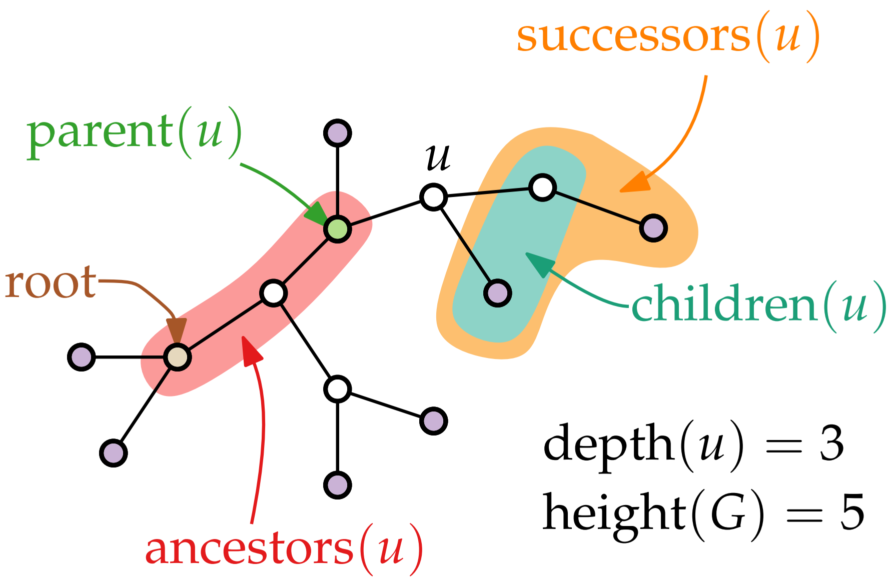

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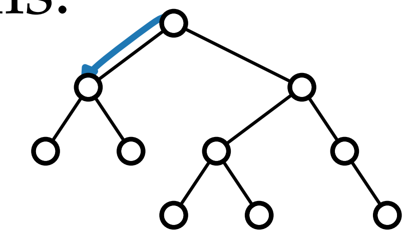

node – left – right

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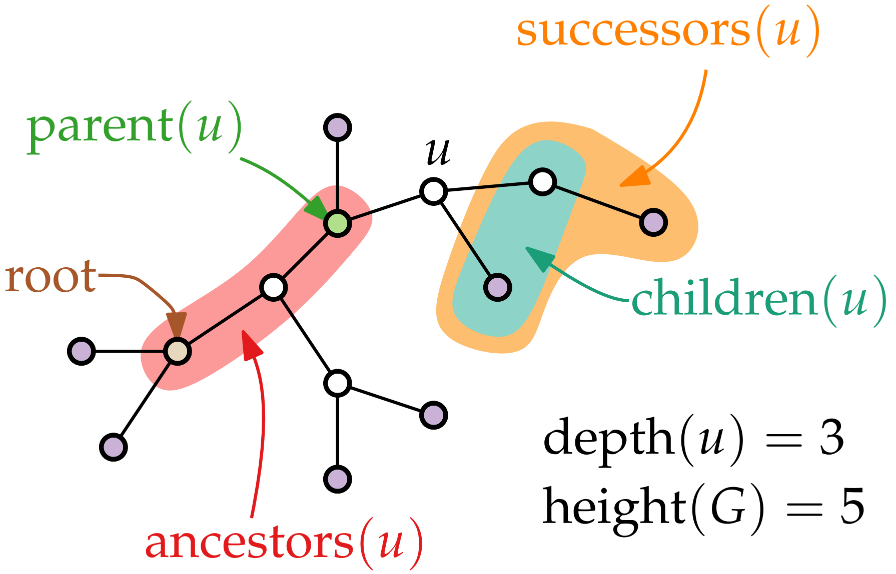

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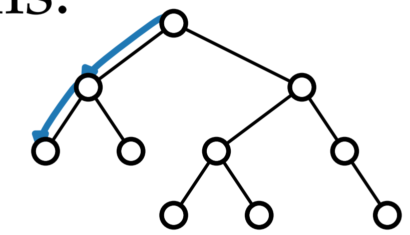

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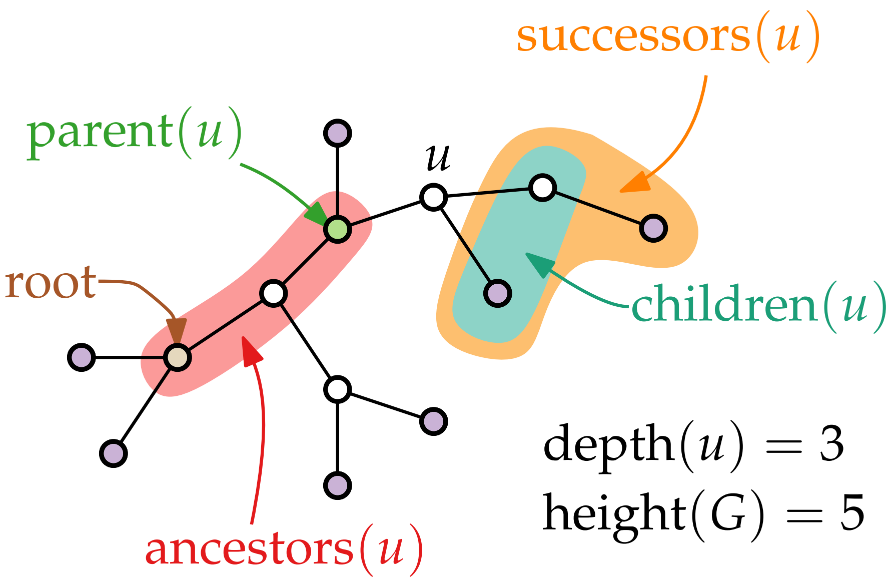

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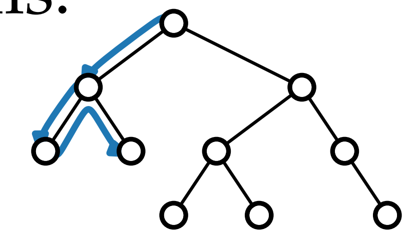

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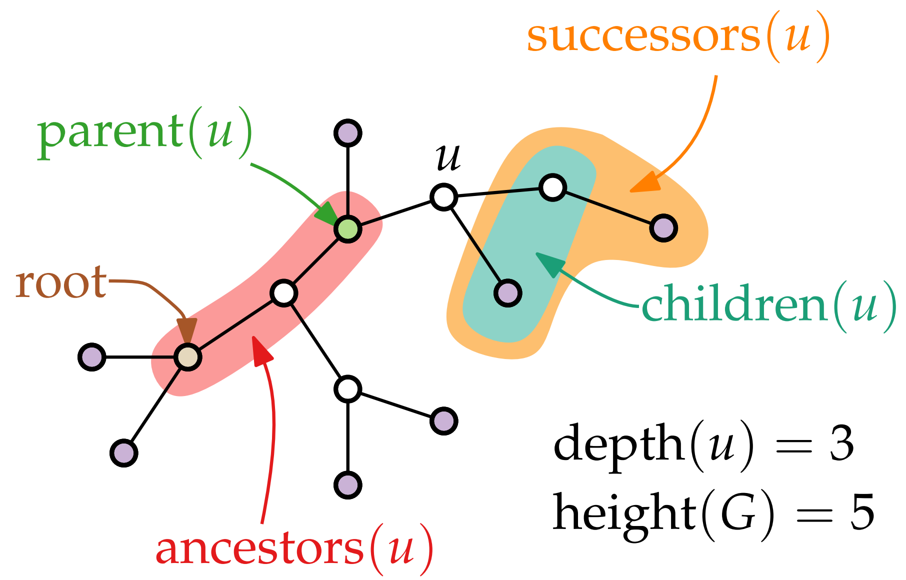

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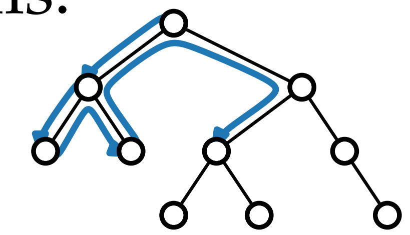
Successor: Vertex not on path to root
Child: Neighbor not on path to root
Depth: Length of path to root
Height: Maximum depth of a leaf

3 traversals:

preorder



node – left – right

Binary Tree: At most two children per vertex (left / right child)

(Rooted) Trees

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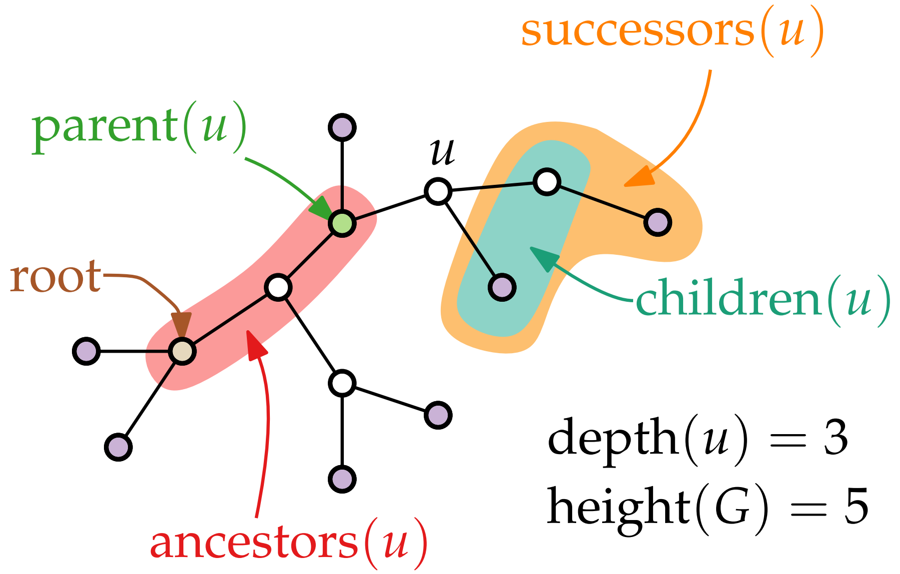

Leaf: Vertex of degree 1
Rooted tree: tree with designated **root**
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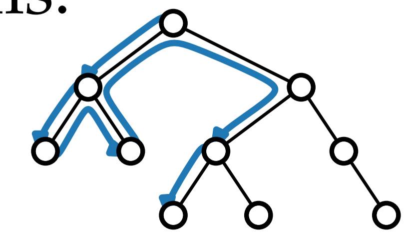
preorder



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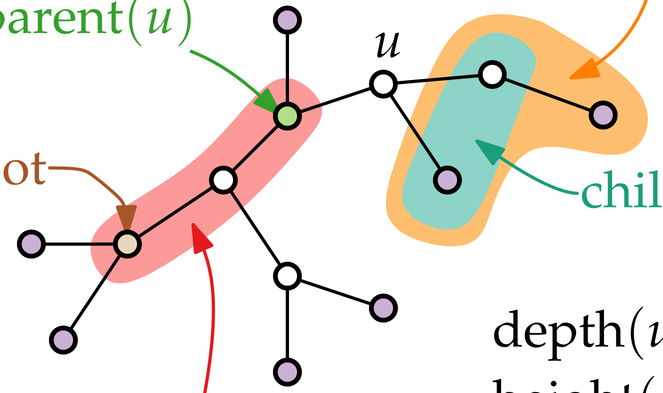

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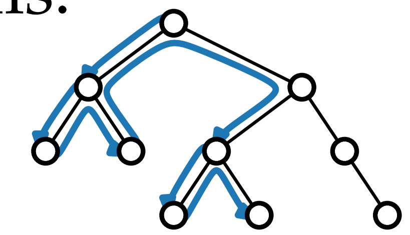
preorder



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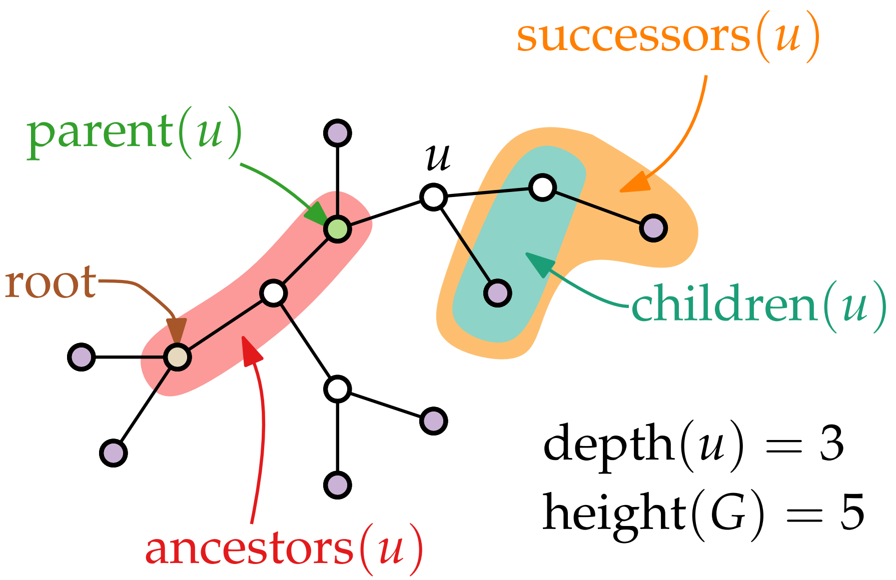

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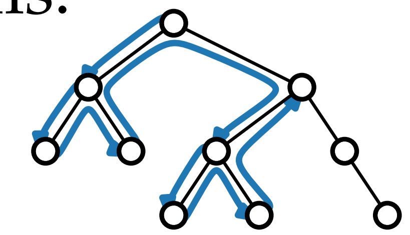
preorder



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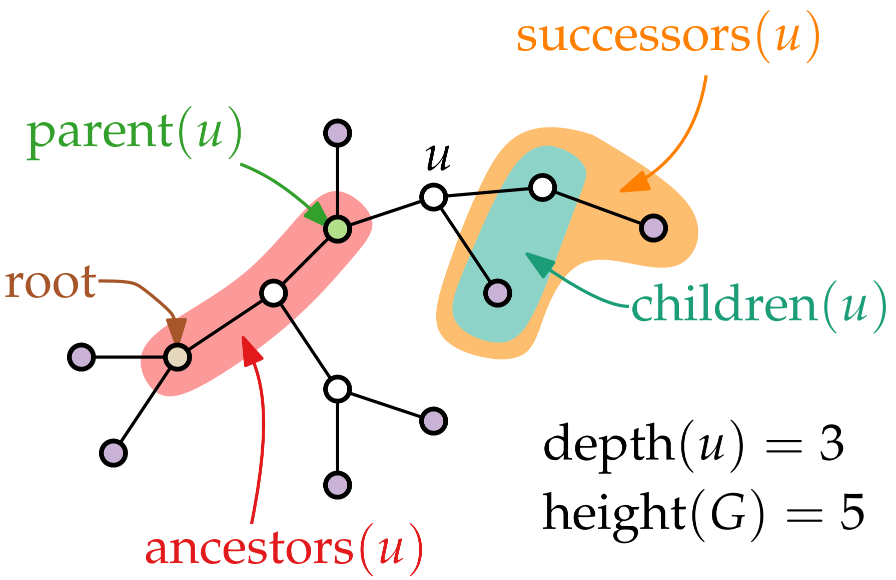

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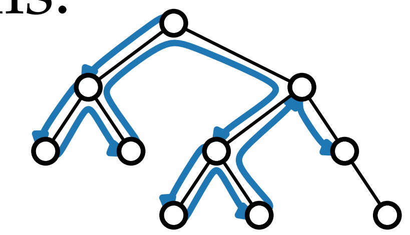

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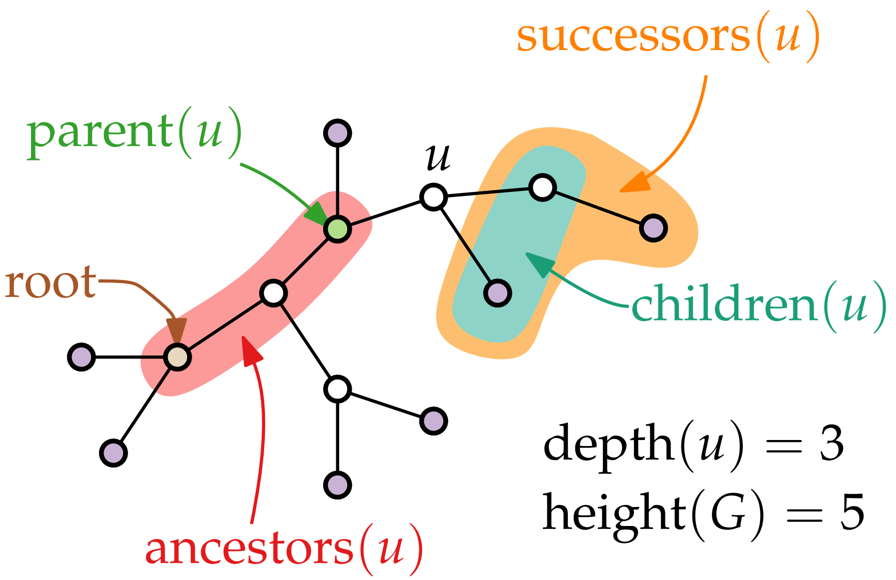

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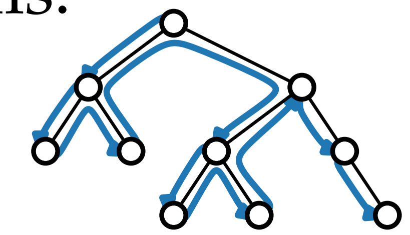

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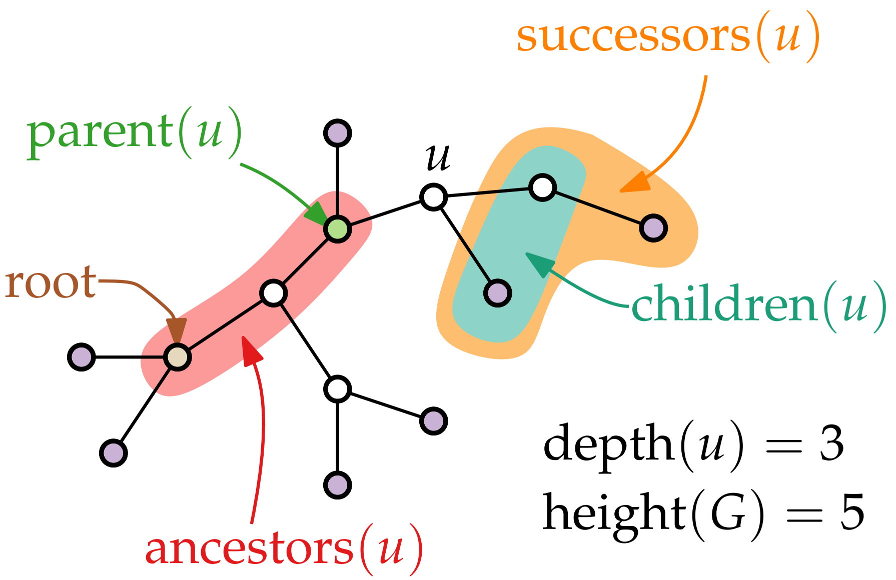

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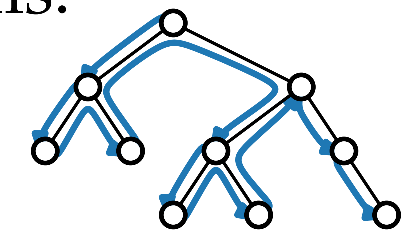
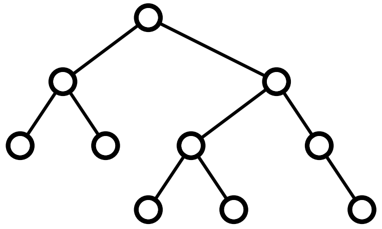


node – left – right

inorder



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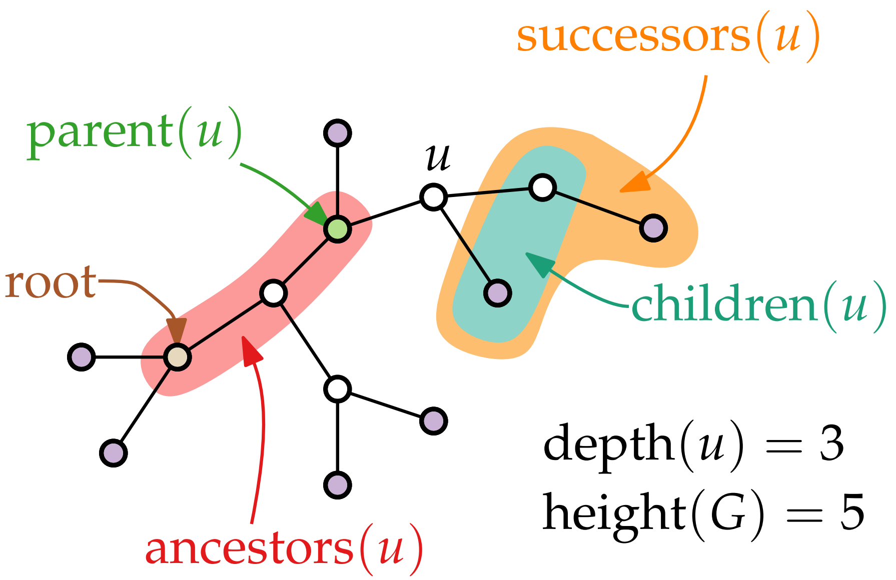

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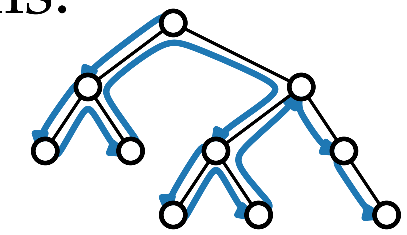
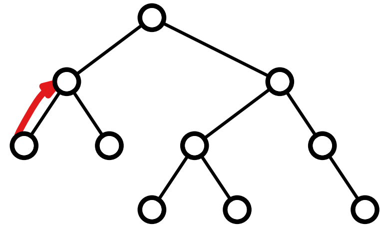
preorder



node – left – right

inorder



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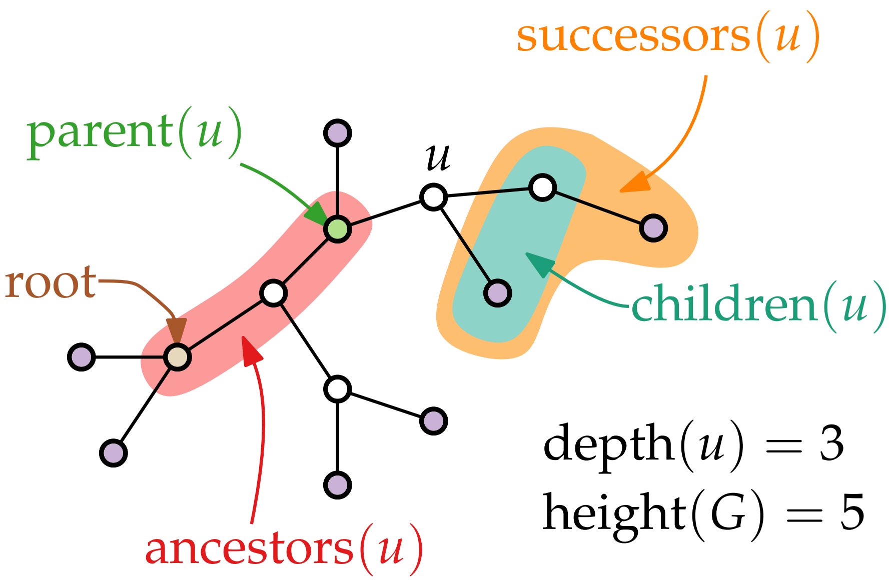

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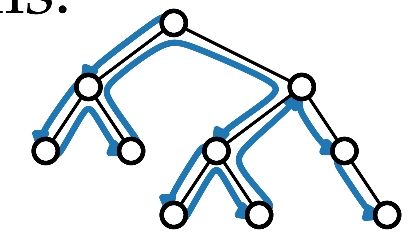
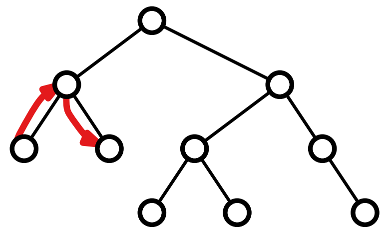


inorder



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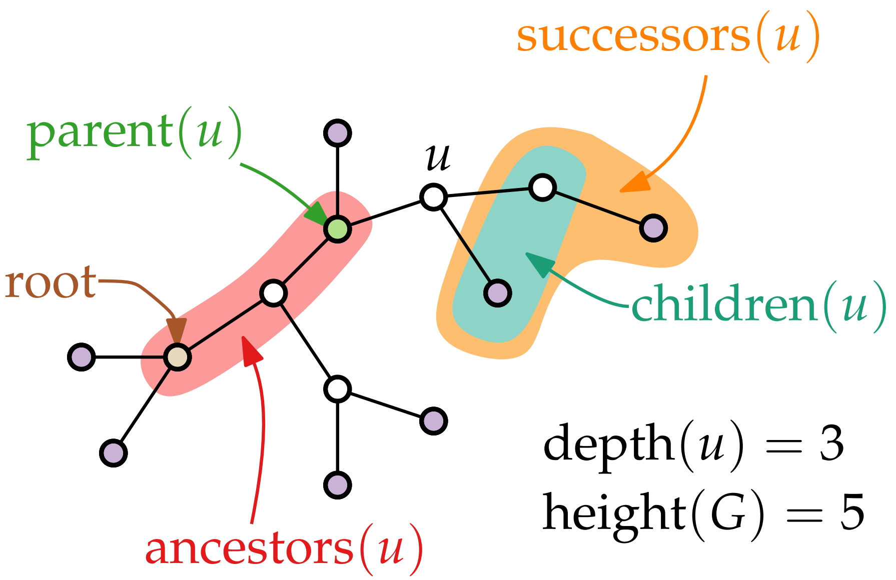

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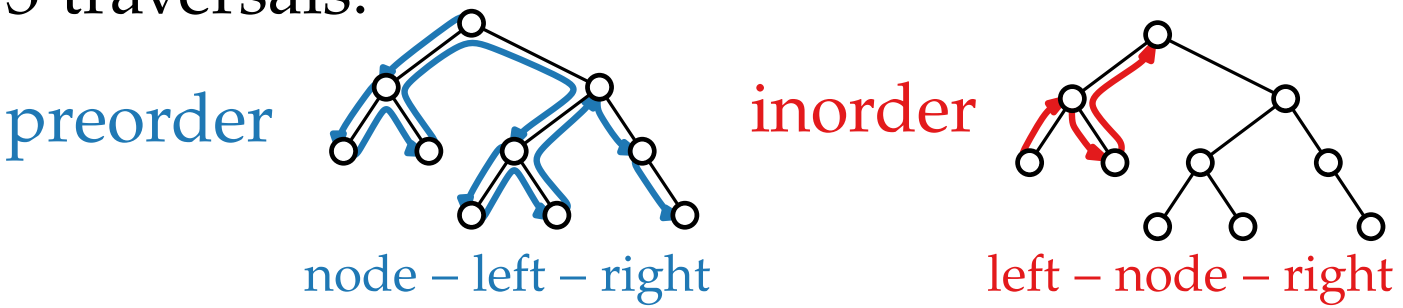

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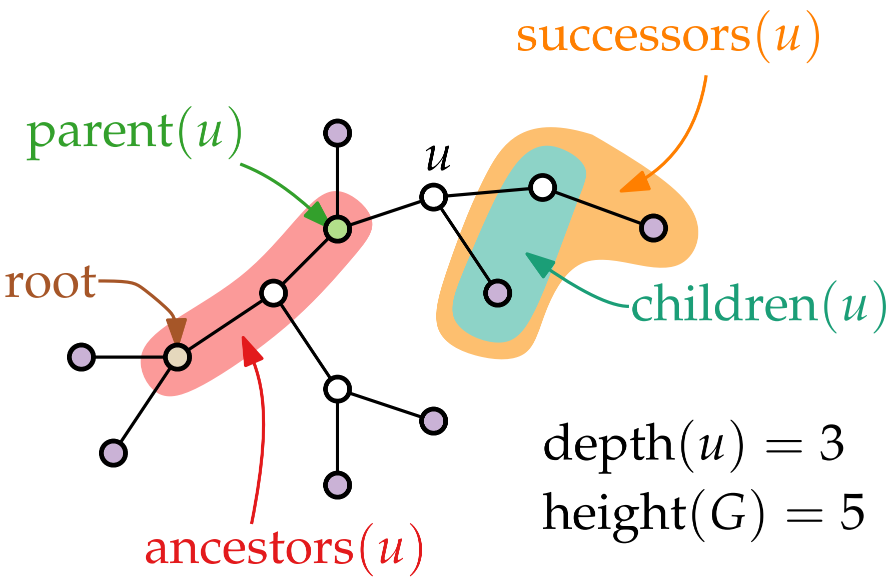

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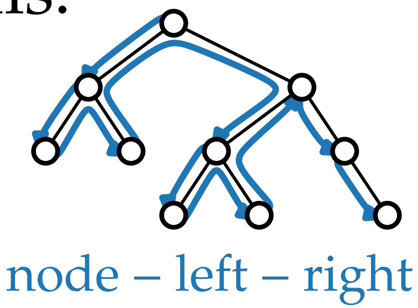

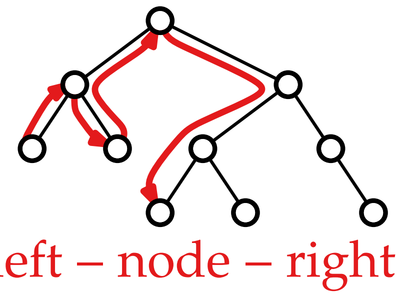

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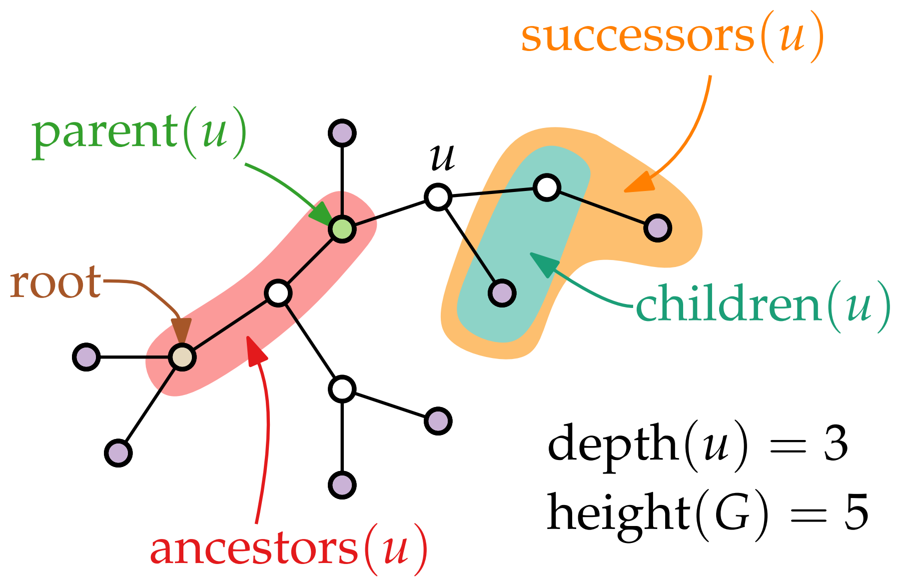

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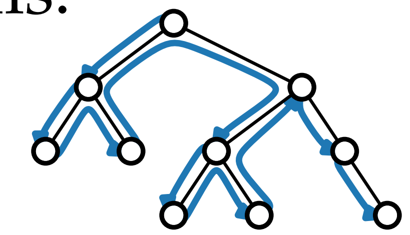
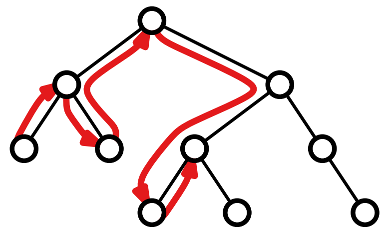
3 traversals:

preorder



node – left – right

inorder



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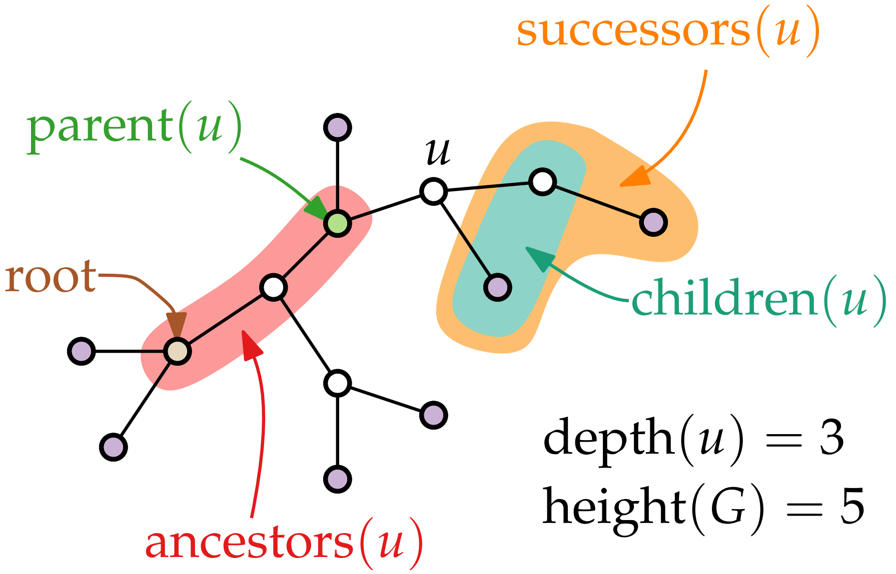

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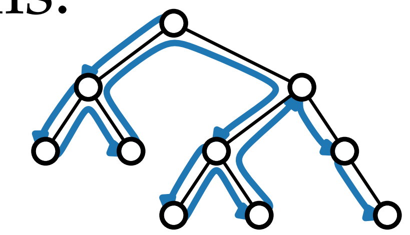

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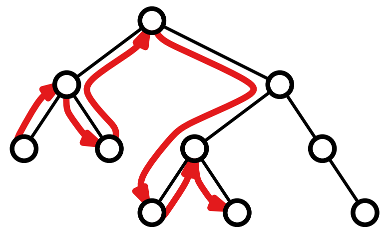

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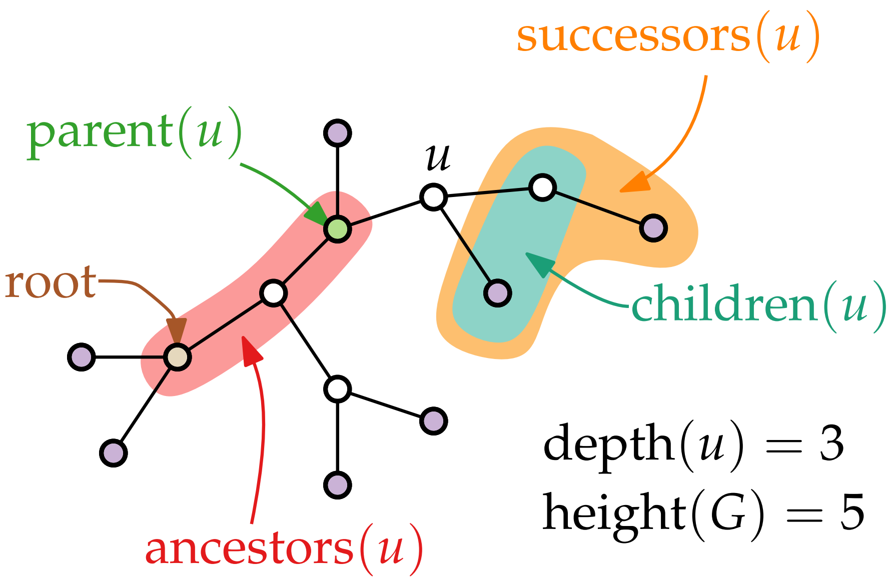

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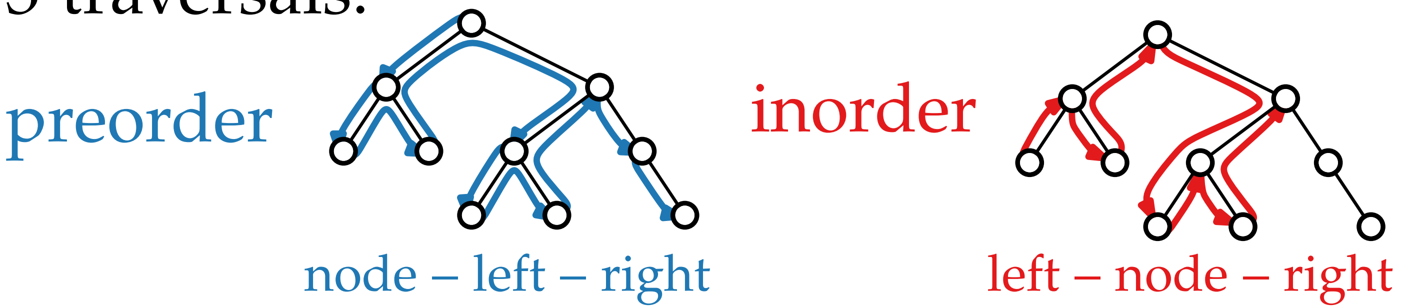

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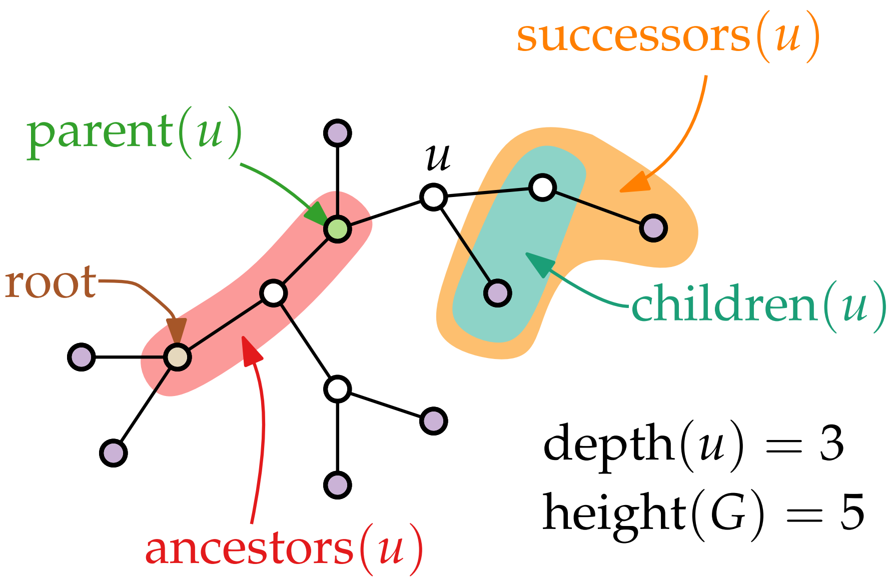

depth(u) $= 3$
height(G) $= 5$

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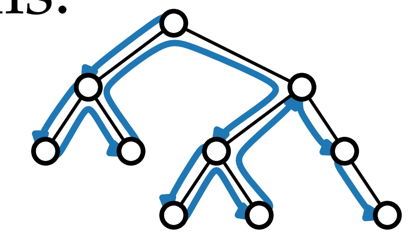
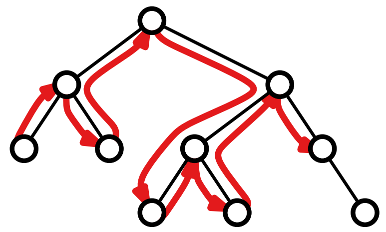
preorder



node – left – right

inorder



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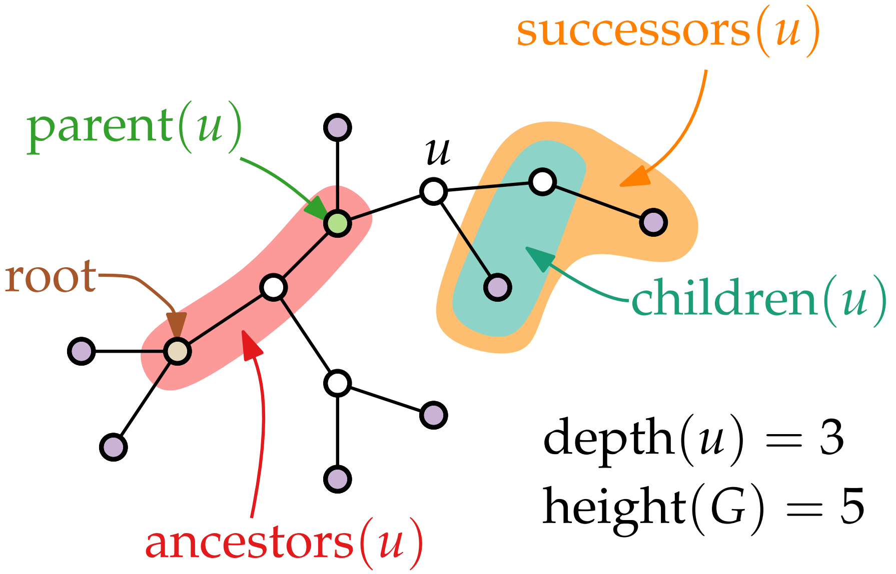

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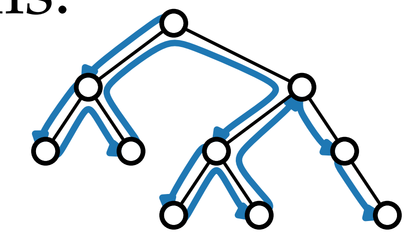

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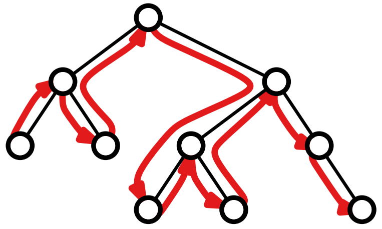

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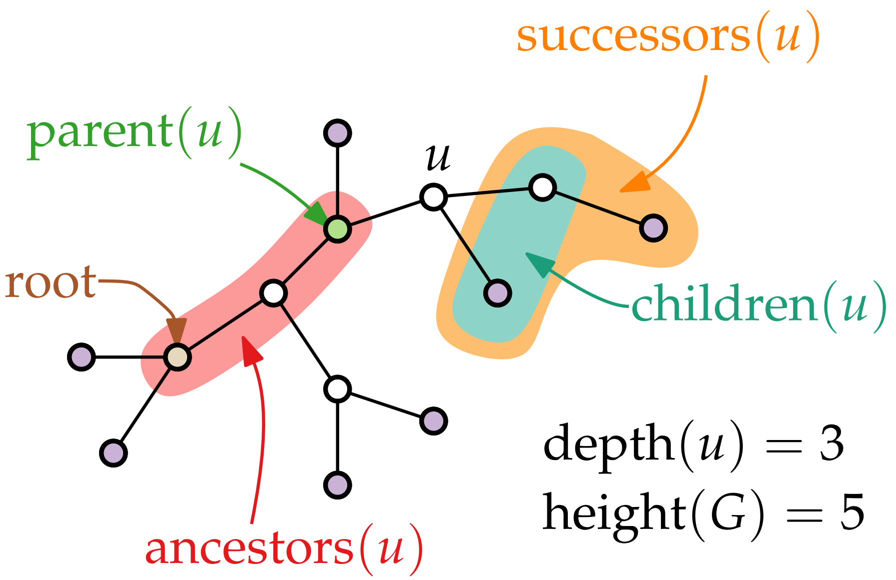

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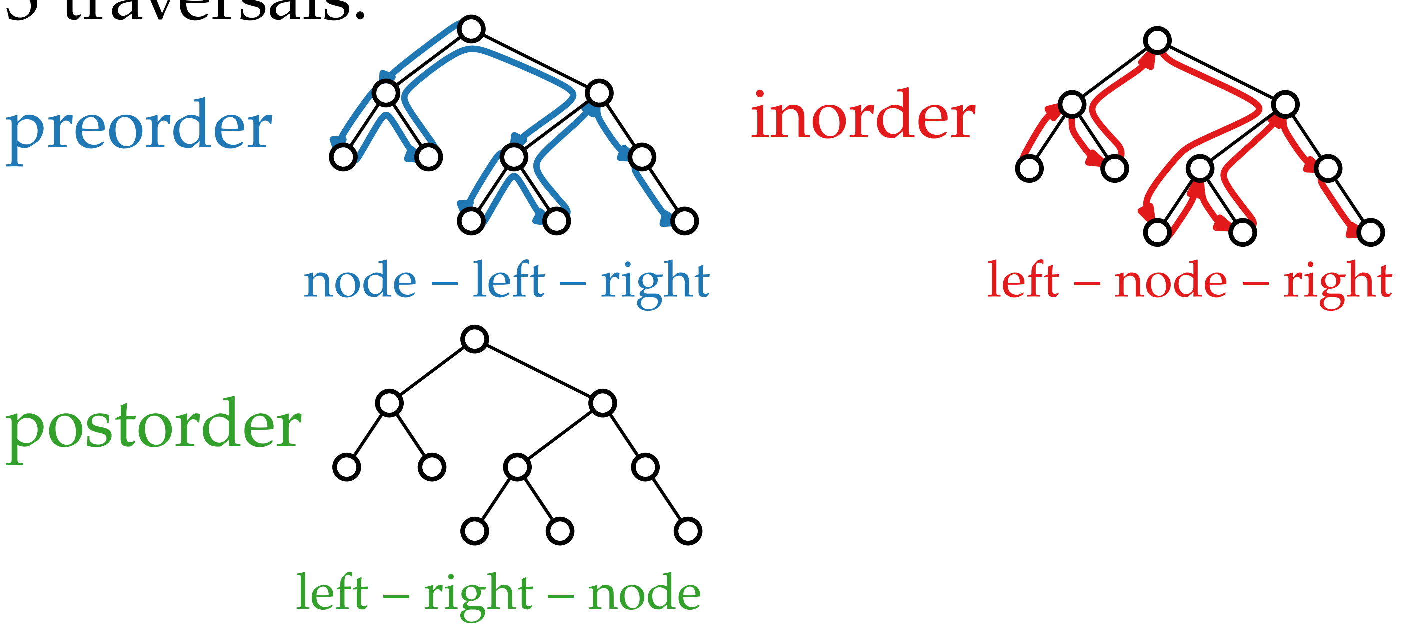

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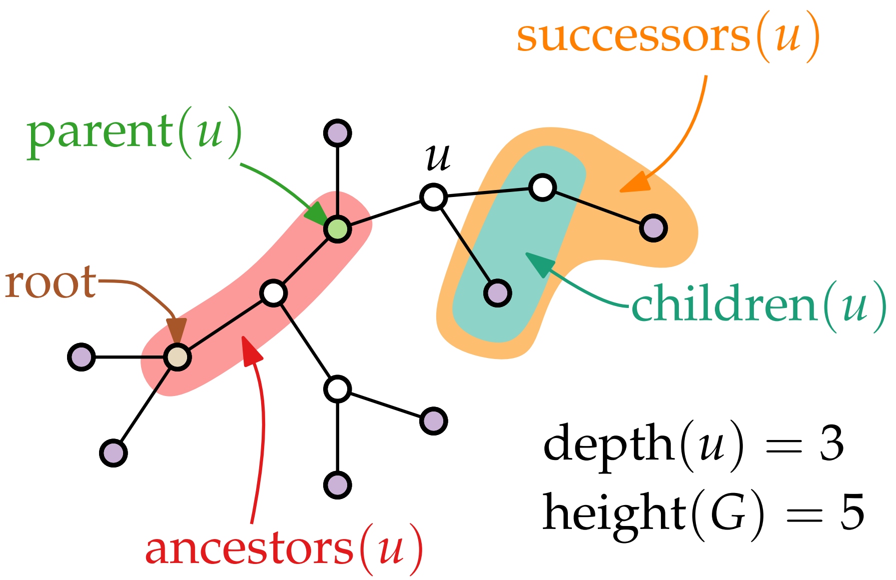

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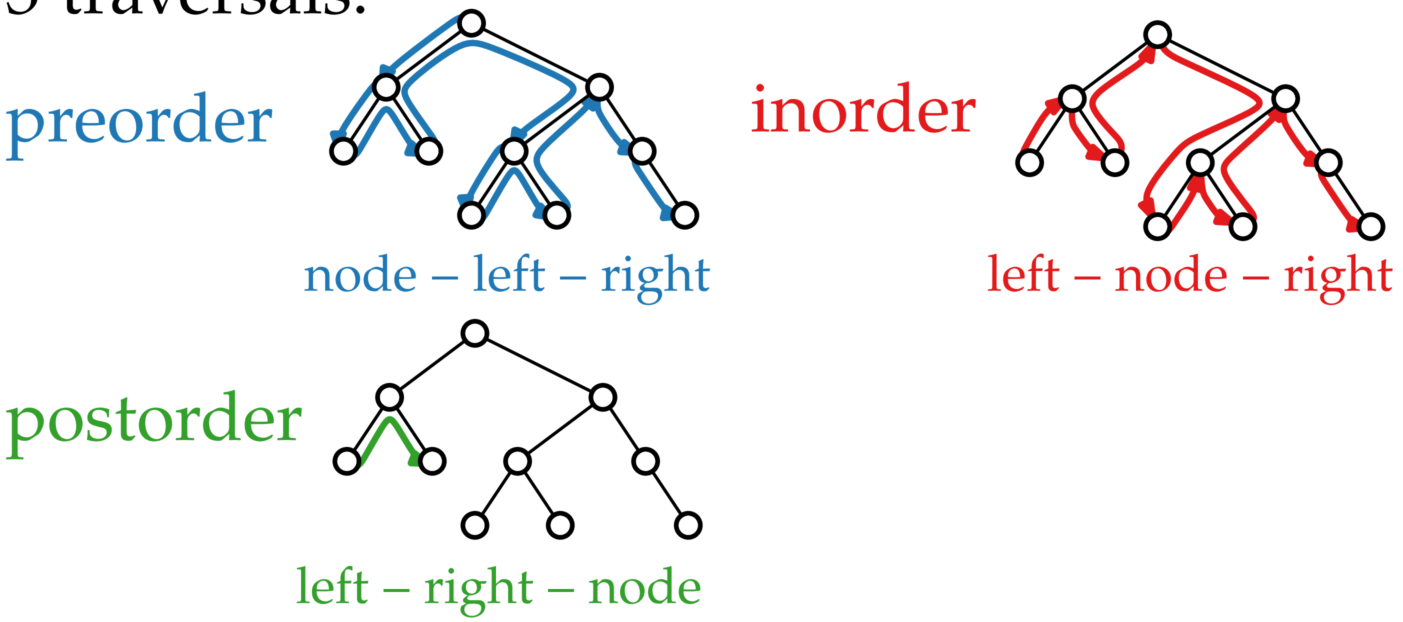

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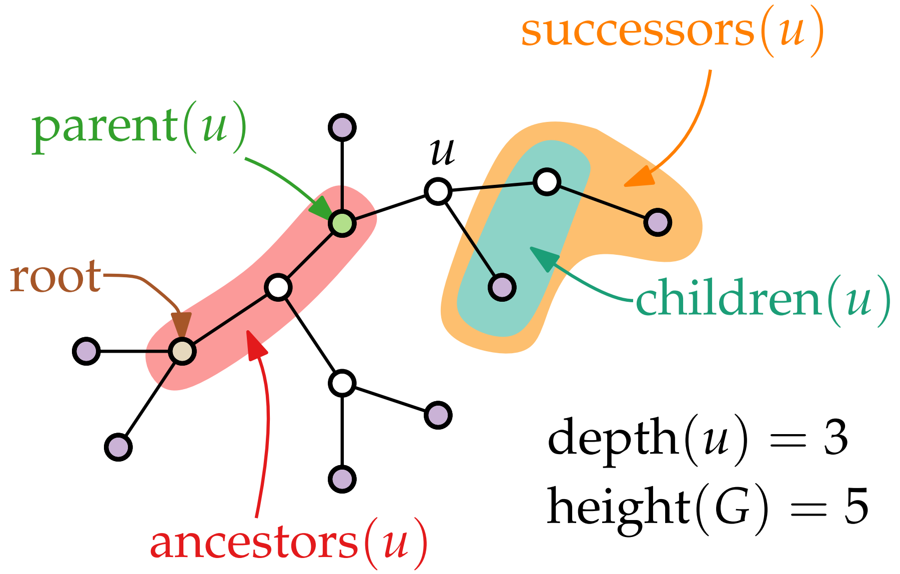

Leaf: Vertex of degree 1
Rooted tree: tree with designated **root**
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Depth: Length of path to root
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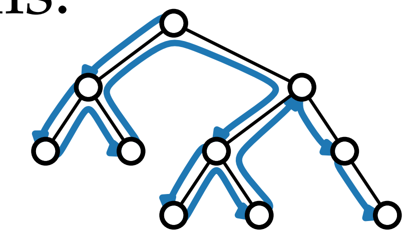
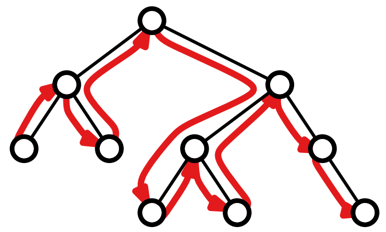
3 traversals:

preorder



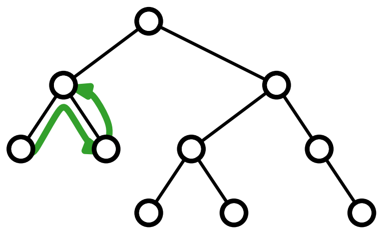
node – left – right

inorder



left – node – right

postorder



left – right – node

Binary Tree: At most two children per vertex (left / right child)

(Rooted) Trees

G is a **tree** if the following equivalent conditions hold:
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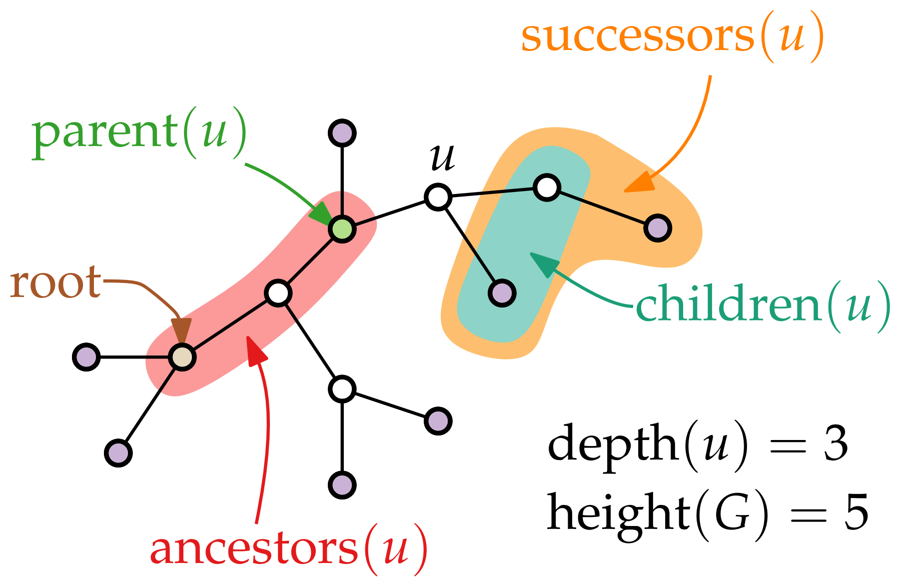

depth(u) $= 3$
height(G) $= 5$

Leaf: Vertex of degree 1
Rooted tree: tree with designated **root**
Ancestor: Vertex on path to root
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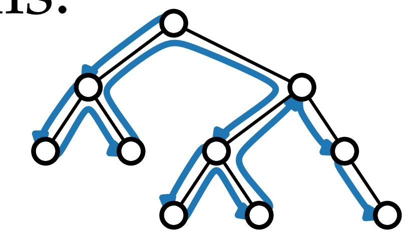
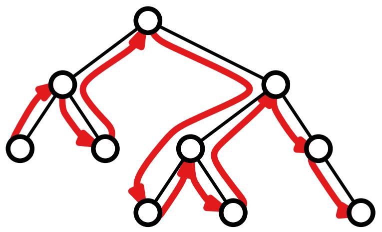


node – left – right

inorder



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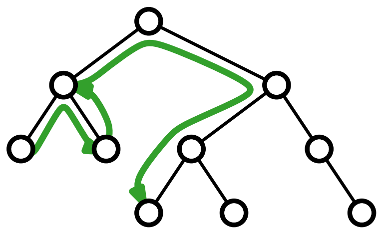

left – right – node

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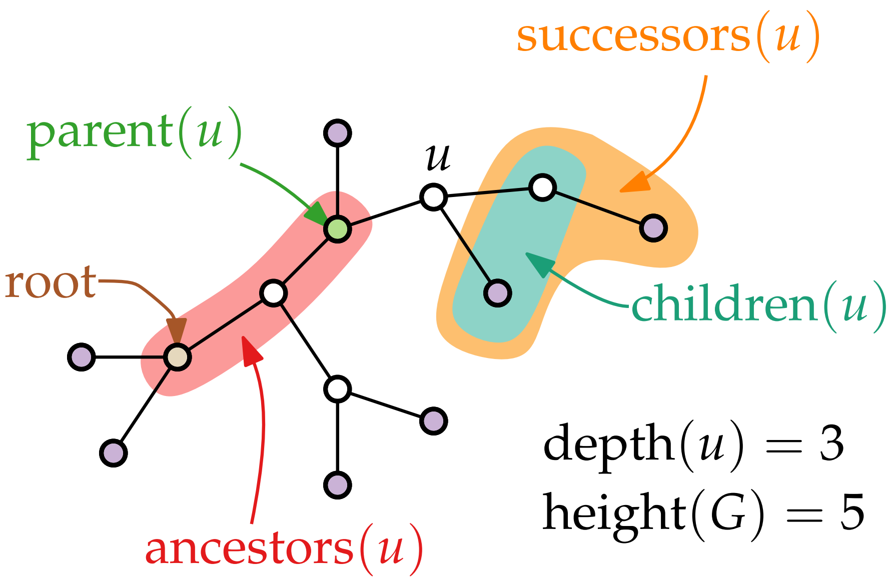

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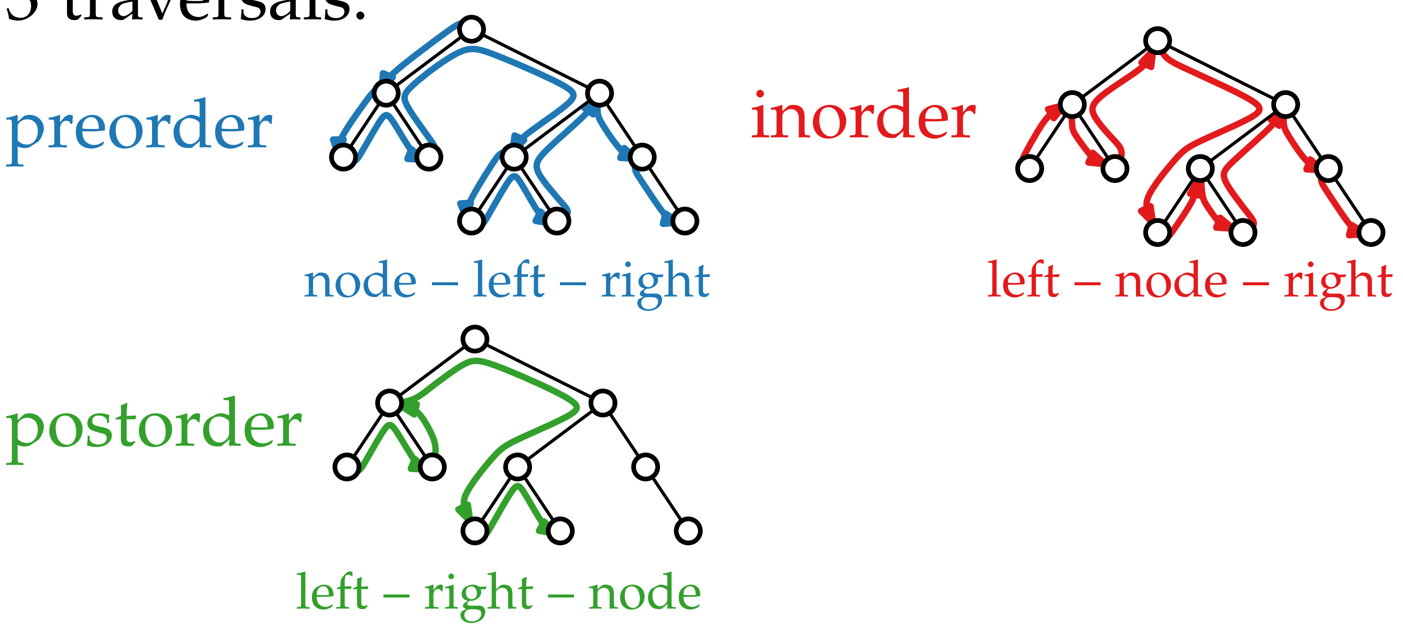

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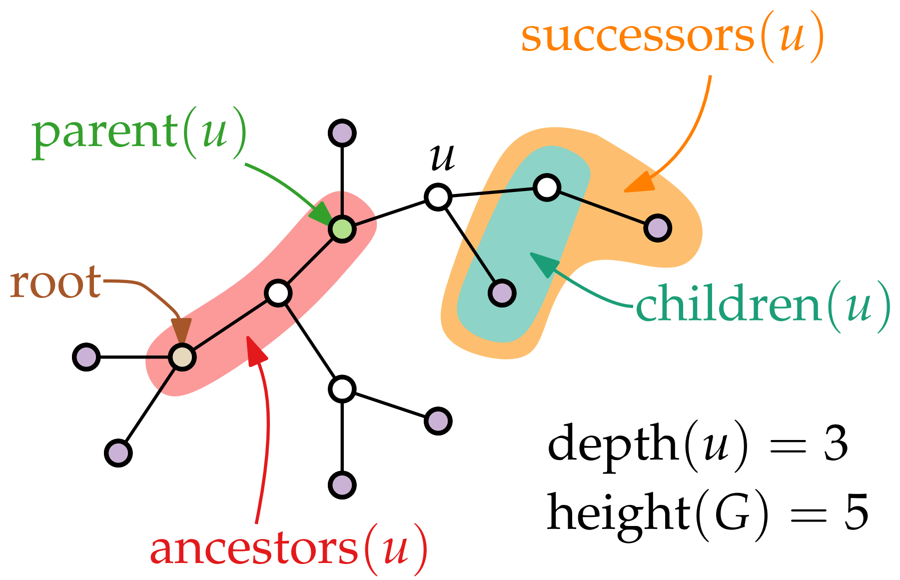

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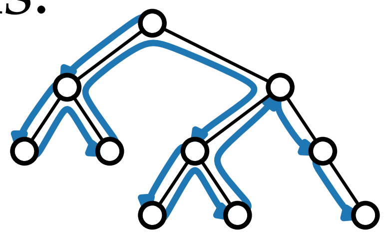
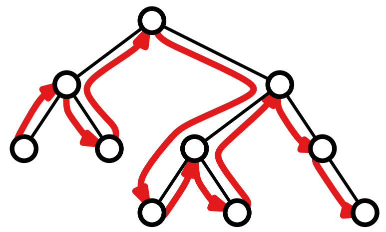
preorder



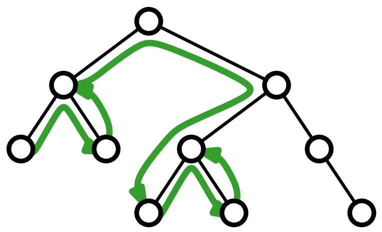
node – left – right

inorder



left – node – right

postorder



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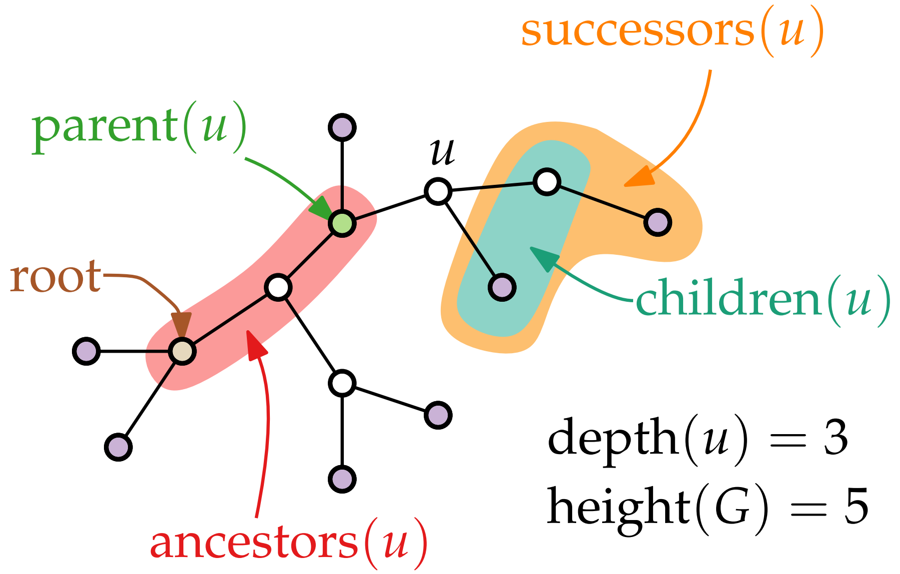

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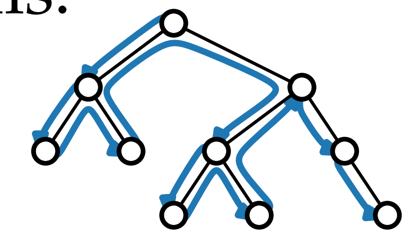

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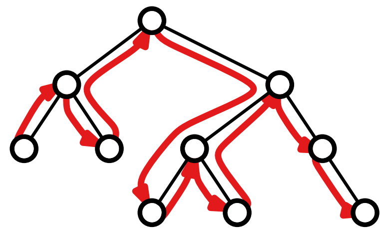

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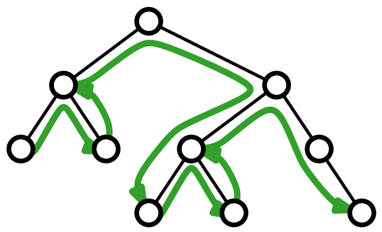

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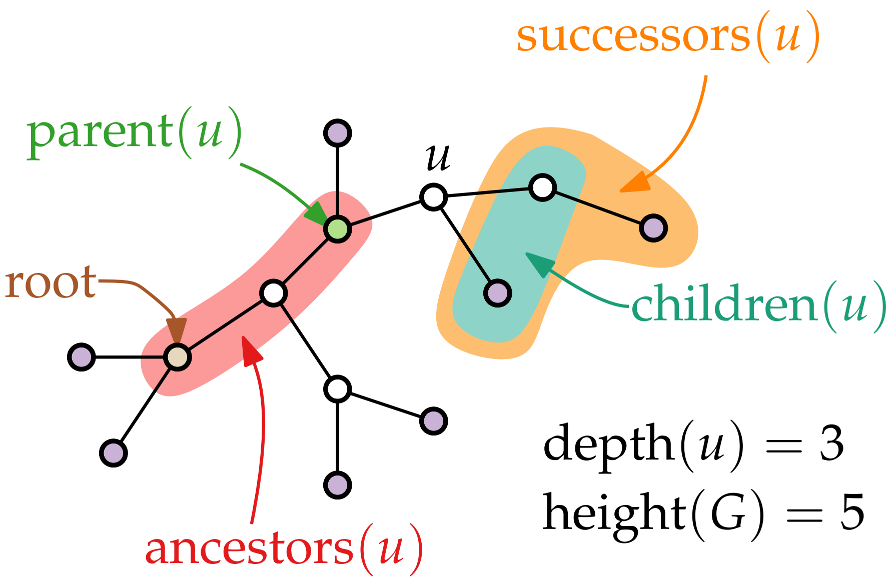

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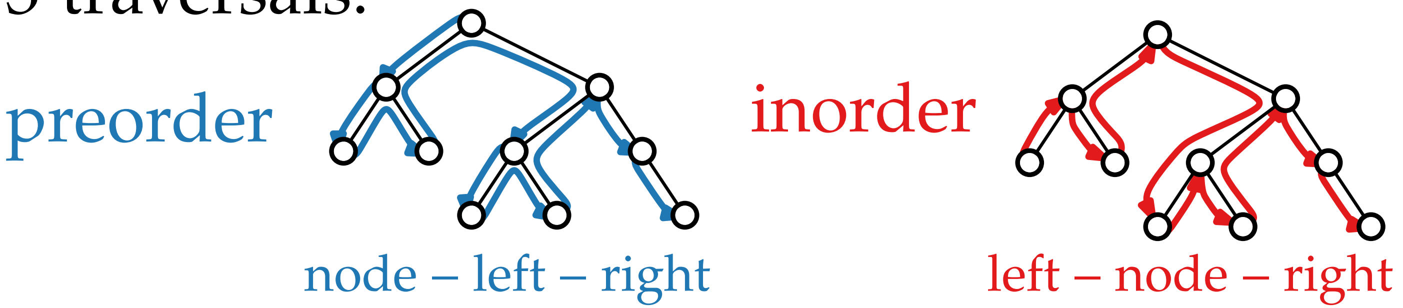

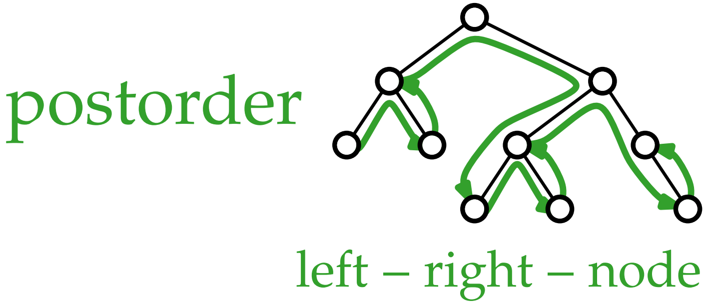

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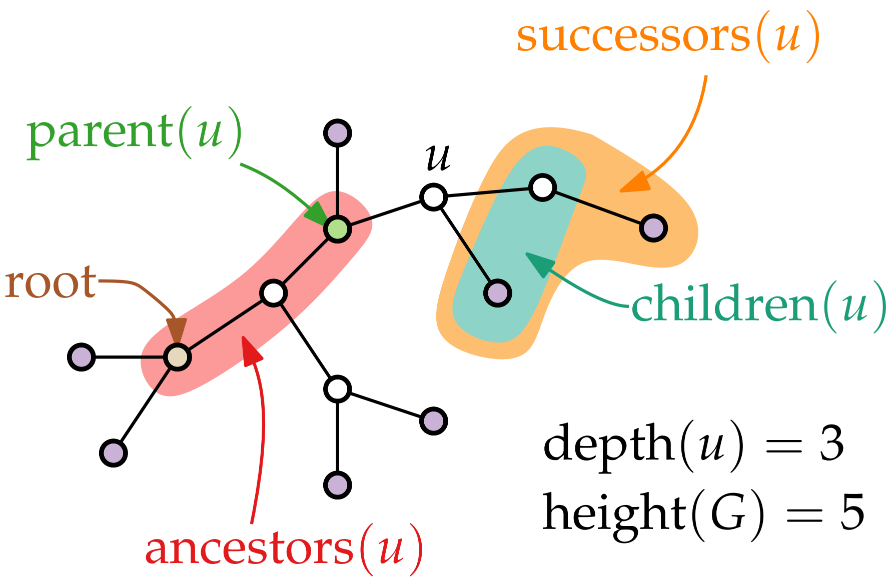

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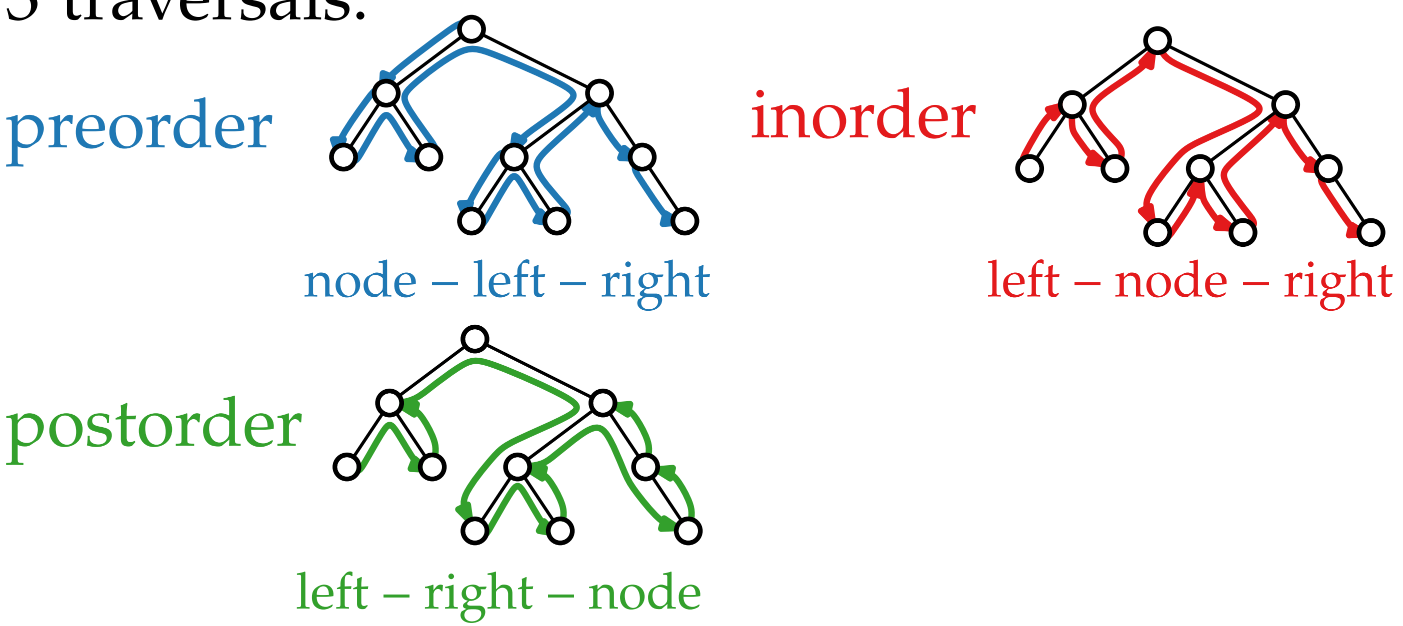

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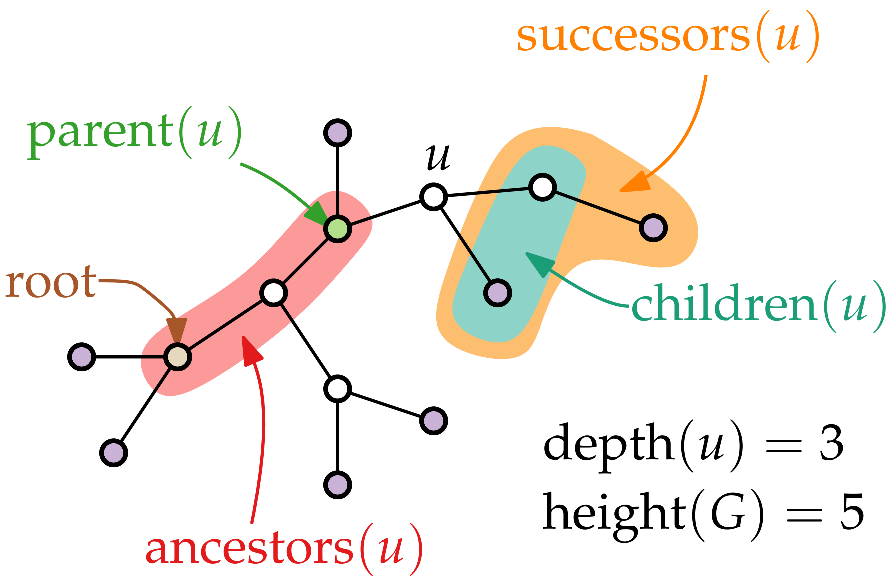

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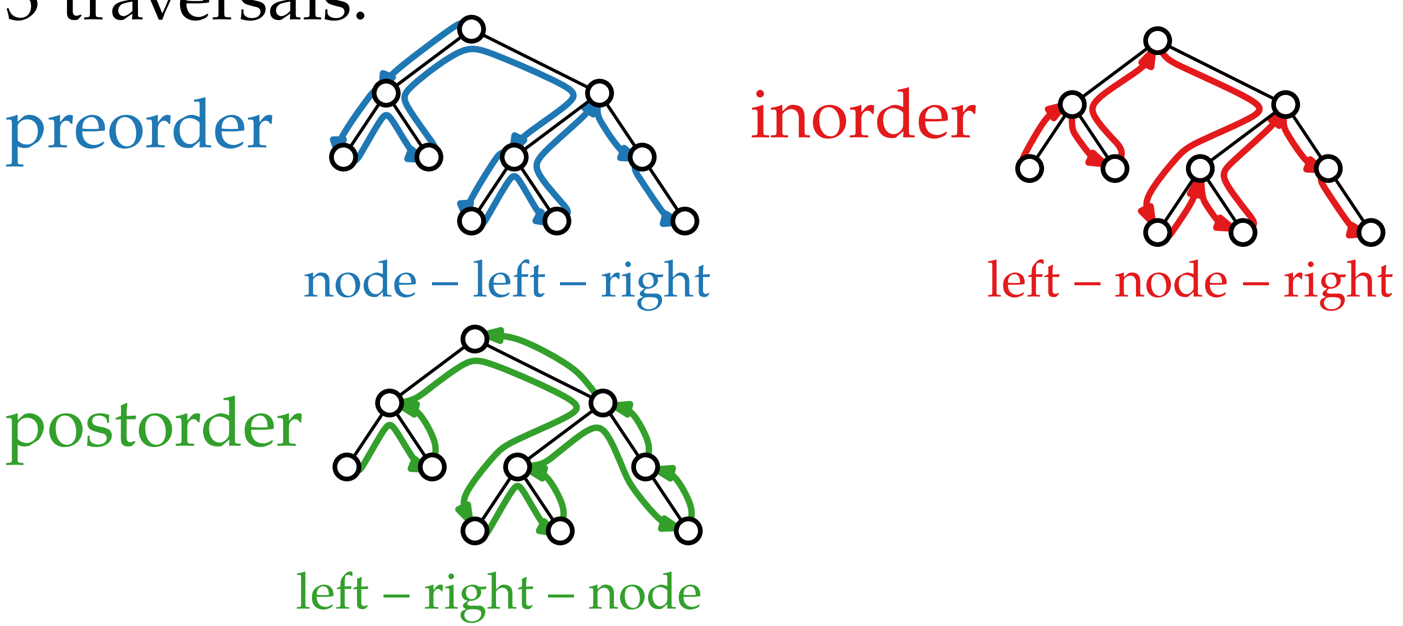

Binary Tree: At most two children per vertex (left / right child)

First Grid Layout of Binary Trees

1. Choose y-coordinates:

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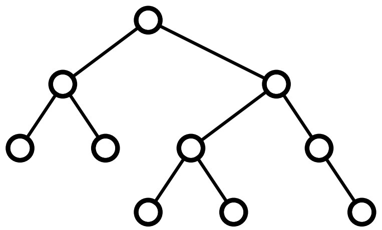

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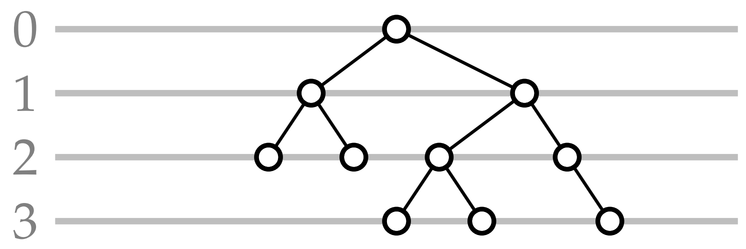

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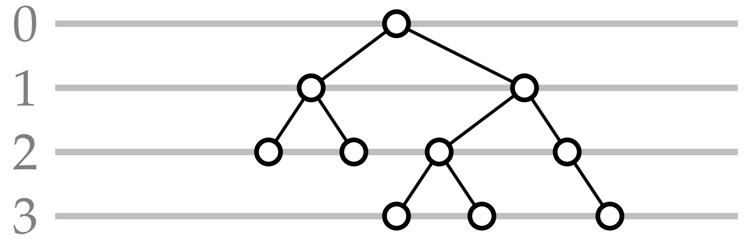

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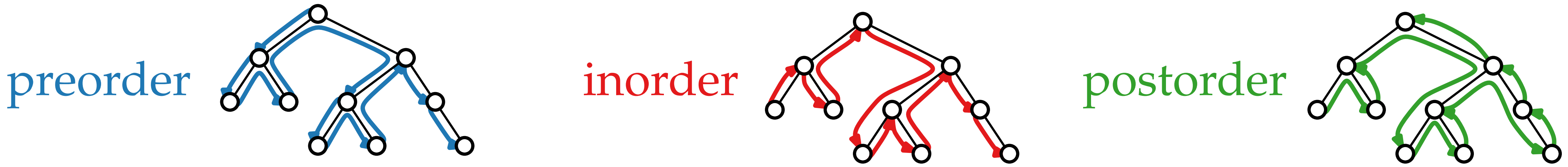

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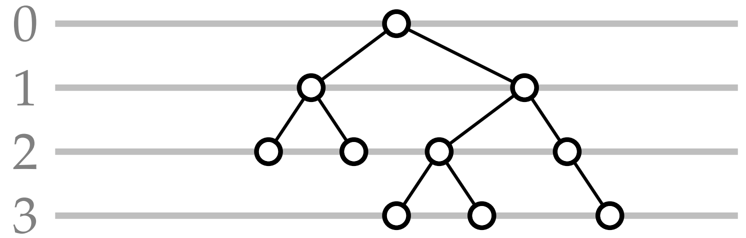

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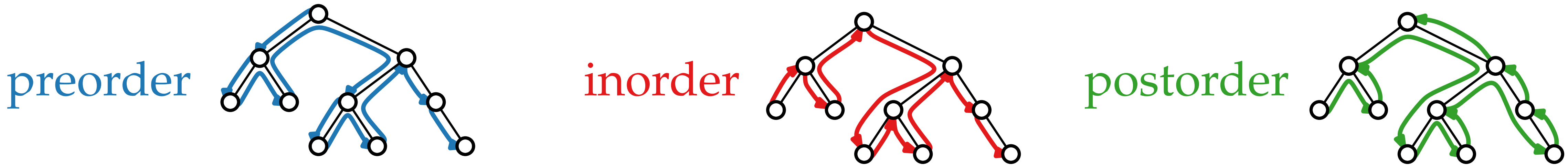

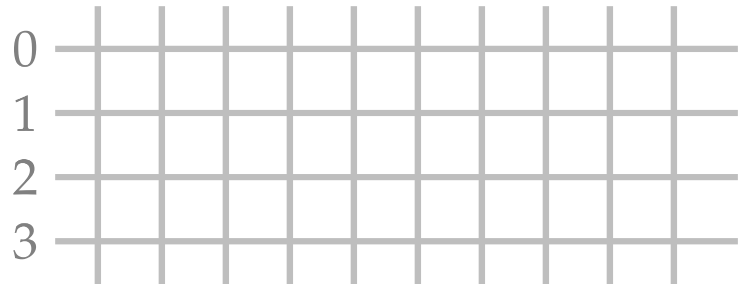

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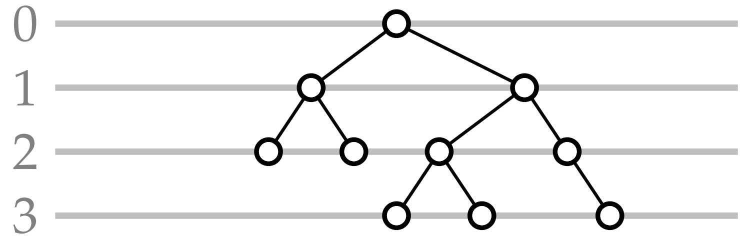

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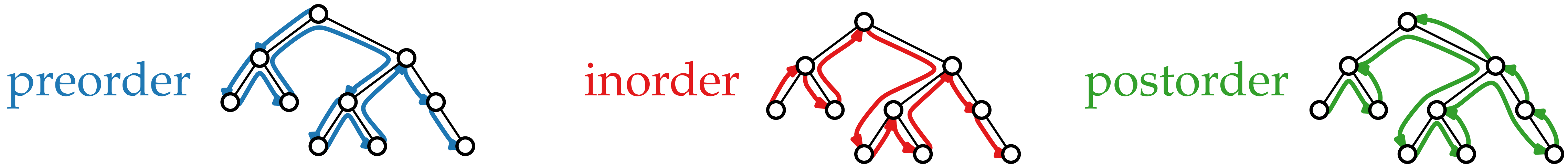

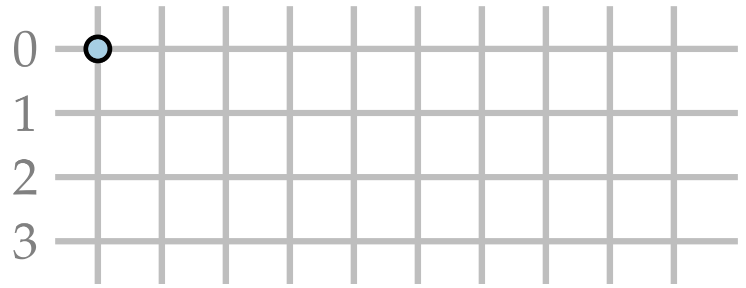

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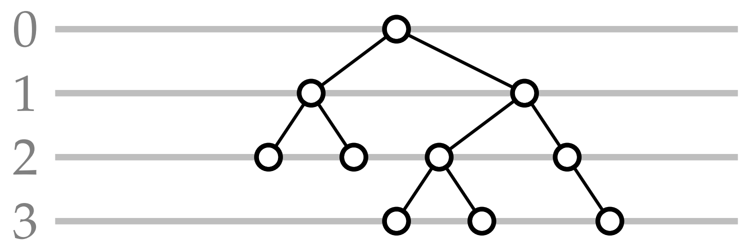

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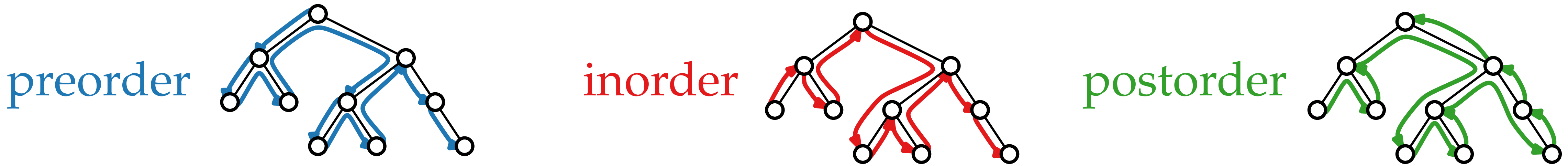

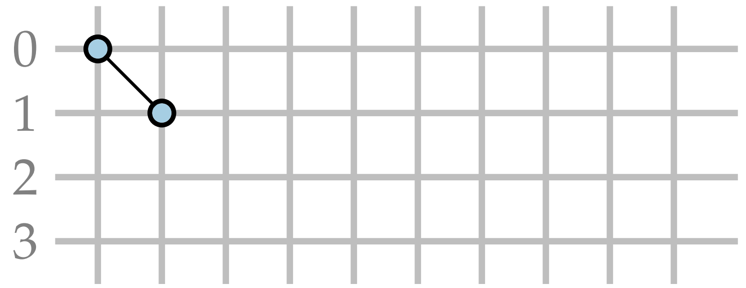

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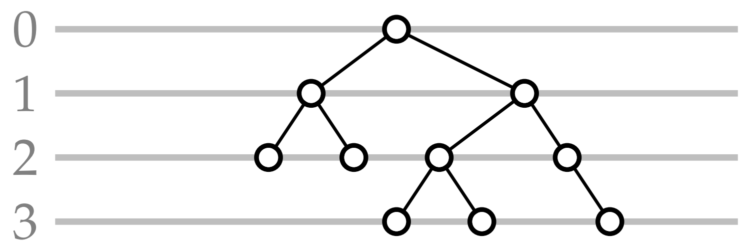

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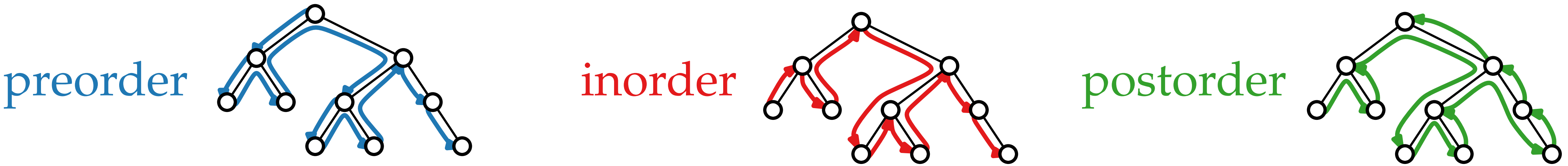

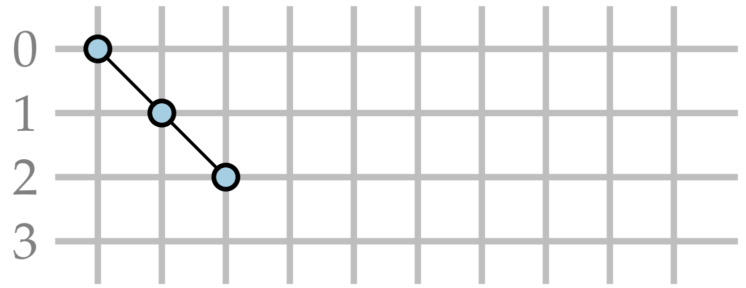

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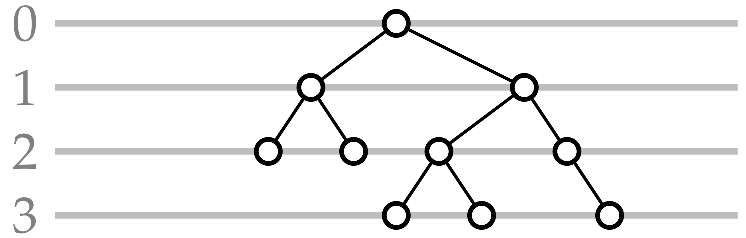

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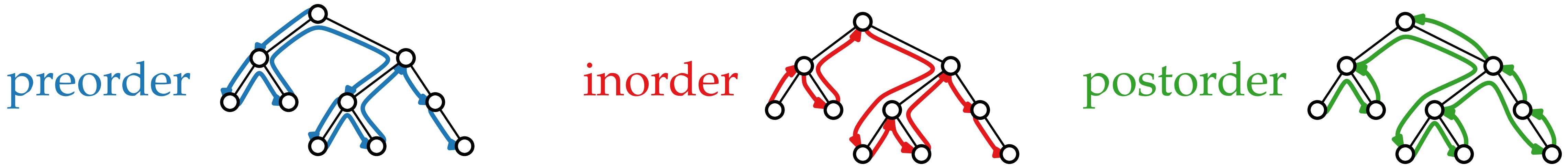

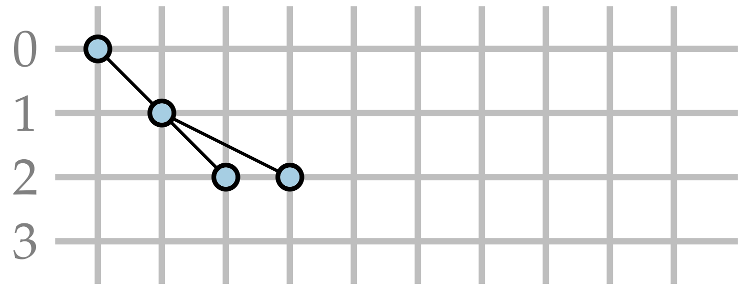

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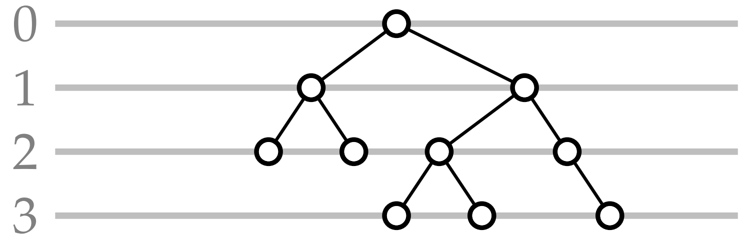

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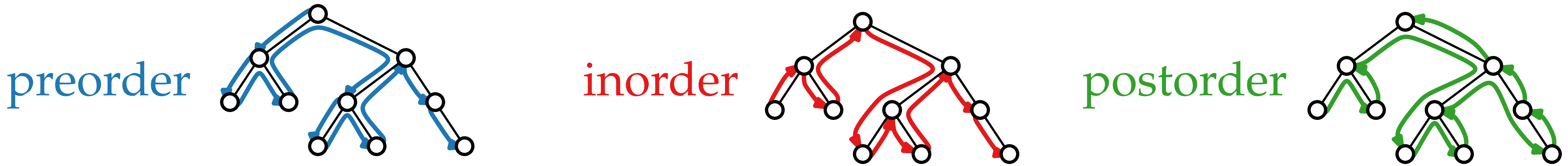

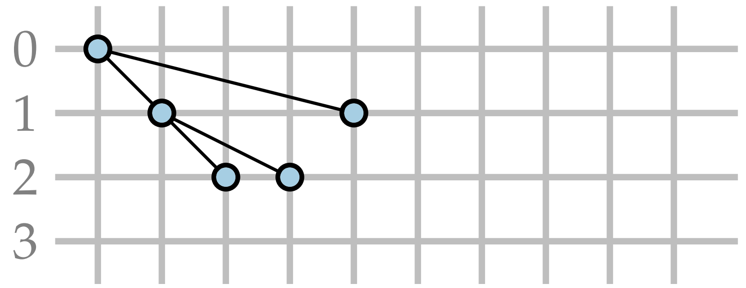

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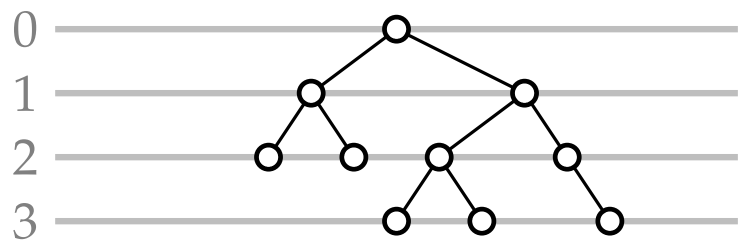

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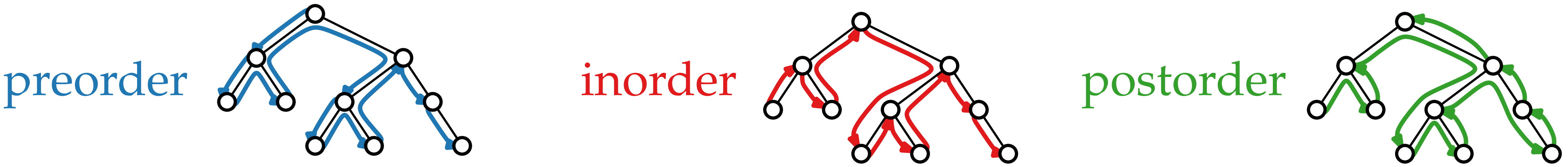

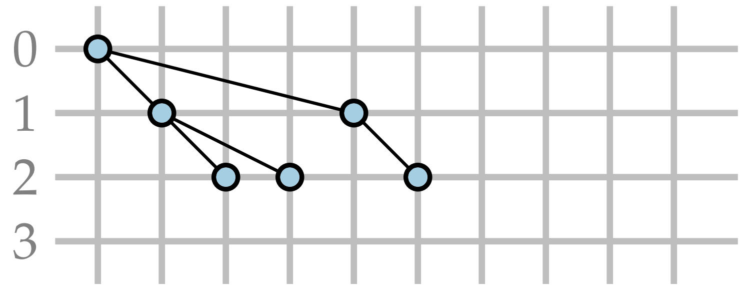

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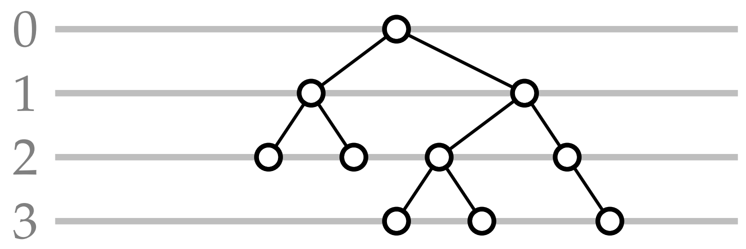

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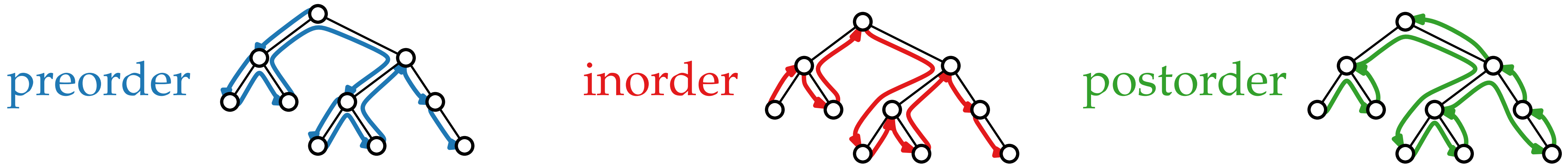

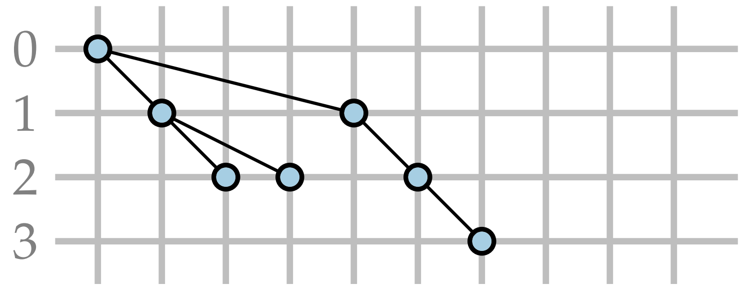

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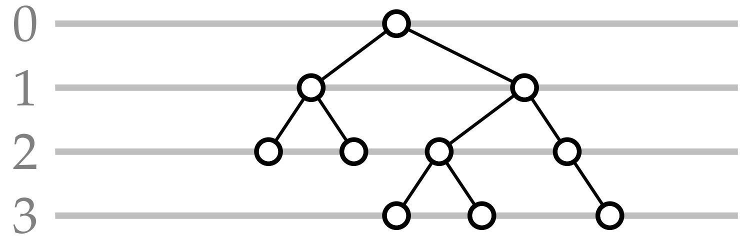

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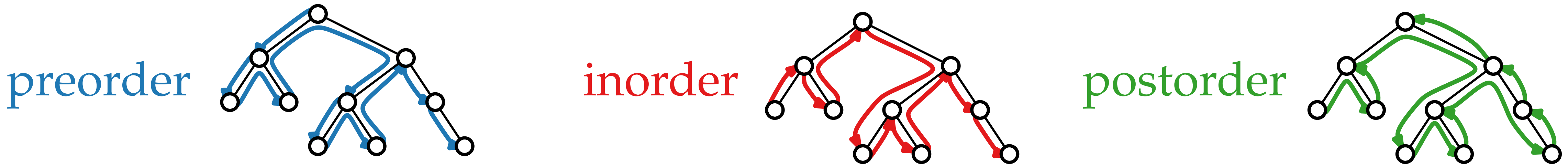

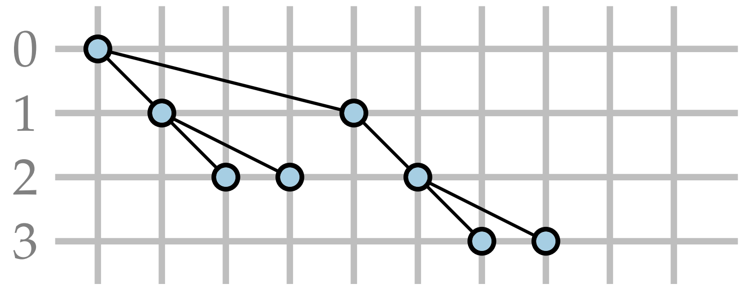

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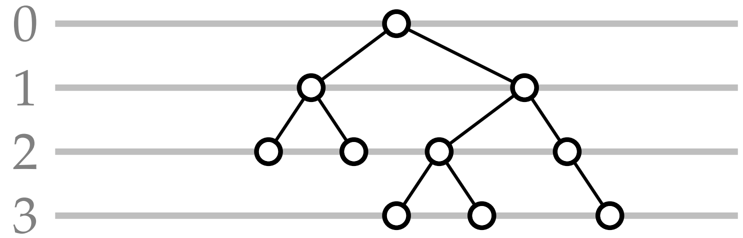

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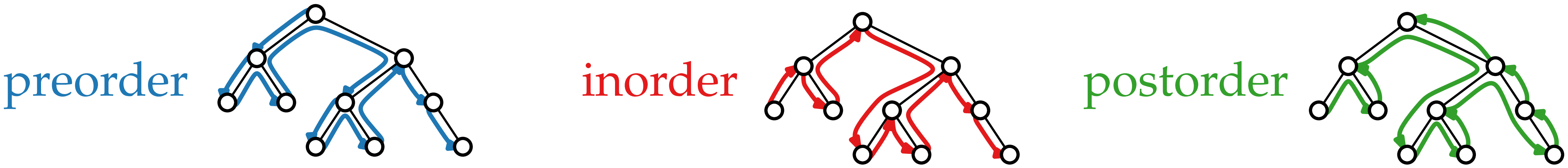
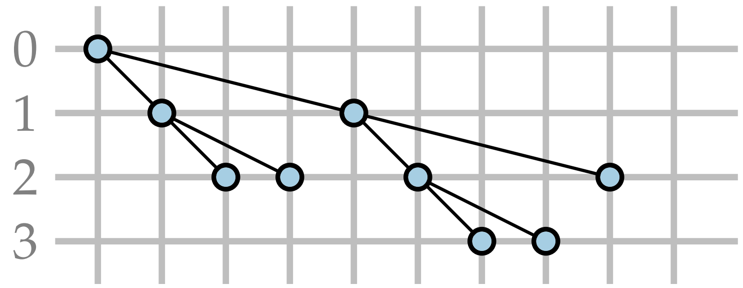
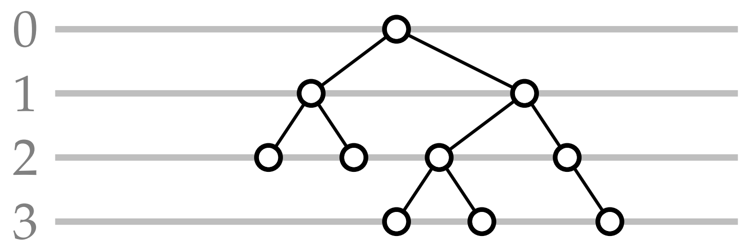




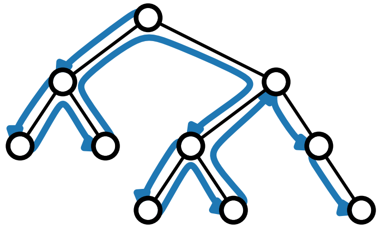
First Grid Layout of Binary Trees

1. Choose y-coordinates: $y(u) = \text{depth}(u)$

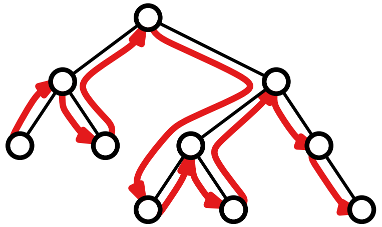


2. Choose x-coordinates:

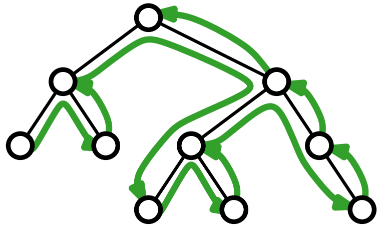
preorder

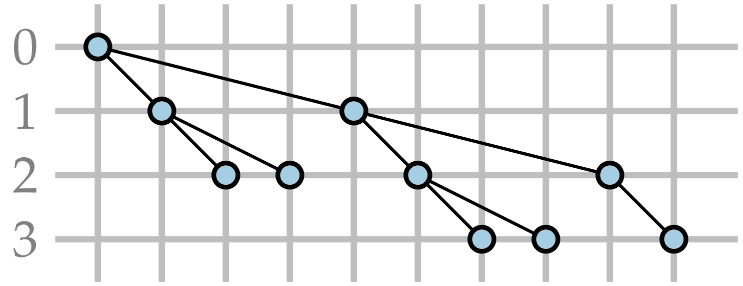


inorder



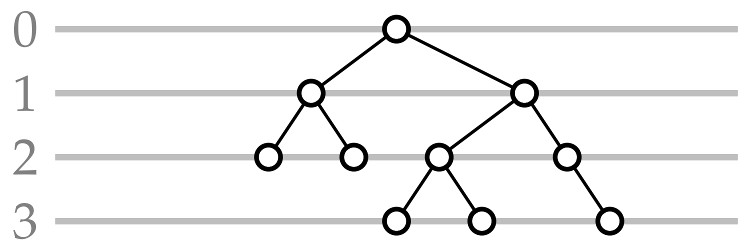
postorder



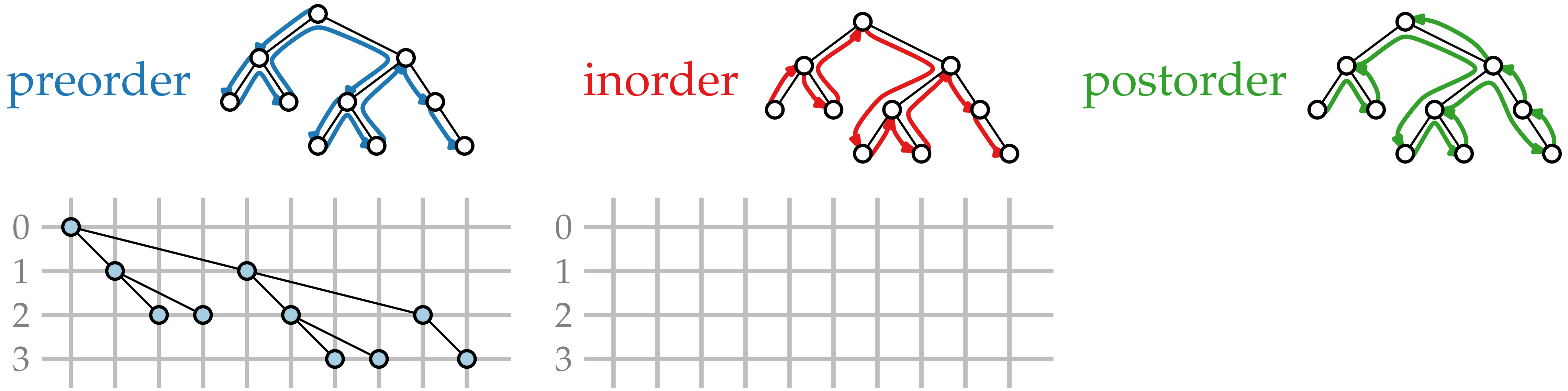


First Grid Layout of Binary Trees

1. Choose y-coordinates: $y(u) =$depth(u)



2. Choose x-coordinates:



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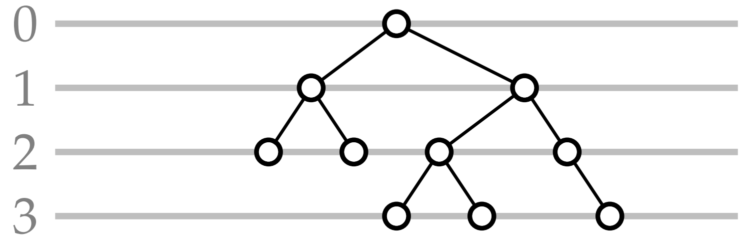
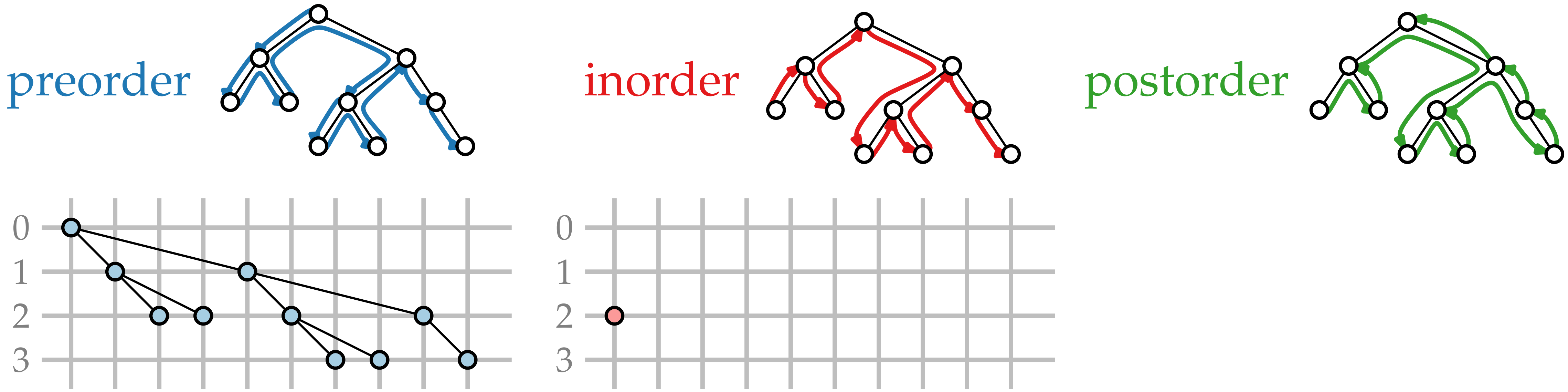


2. Choose x-coordinates:



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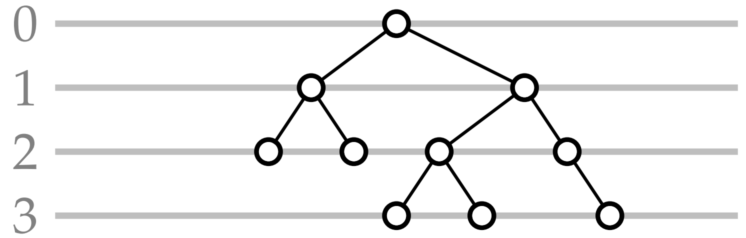
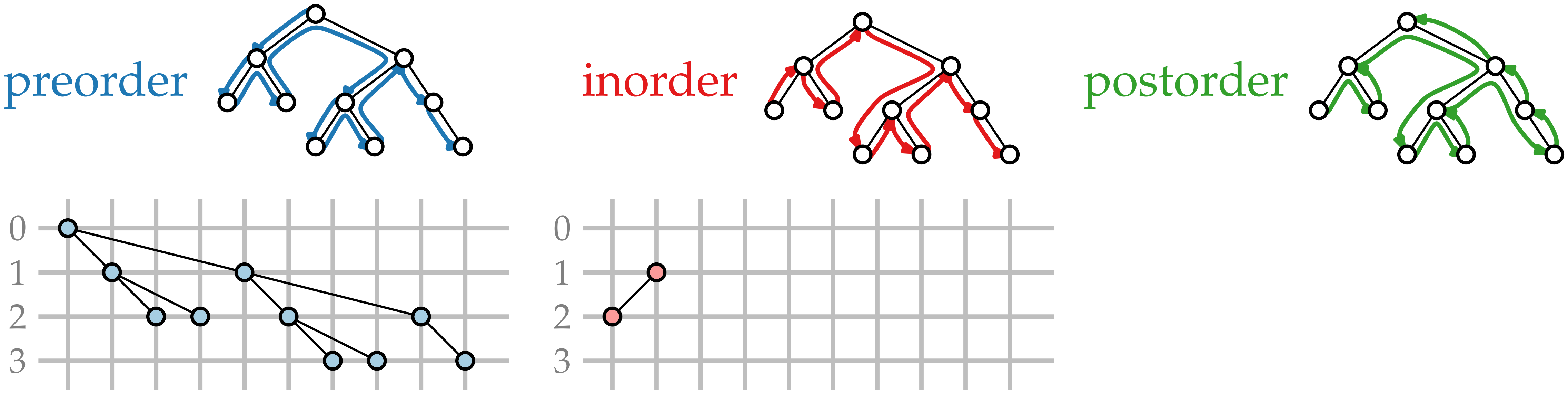


2. Choose x-coordinates:



First Grid Layout of Binary Trees

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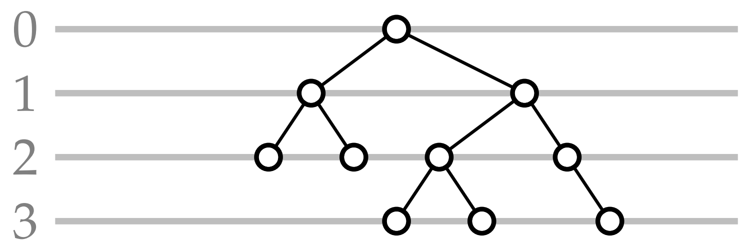
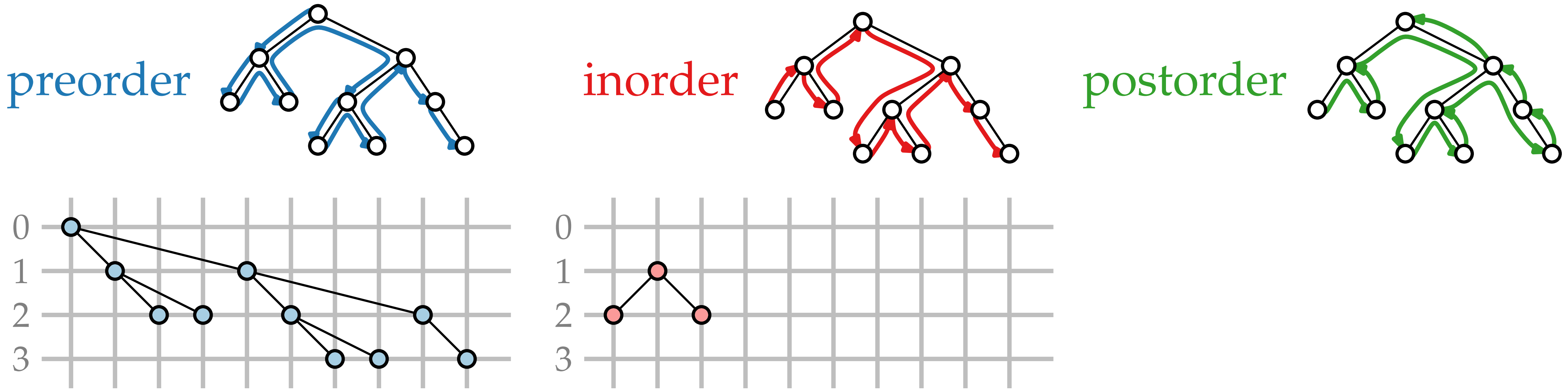
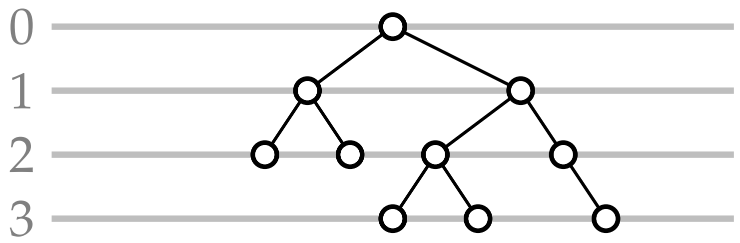


2. Choose x-coordinates:



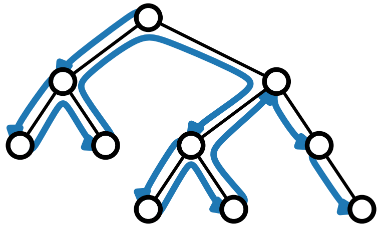
First Grid Layout of Binary Trees

1. Choose y-coordinates: $y(u) = \text{depth}(u)$

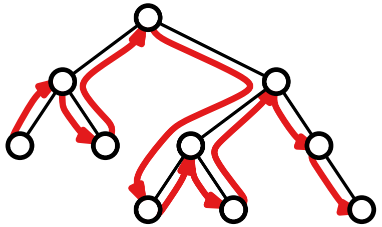


2. Choose x-coordinates:

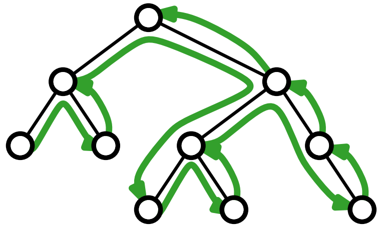
preorder

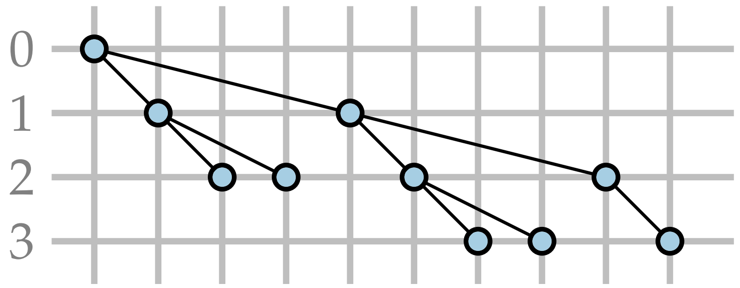


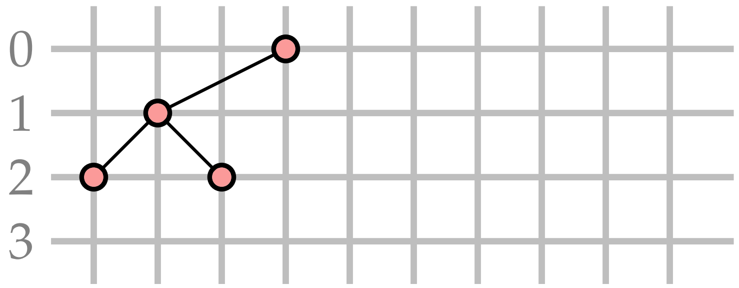
inorder



postorder

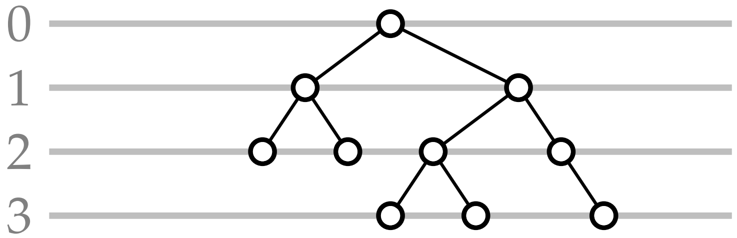




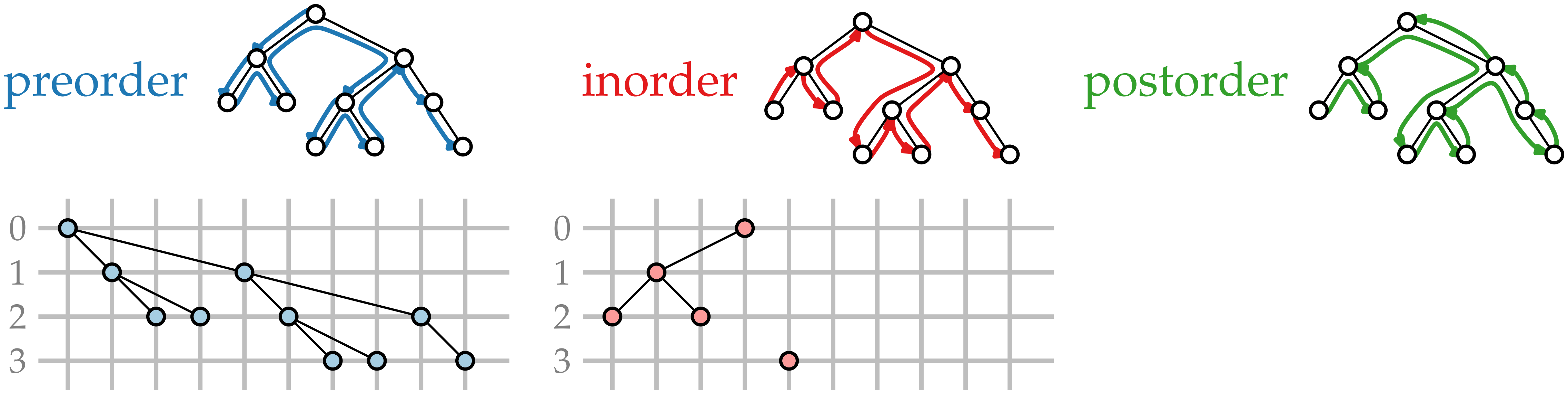


First Grid Layout of Binary Trees

1. Choose y-coordinates: $y(u) = \text{depth}(u)$

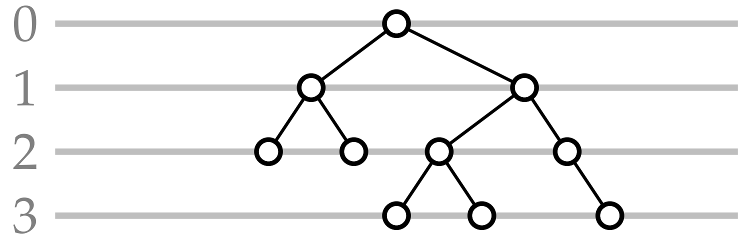


2. Choose x-coordinates:



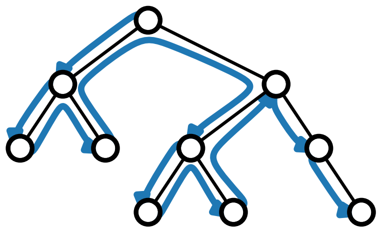
First Grid Layout of Binary Trees

1. Choose y-coordinates: $y(u)$ =depth(u)

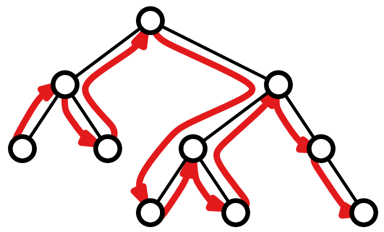


2. Choose x-coordinates:

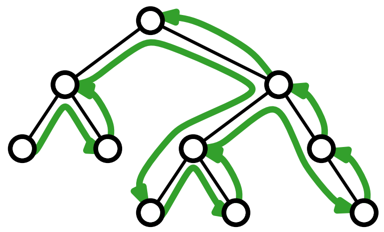
preorder

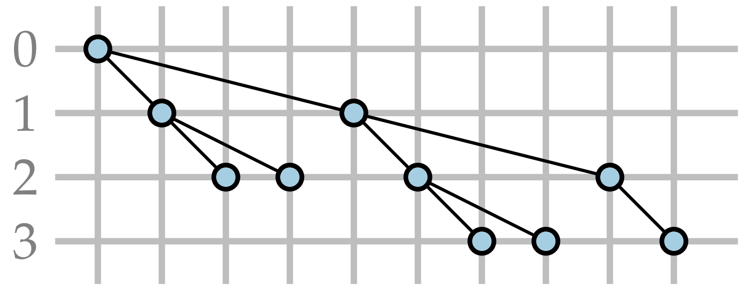


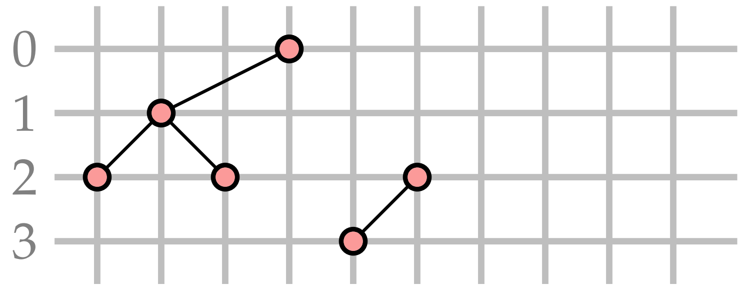
inorder



postorder

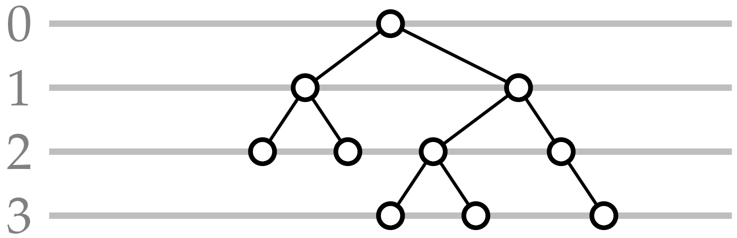






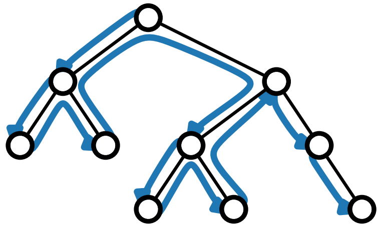
First Grid Layout of Binary Trees

1. Choose y-coordinates: $y(u) = \text{depth}(u)$

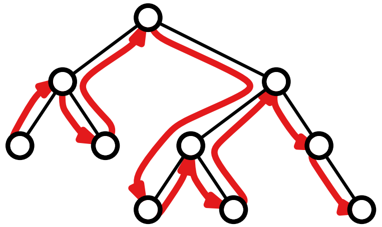


2. Choose x-coordinates:

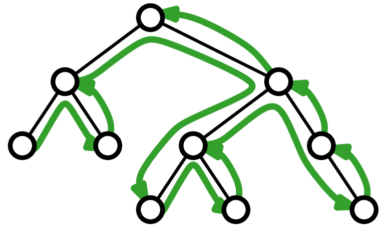
preorder

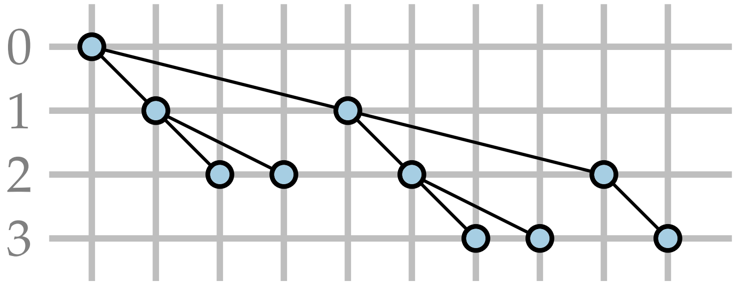


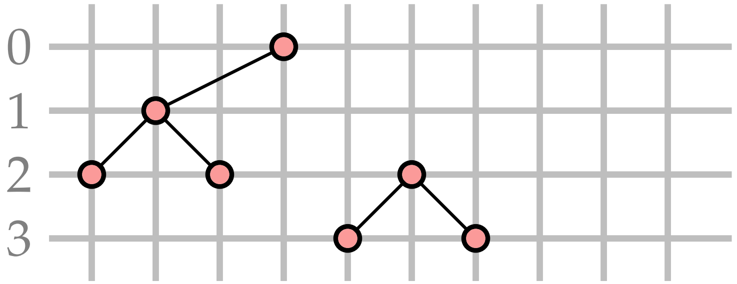
inorder



postorder

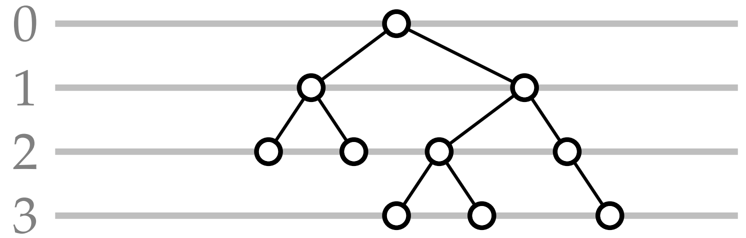




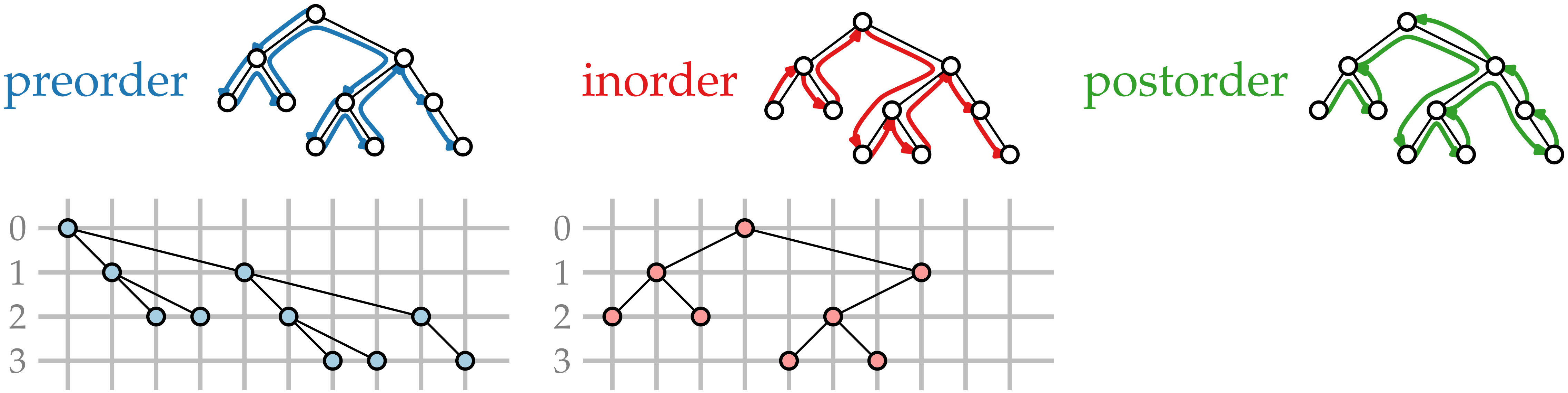


First Grid Layout of Binary Trees

1. Choose y-coordinates: $y(u) =$depth(u)



2. Choose x-coordinates:



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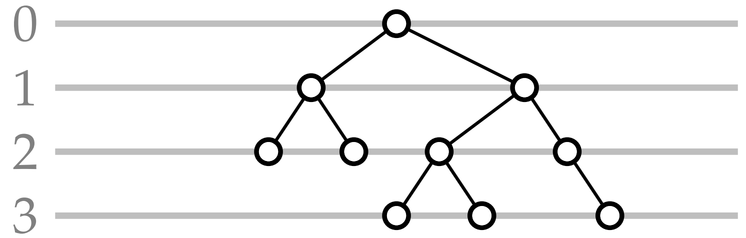

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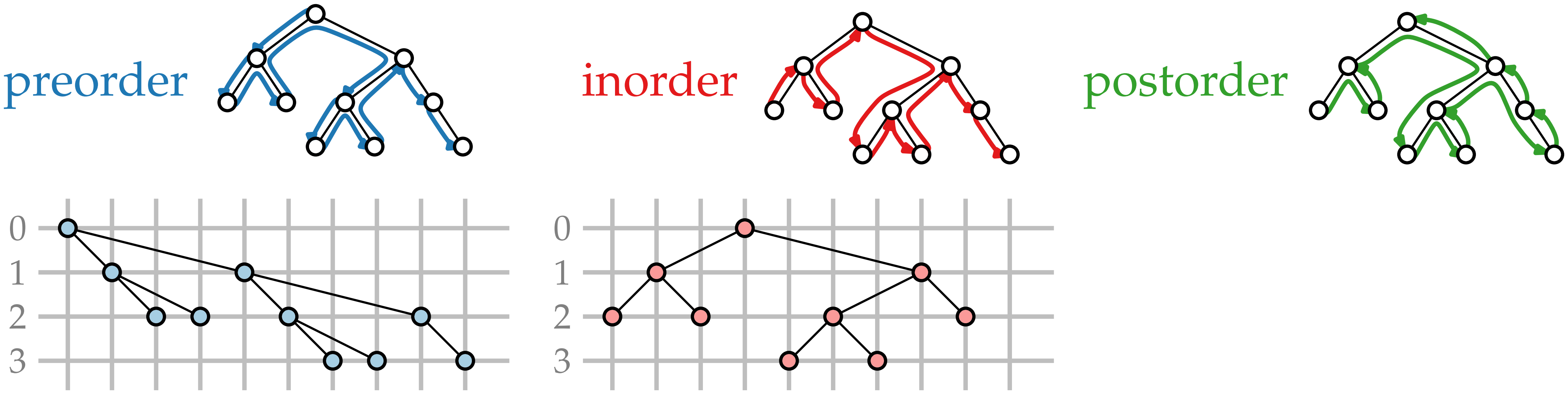

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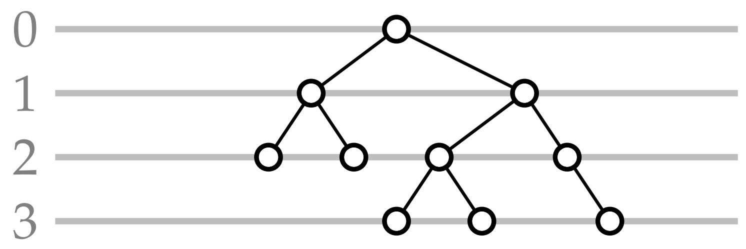

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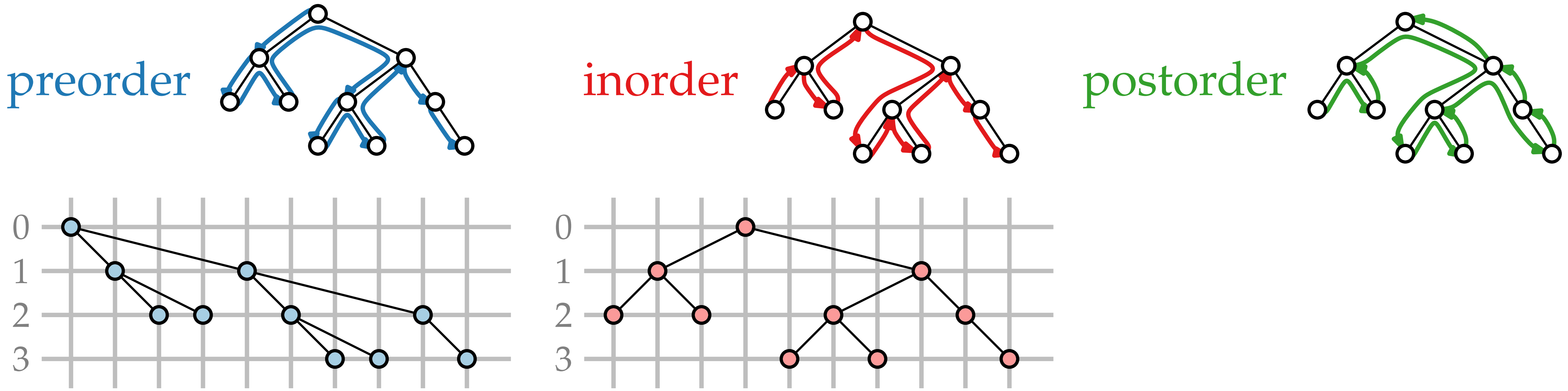

First Grid Layout of Binary Trees

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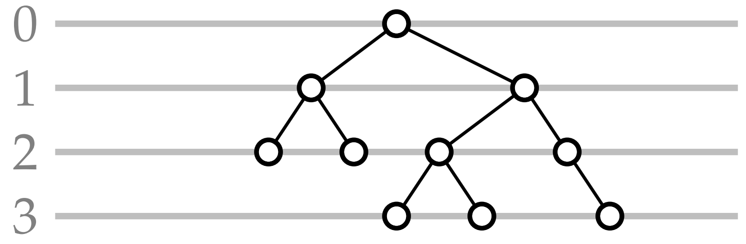
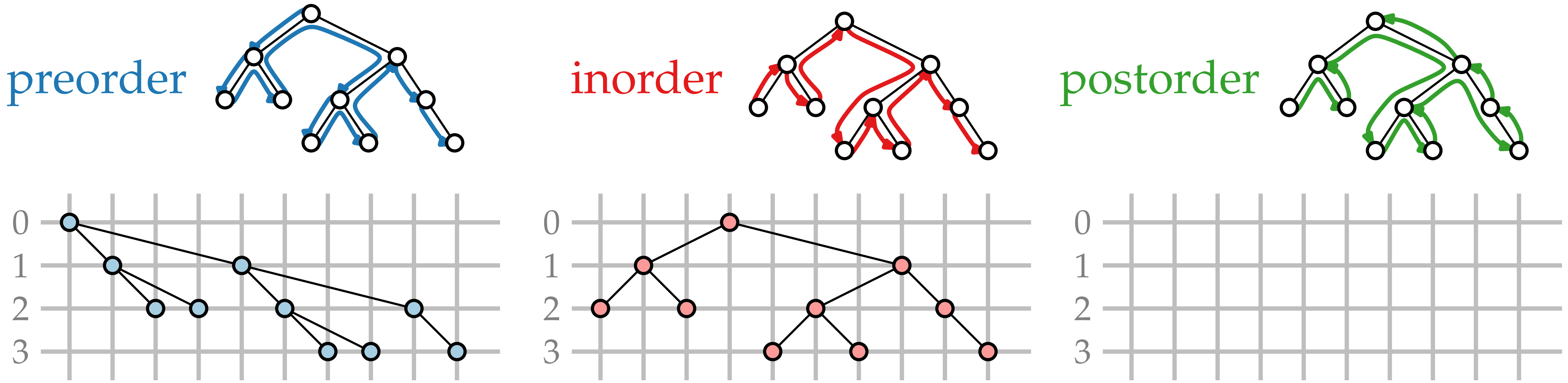


2. Choose x-coordinates:



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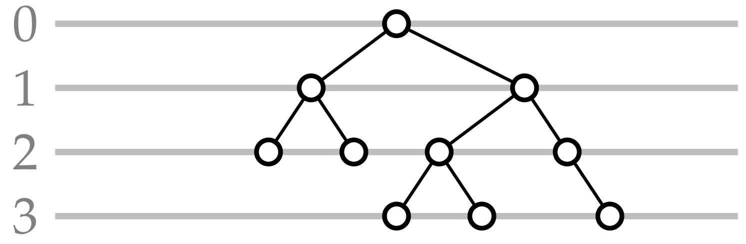

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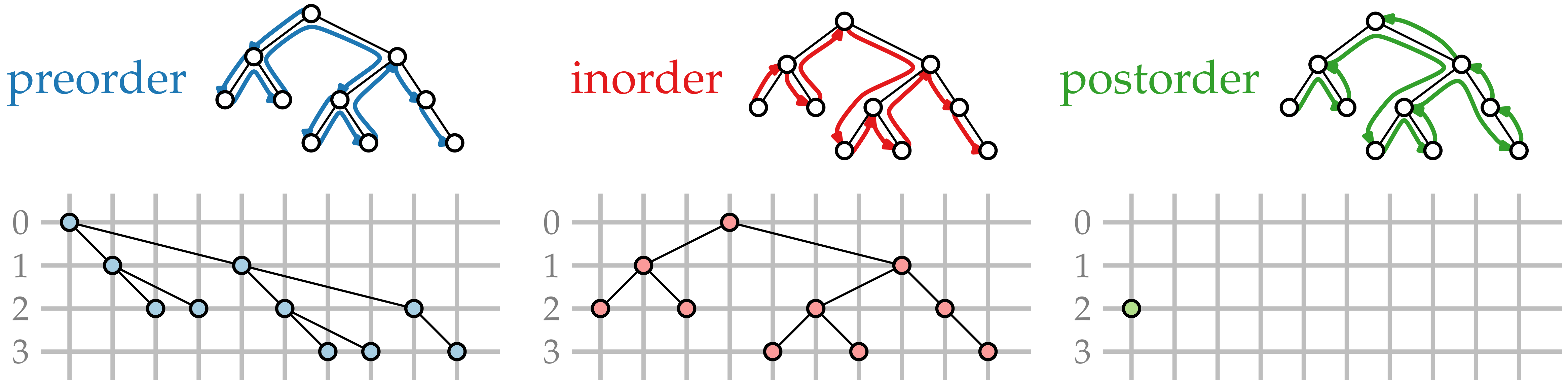

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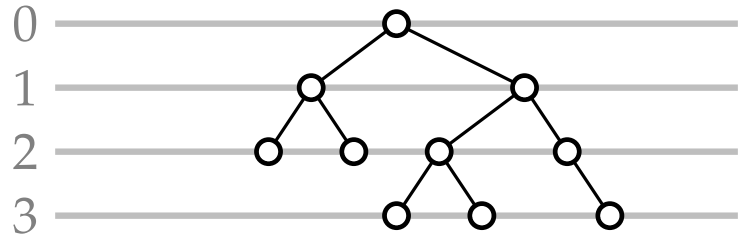

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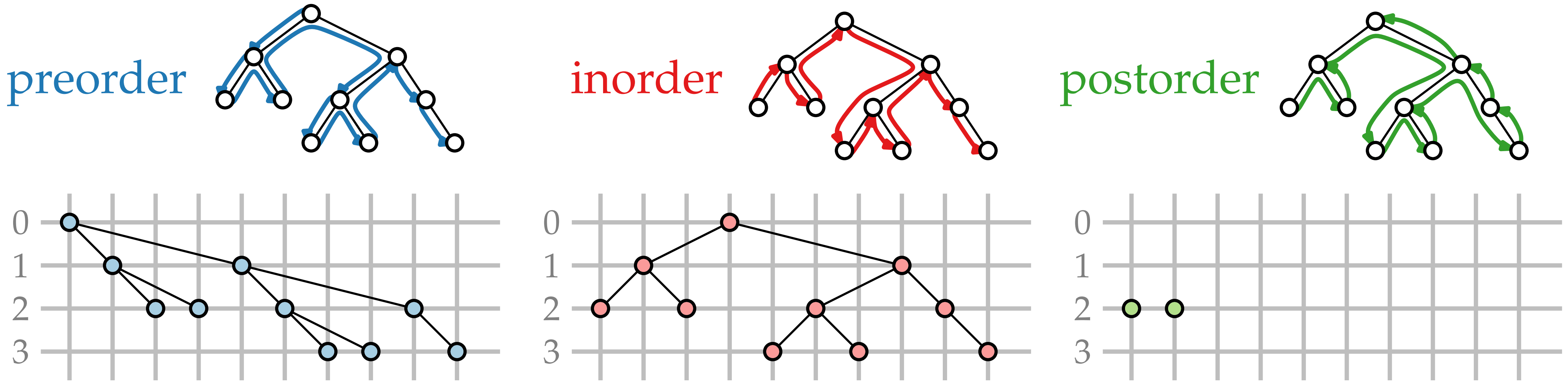

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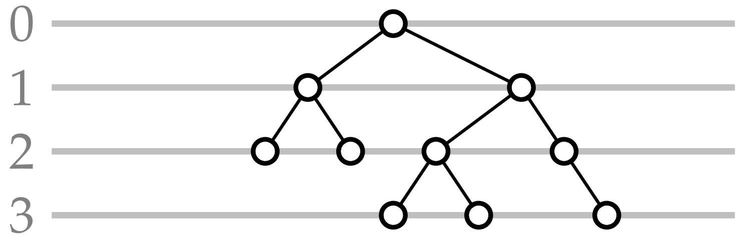
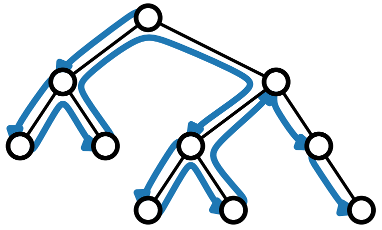
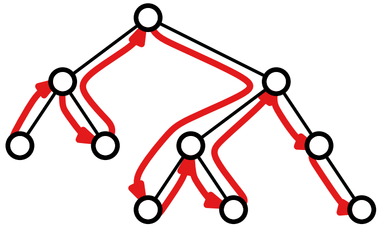


2. Choose x-coordinates:

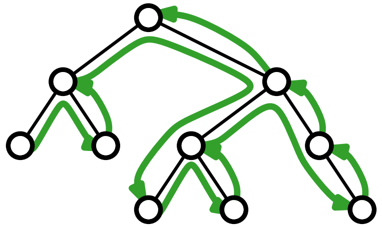
preorder

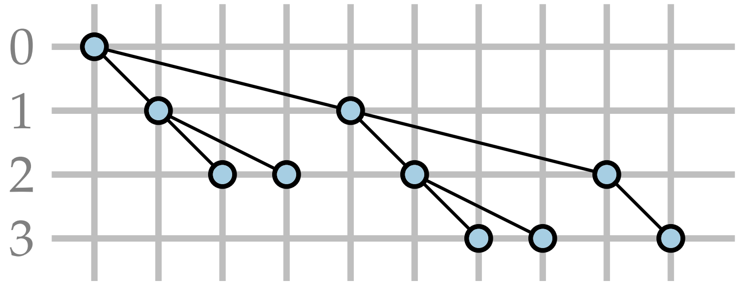


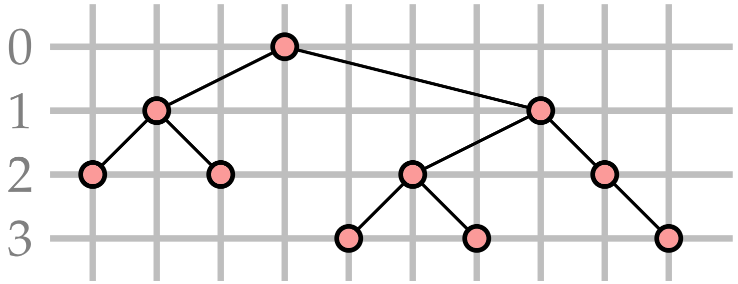
inorder

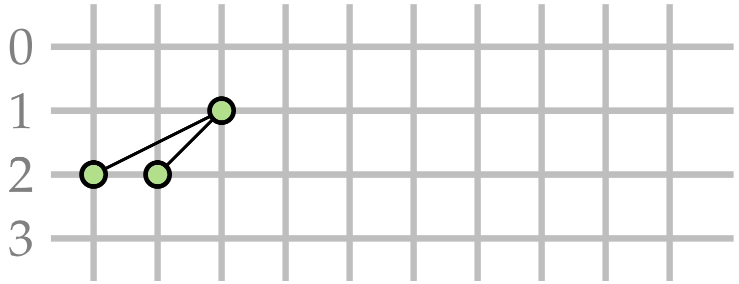


postorder









First Grid Layout of Binary Trees

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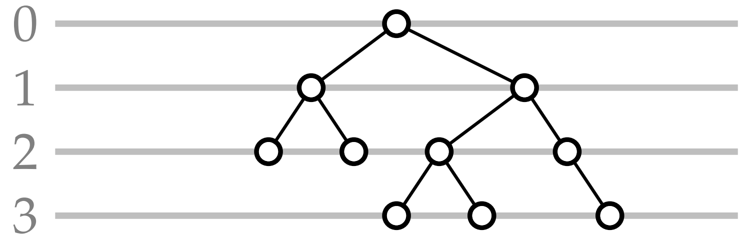
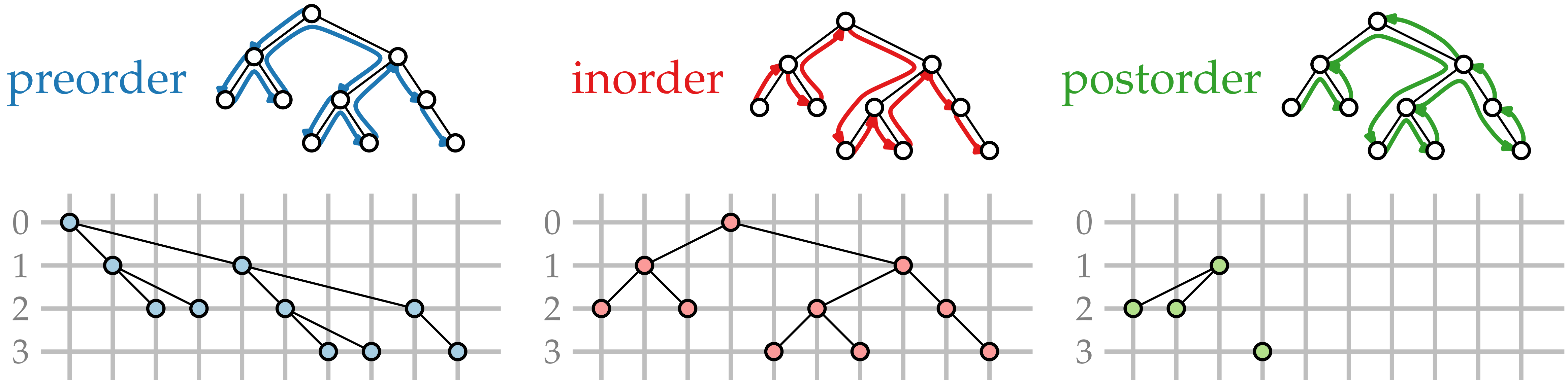


2. Choose x-coordinates:



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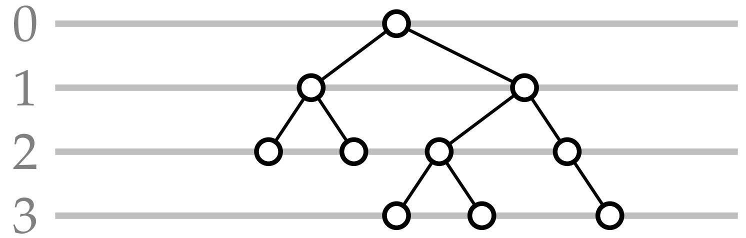
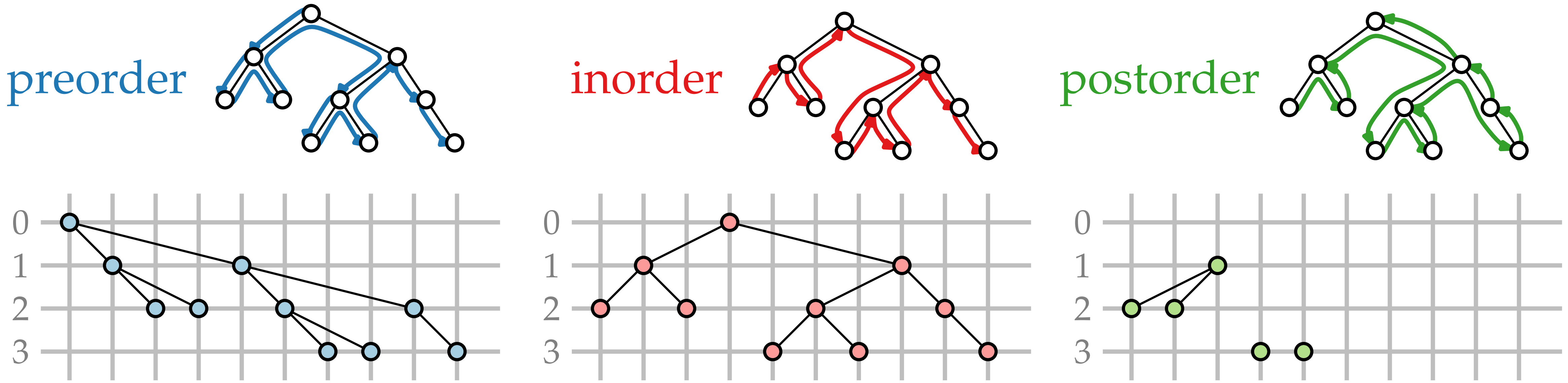


2. Choose x-coordinates:



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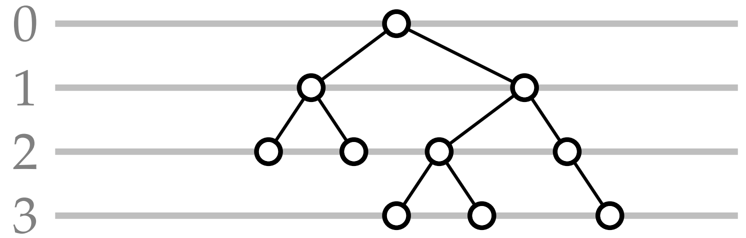

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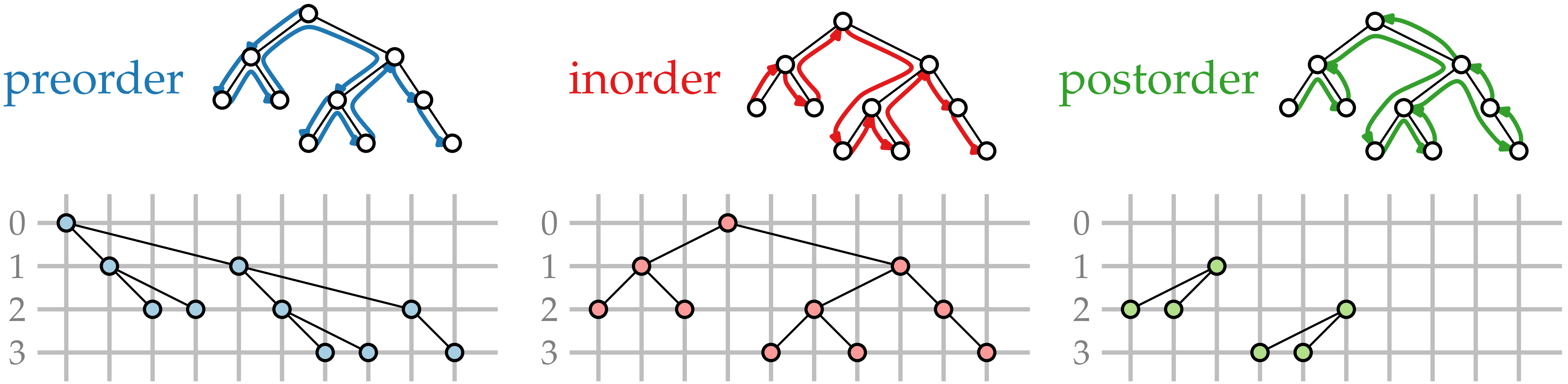

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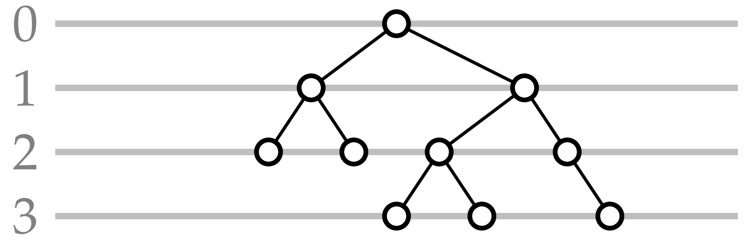
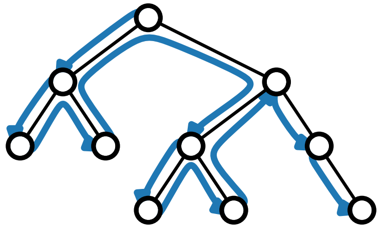
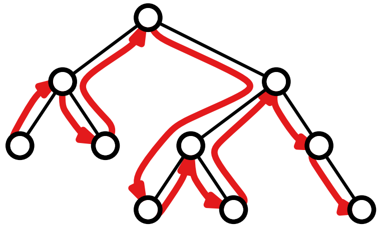


2. Choose x-coordinates:

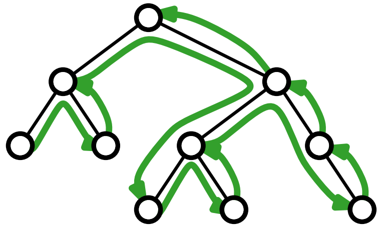
preorder

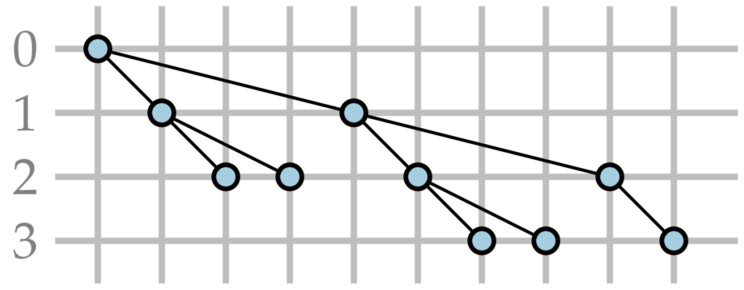


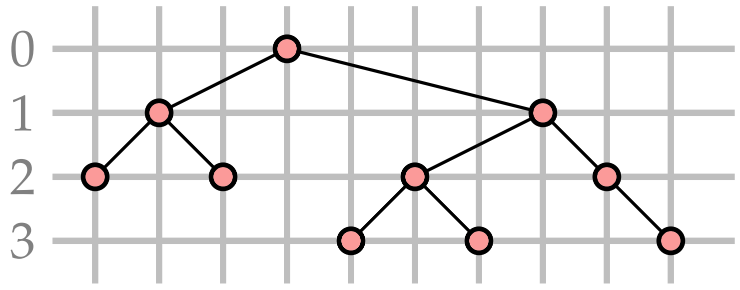
inorder

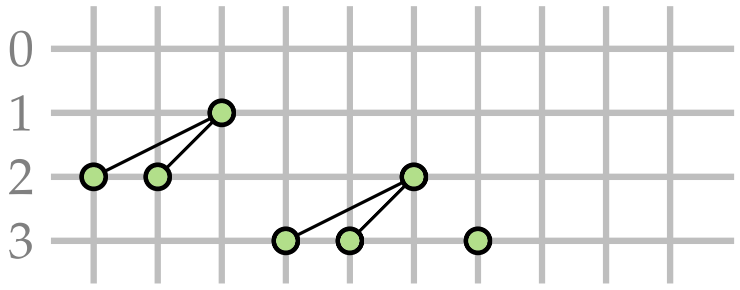


postorder









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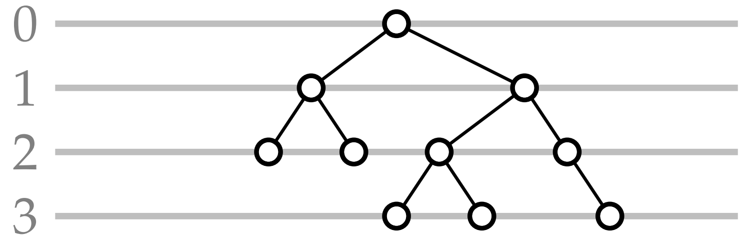
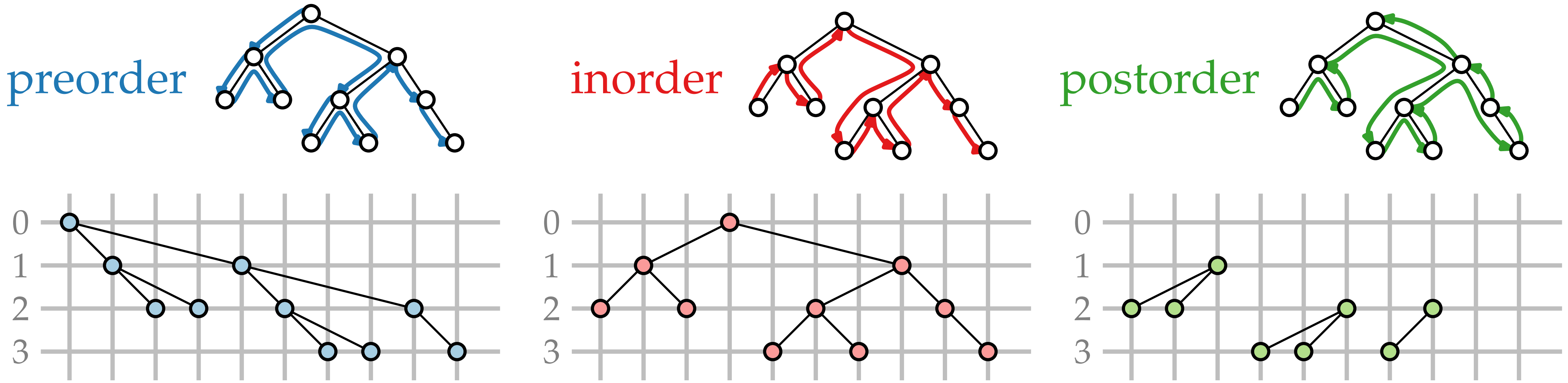
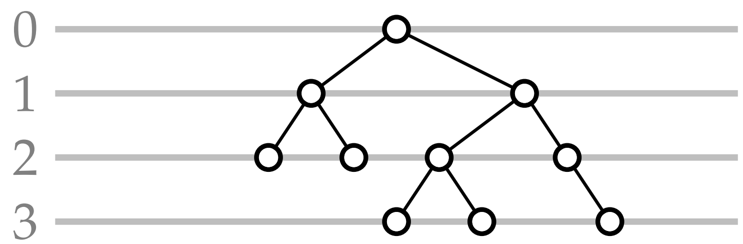


2. Choose x-coordinates:



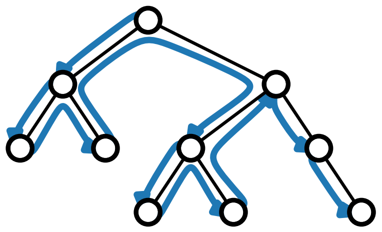
First Grid Layout of Binary Trees

1. Choose y-coordinates: $y(u)$ =depth(u)

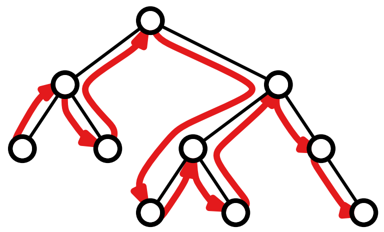


2. Choose x-coordinates:

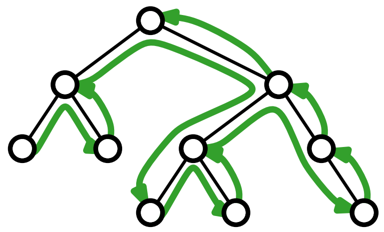
preorder

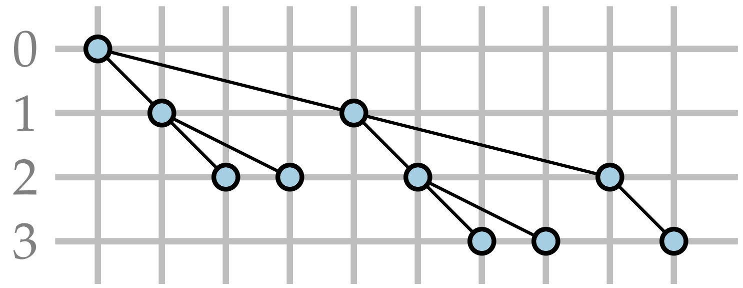


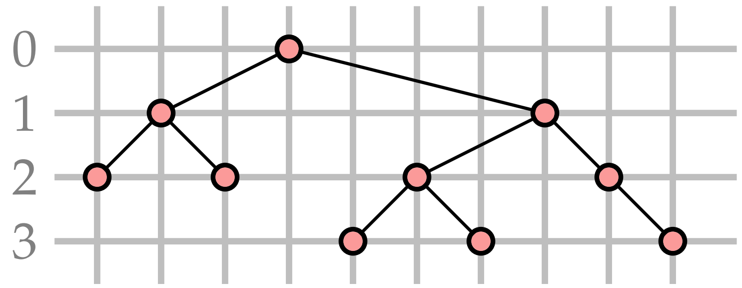
inorder

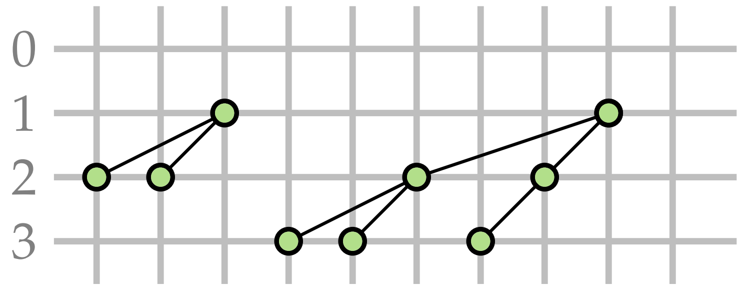


postorder



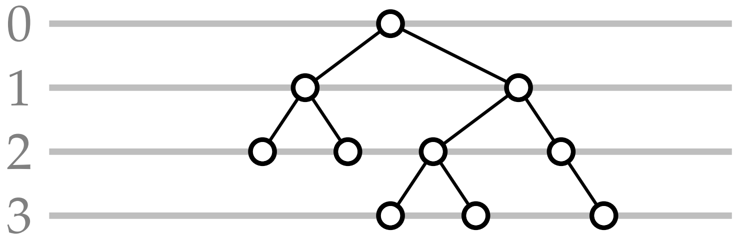






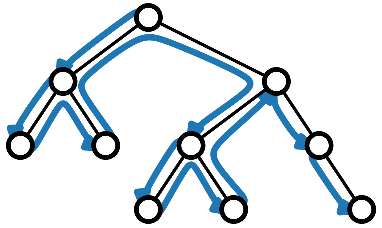
First Grid Layout of Binary Trees

1. Choose y-coordinates: $y(u) = \text{depth}(u)$

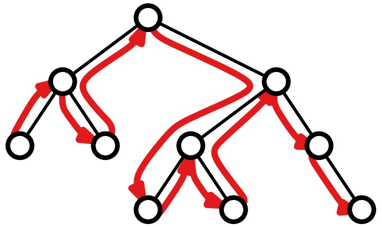


2. Choose x-coordinates:

preorder



inorder



postorder