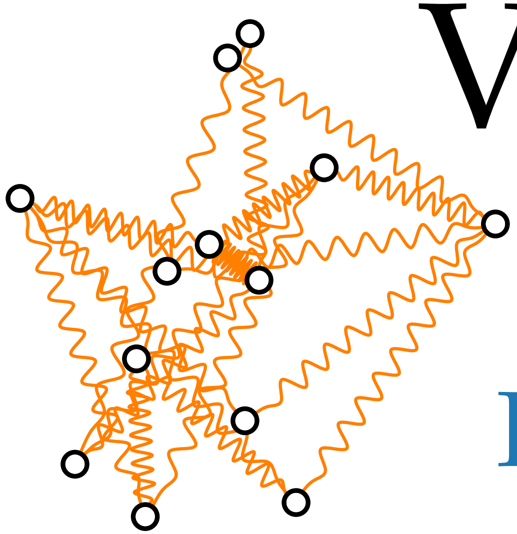
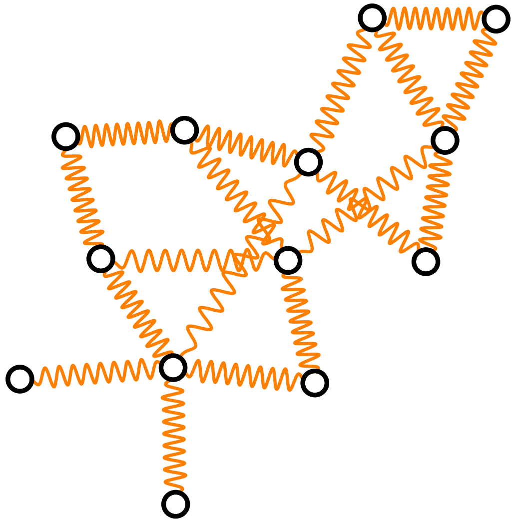
Universität Trier



Visualization of Graphs

Lecture 3:
Force-Directed Drawing Algorithms

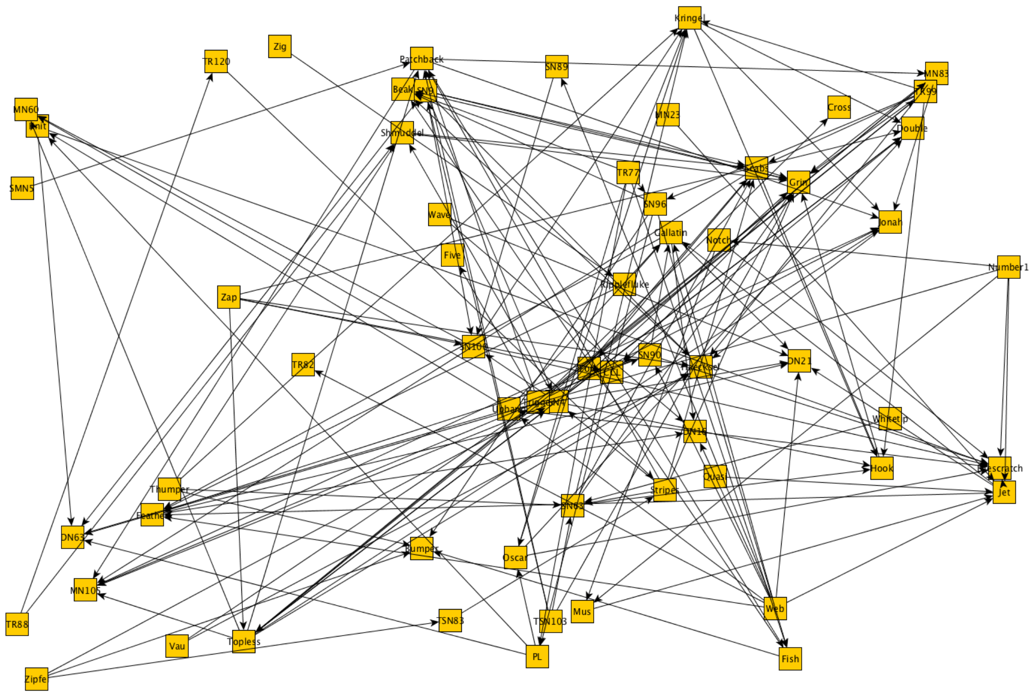


Part I:
Algorithm Framework

Philipp Kindermann

General Layout Problem

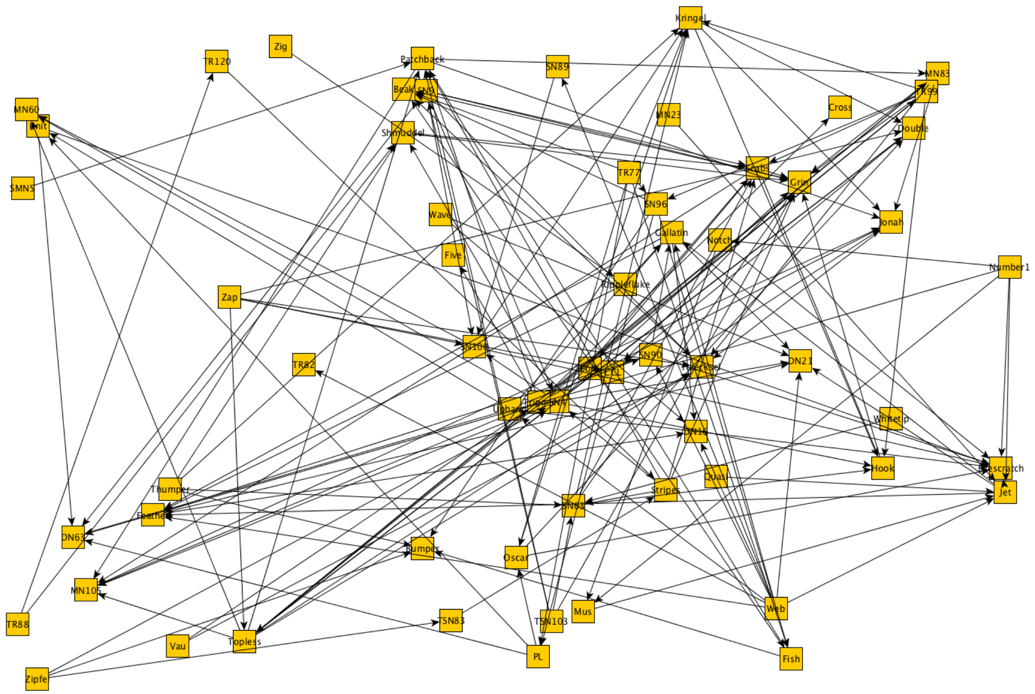
Input: Graph $G = (V, E)$

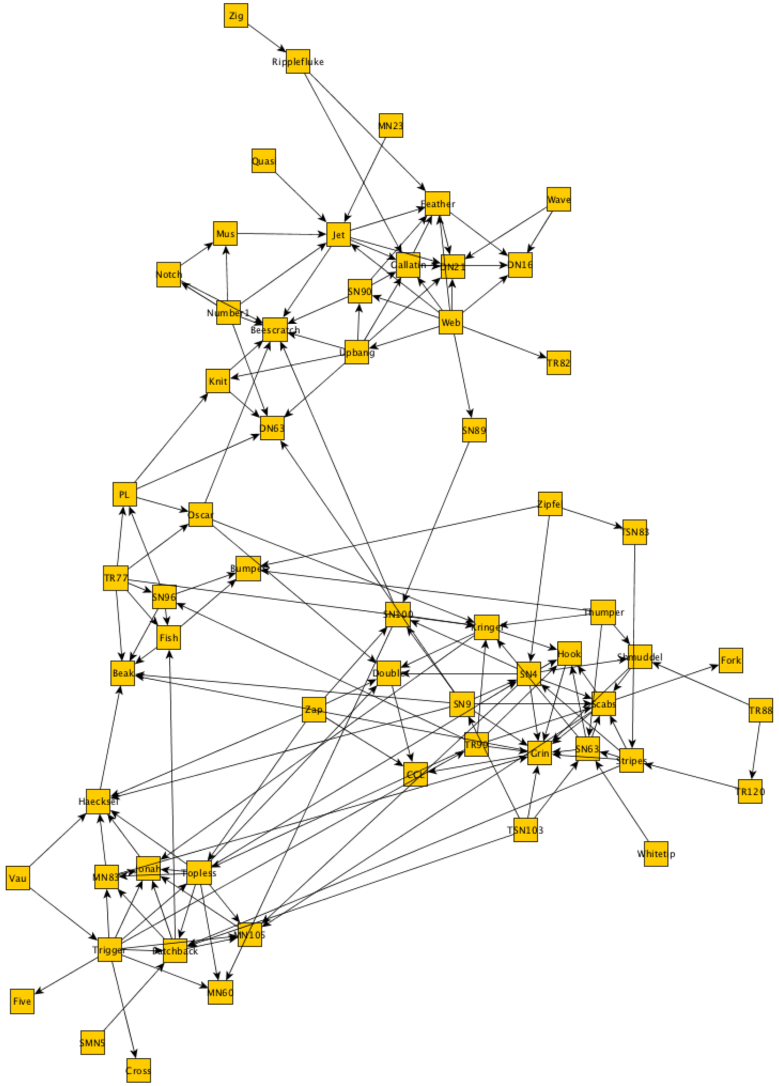


General Layout Problem

Input: Graph $G = (V, E)$

Output: Clear and readable straight-line drawing of G





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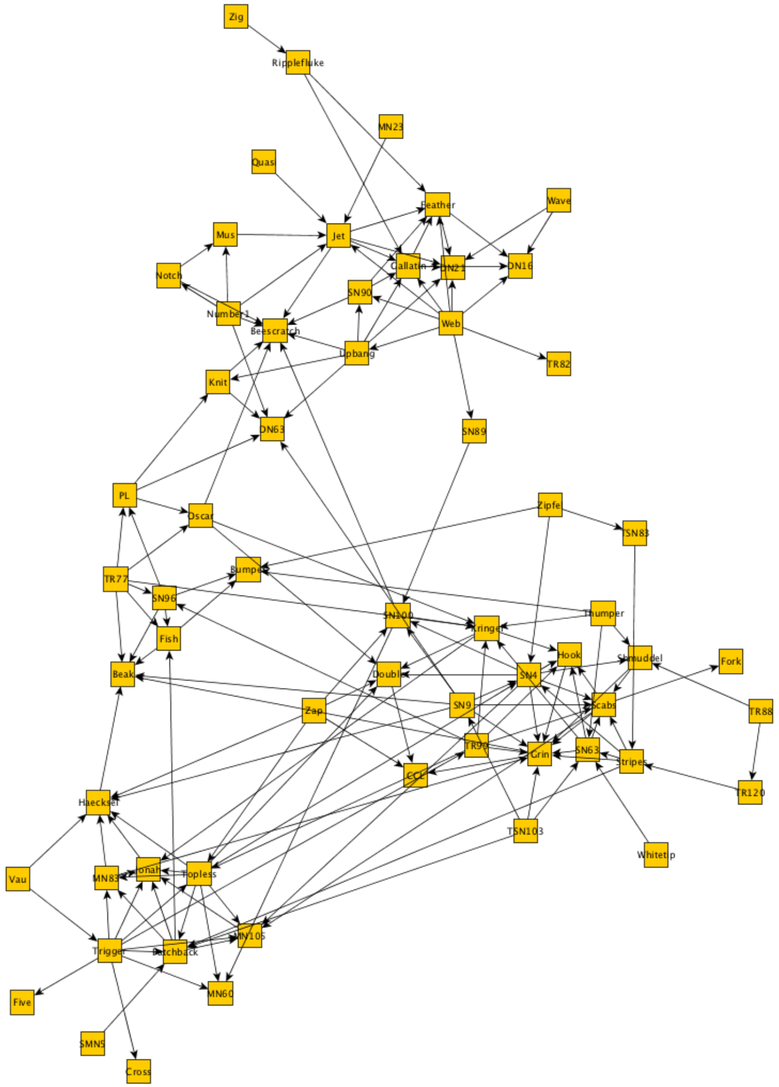
Drawing aesthetics:



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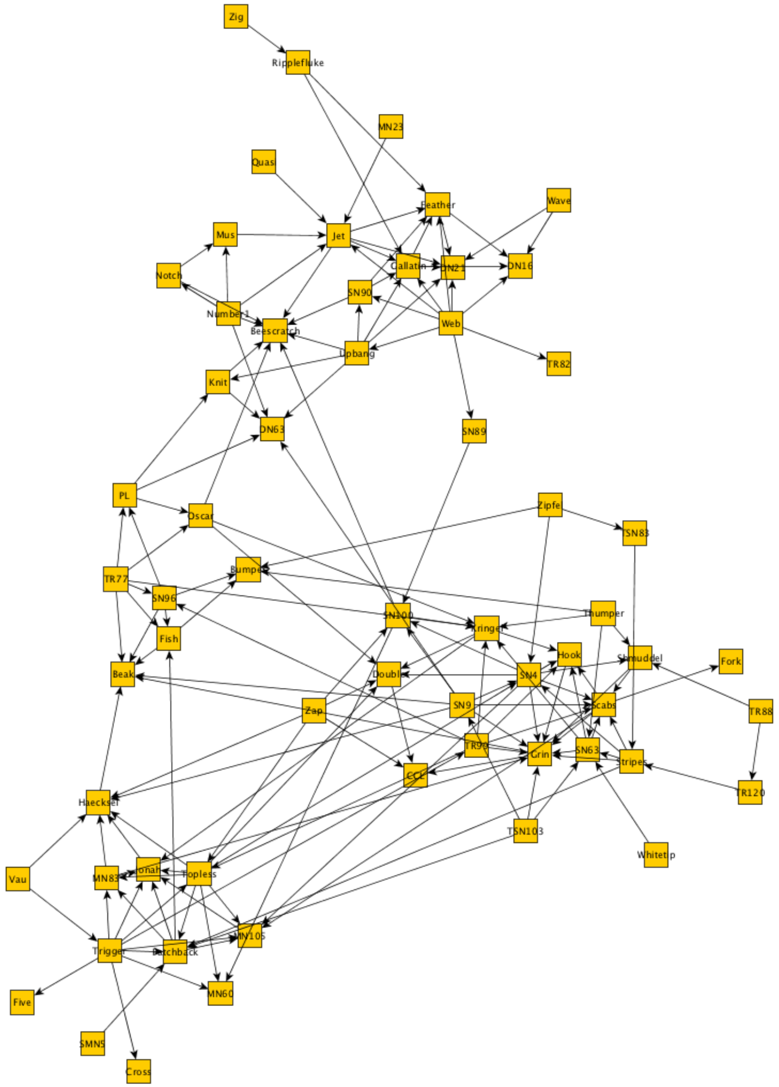
- adjacent vertices are close



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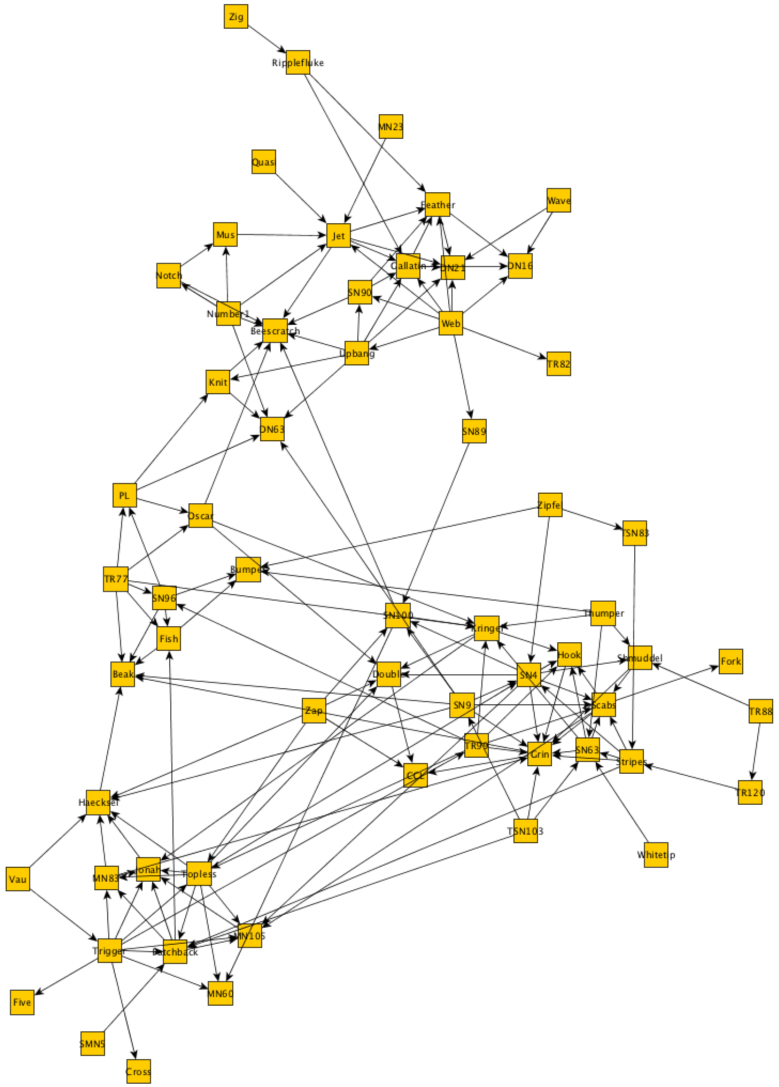

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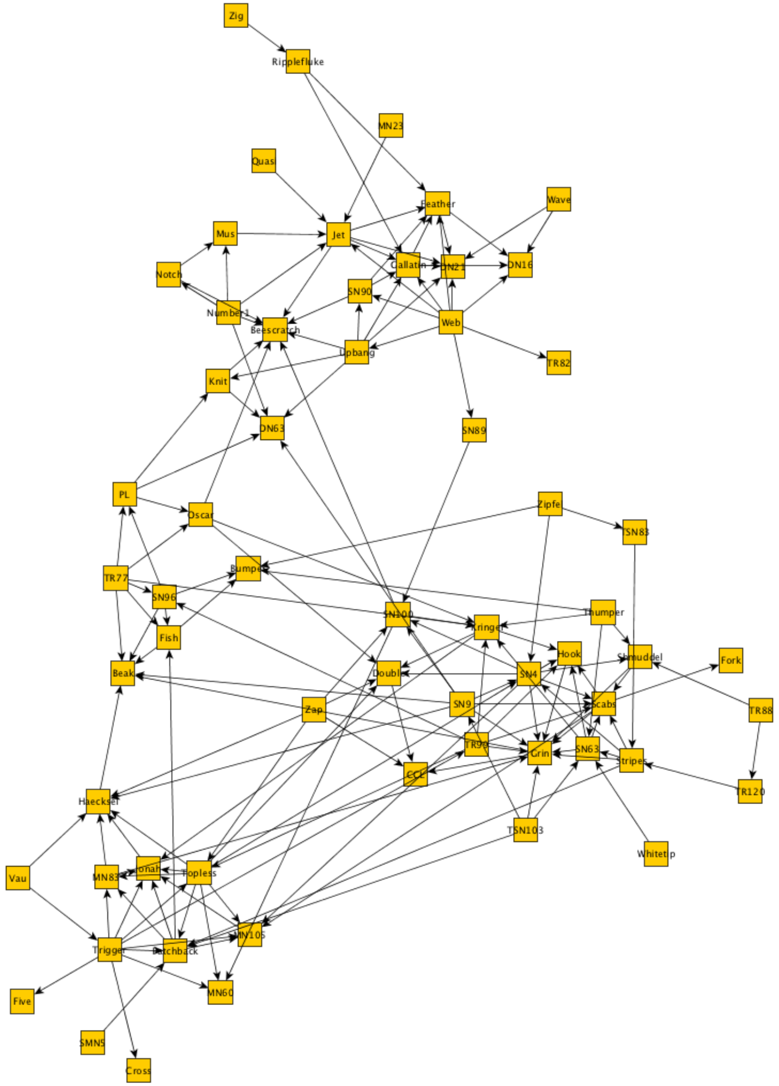
Drawing aesthetics:

- adjacent vertices are close
- non-adjacent vertices are far apart
- edges short, straight-line, **similar length**



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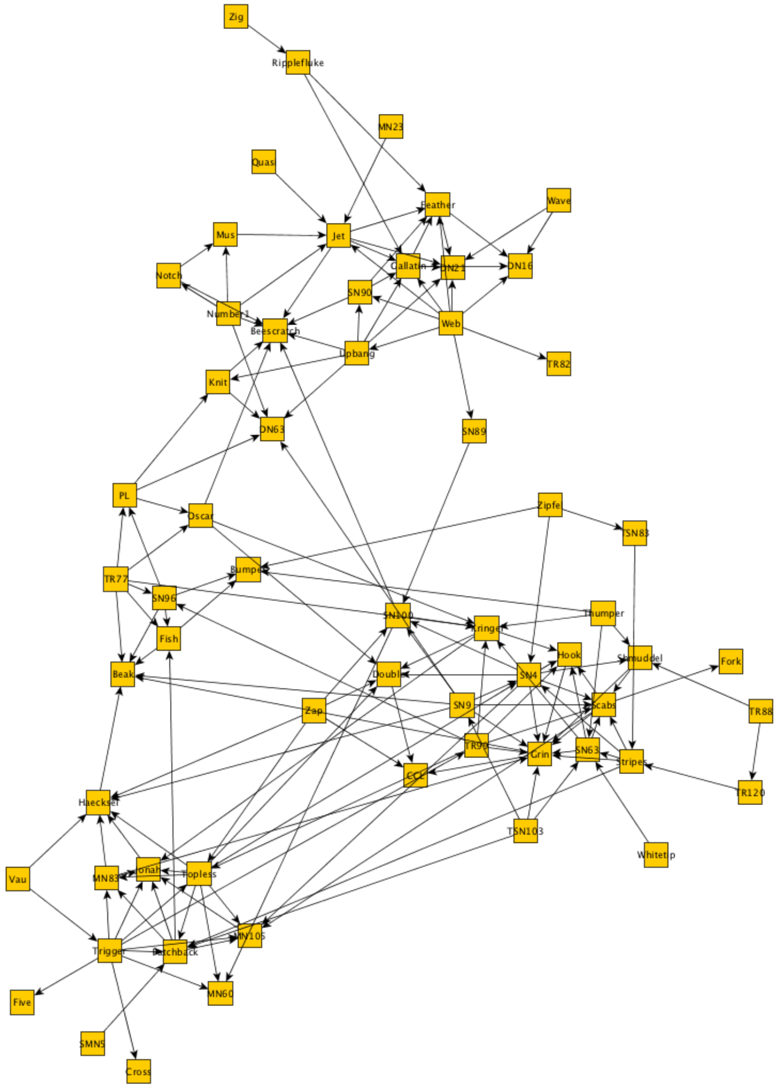

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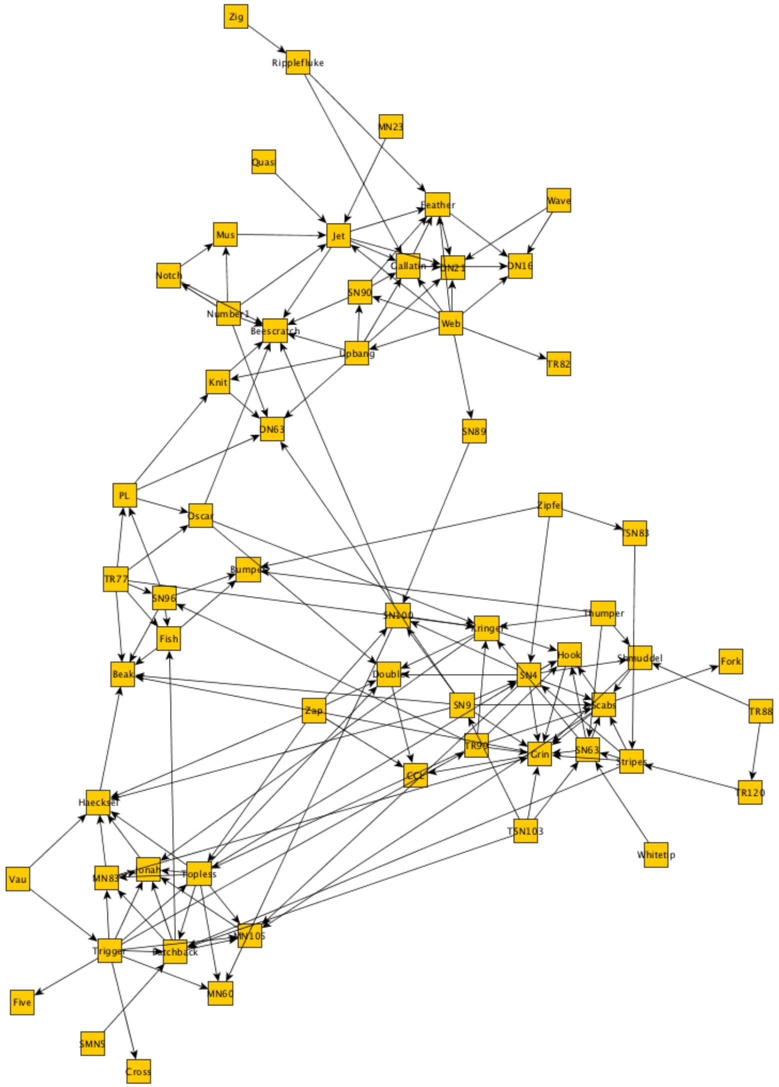

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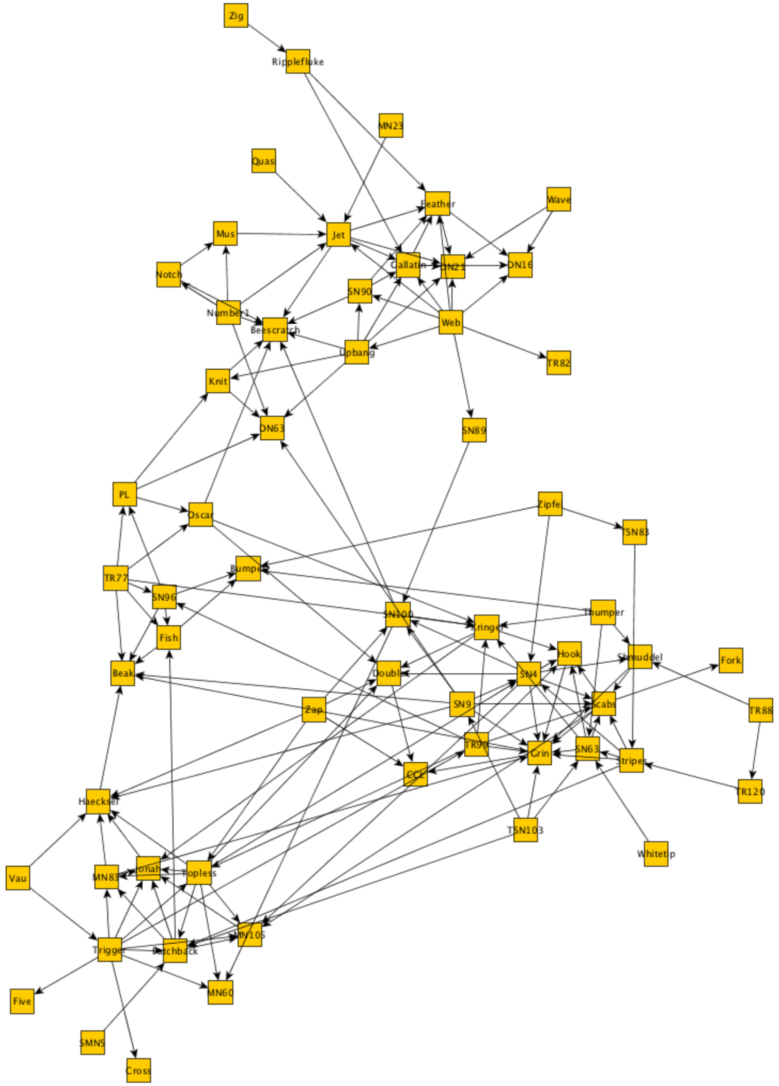

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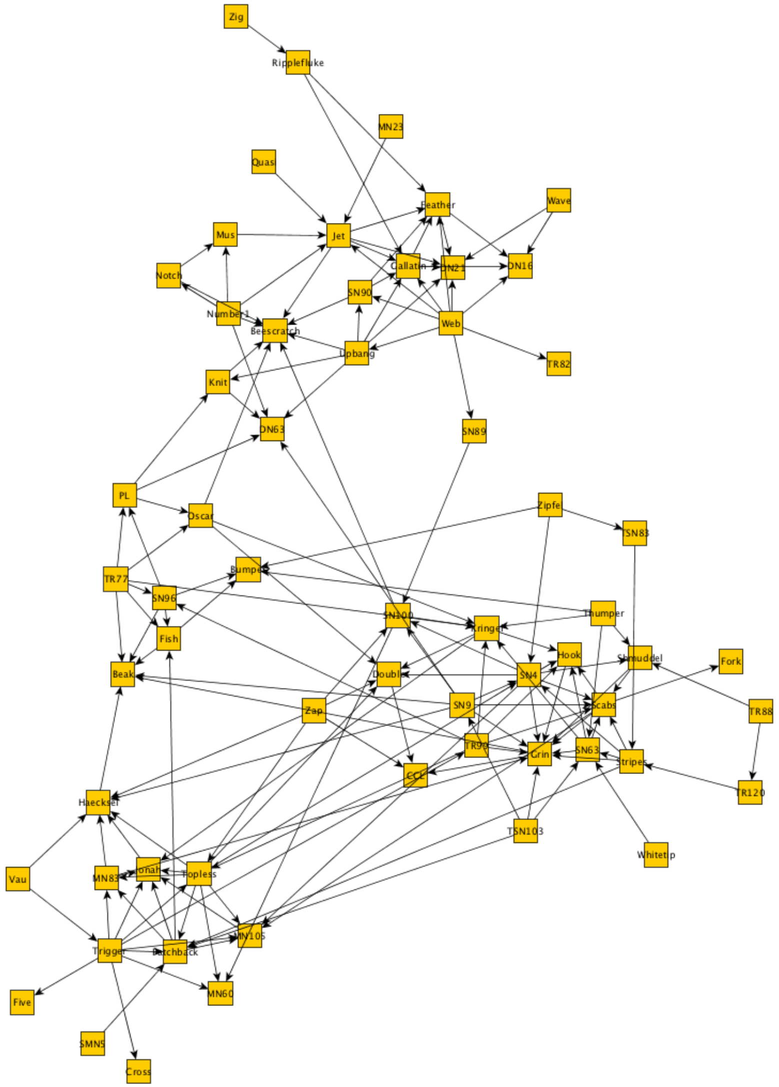
Optimization criteria partially contradict each other



Fixed Edge Lengths?

Input: Graph $G = (V, E)$, required edge length $\ell(e)$, $\forall e \in E$

Output: Drawing of G which realizes all the edge lengths

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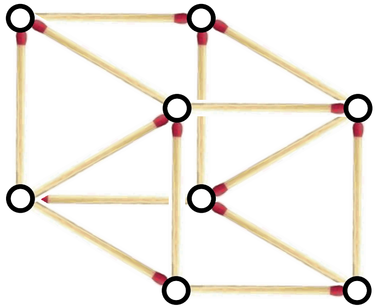

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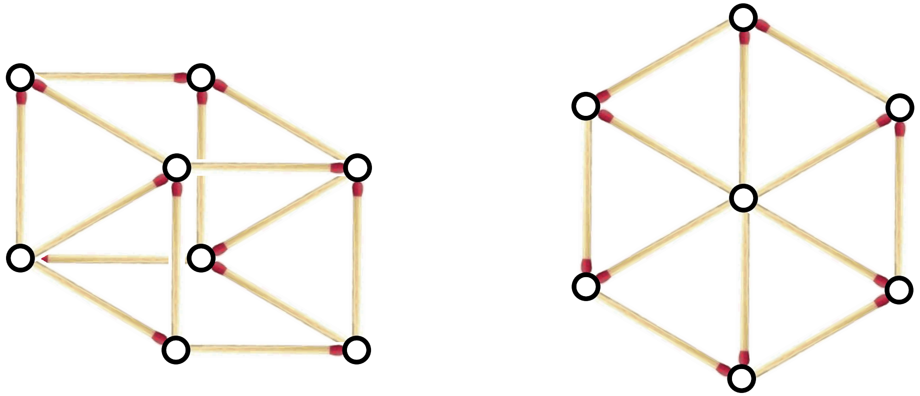

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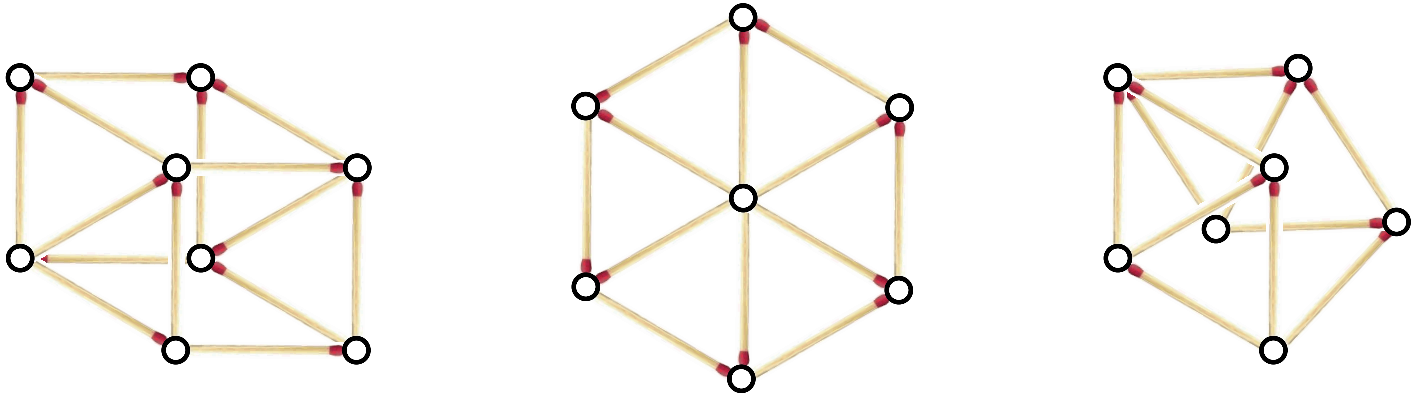

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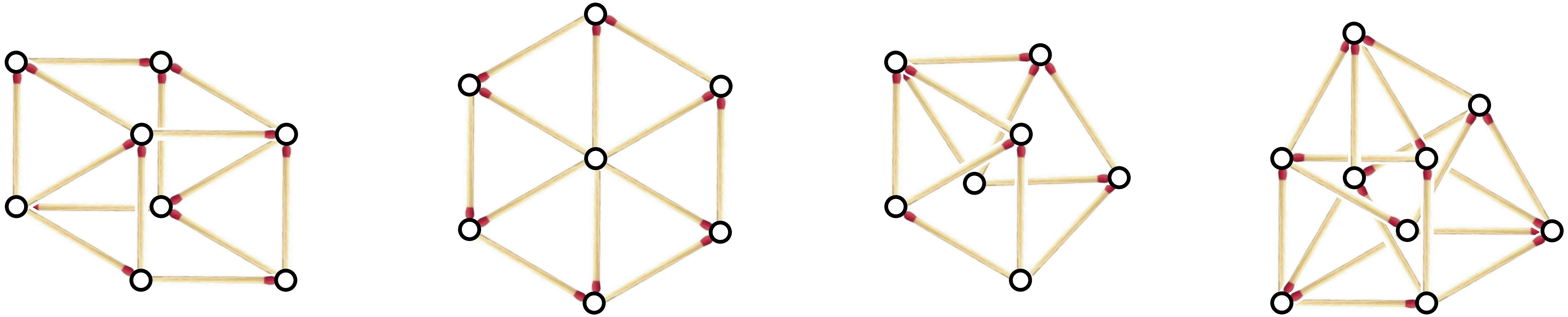

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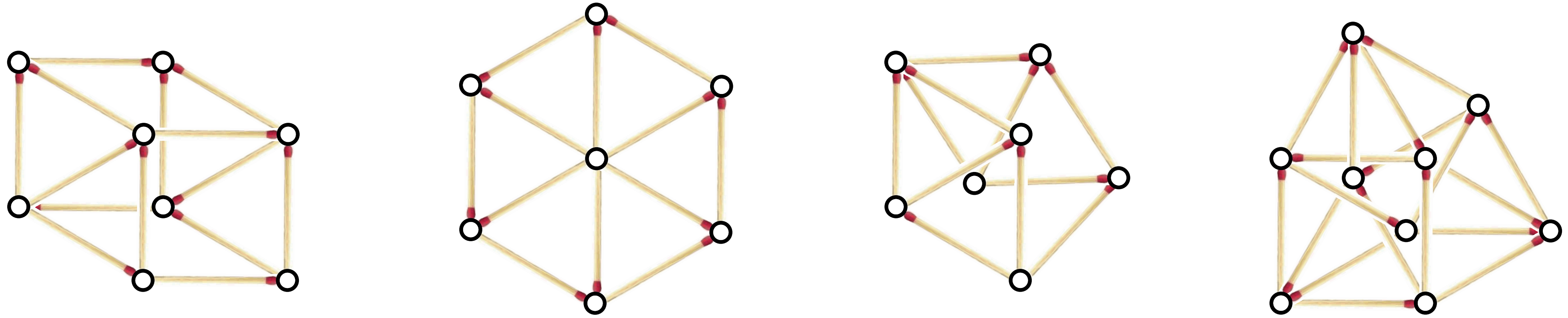

NP-hard for

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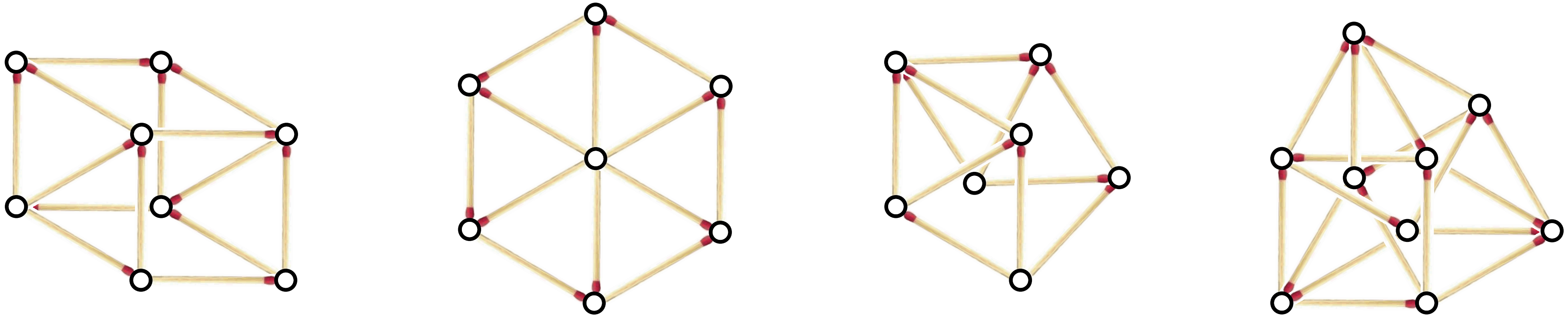

NP-hard for

- uniform edge lengths in any dimension [Johnson ’82]

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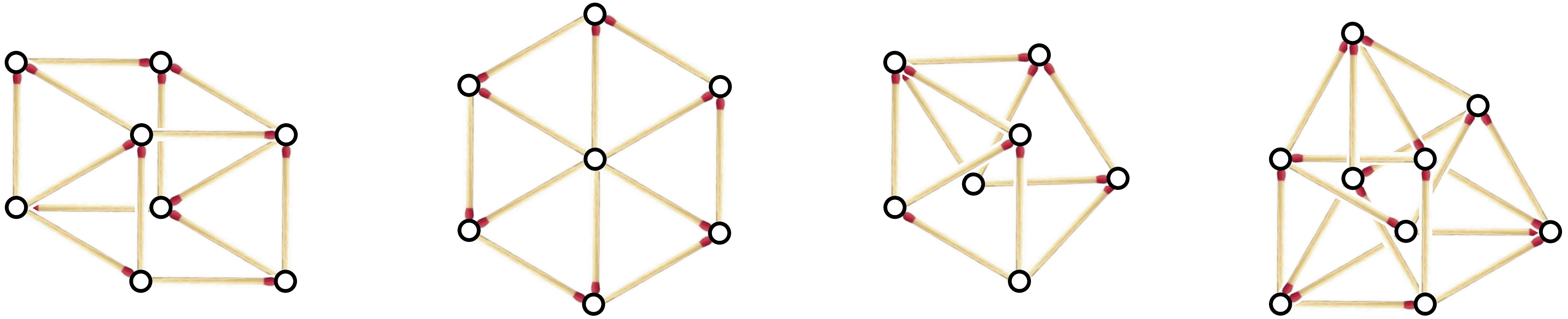

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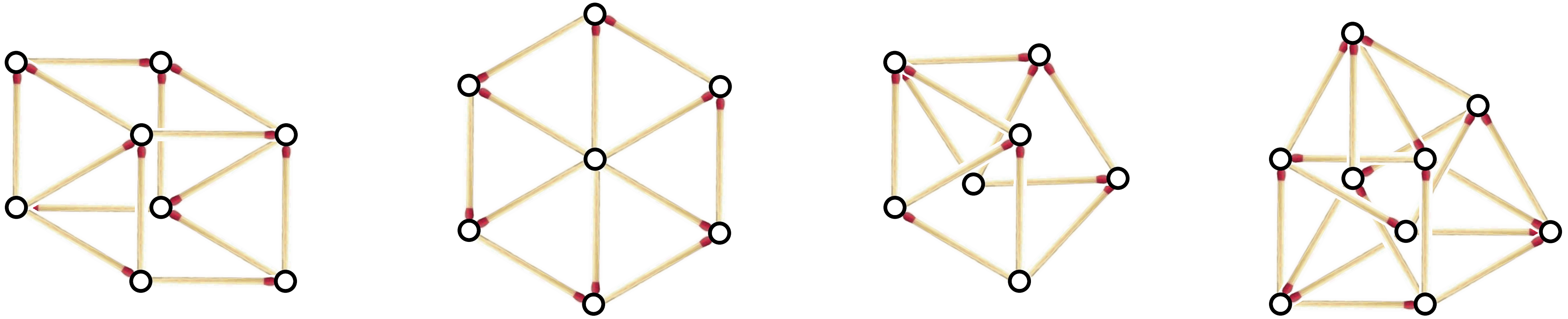

NP-hard for

- uniform edge lengths in any dimension [Johnson '82]
- uniform edge lengths in planar drawings [Eades, Wormald '90]
- edge lengths $\{1, 2\}$ [Saxe '80]

Physical Analogy

Idea. [Eades ’84]

“To embed a graph we replace the vertices by steel rings and replace each edge with a **spring** to form a mechanical system .

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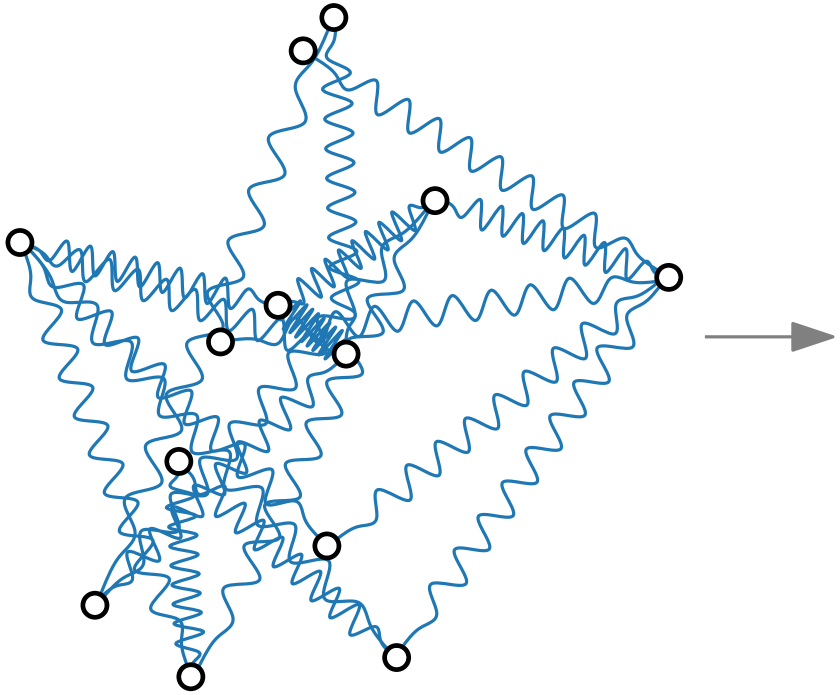

Physical Analogy

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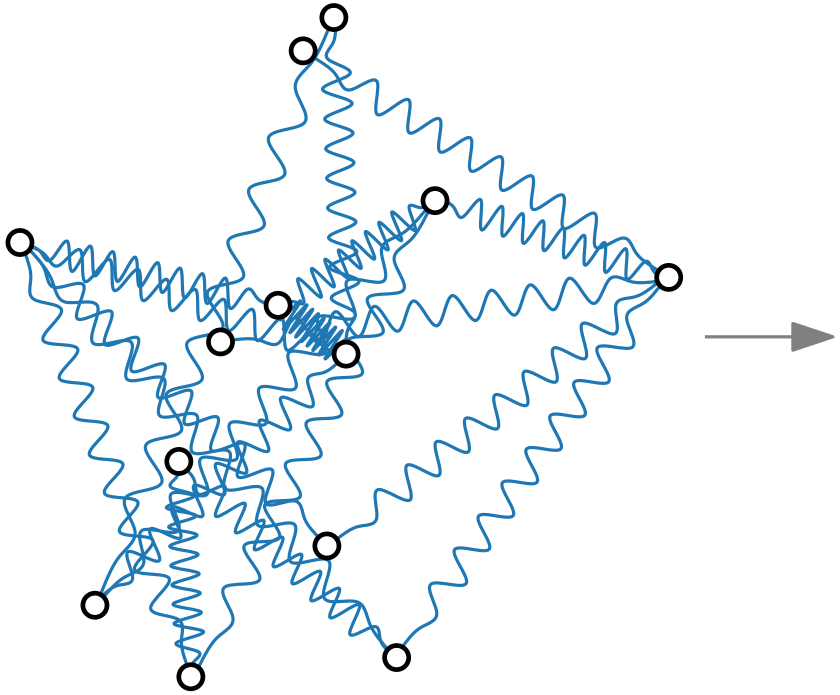
"To embed a graph we replace the vertices by steel rings and replace each edge with a **spring** to form a mechanical system . . . The vertices are placed in some initial layout and let go so that the spring forces on the rings move the system to a minimal energy state."



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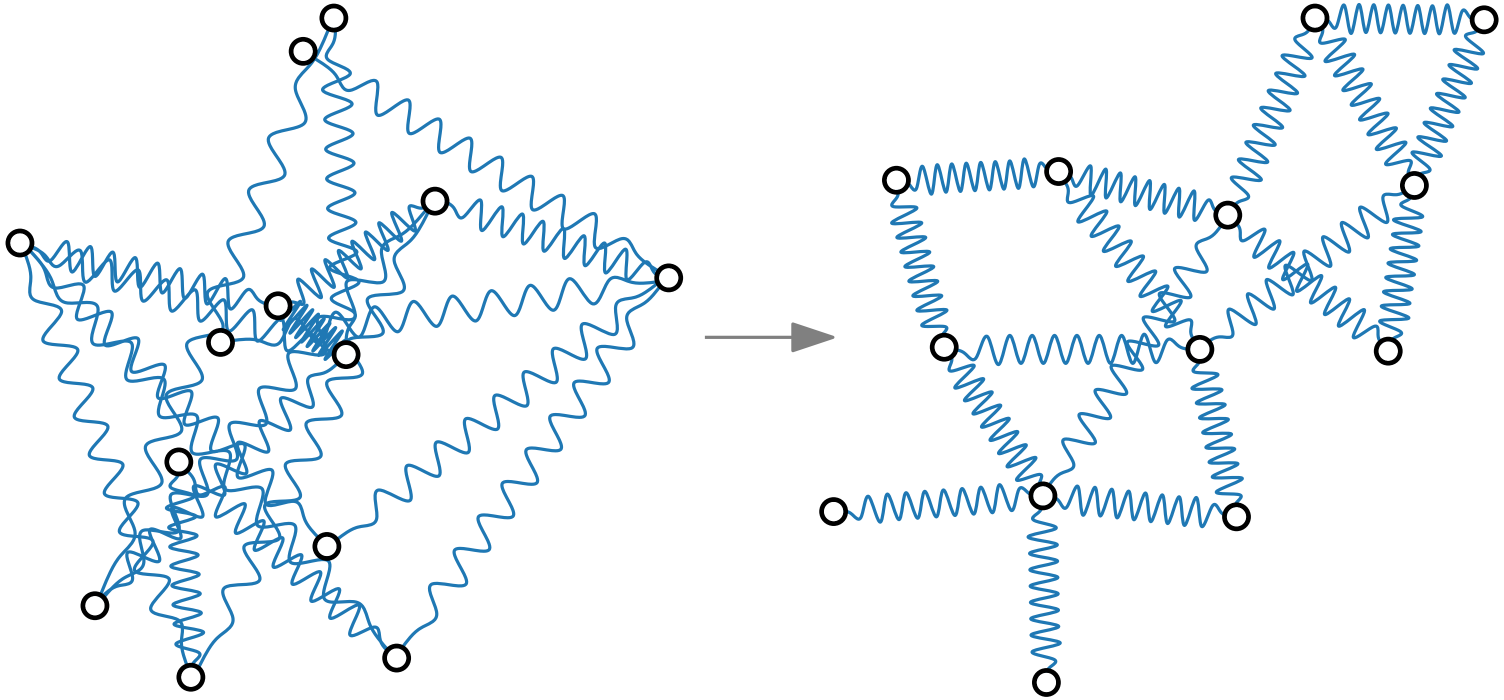

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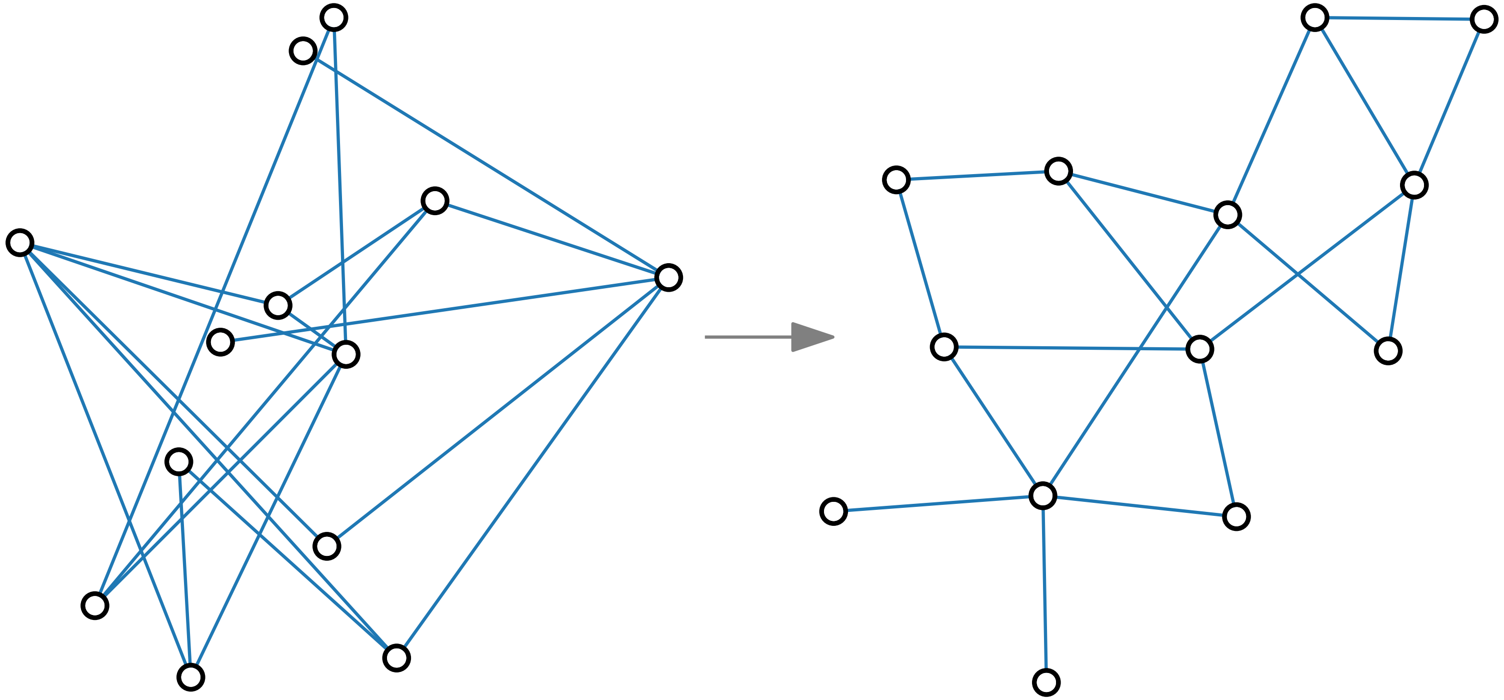

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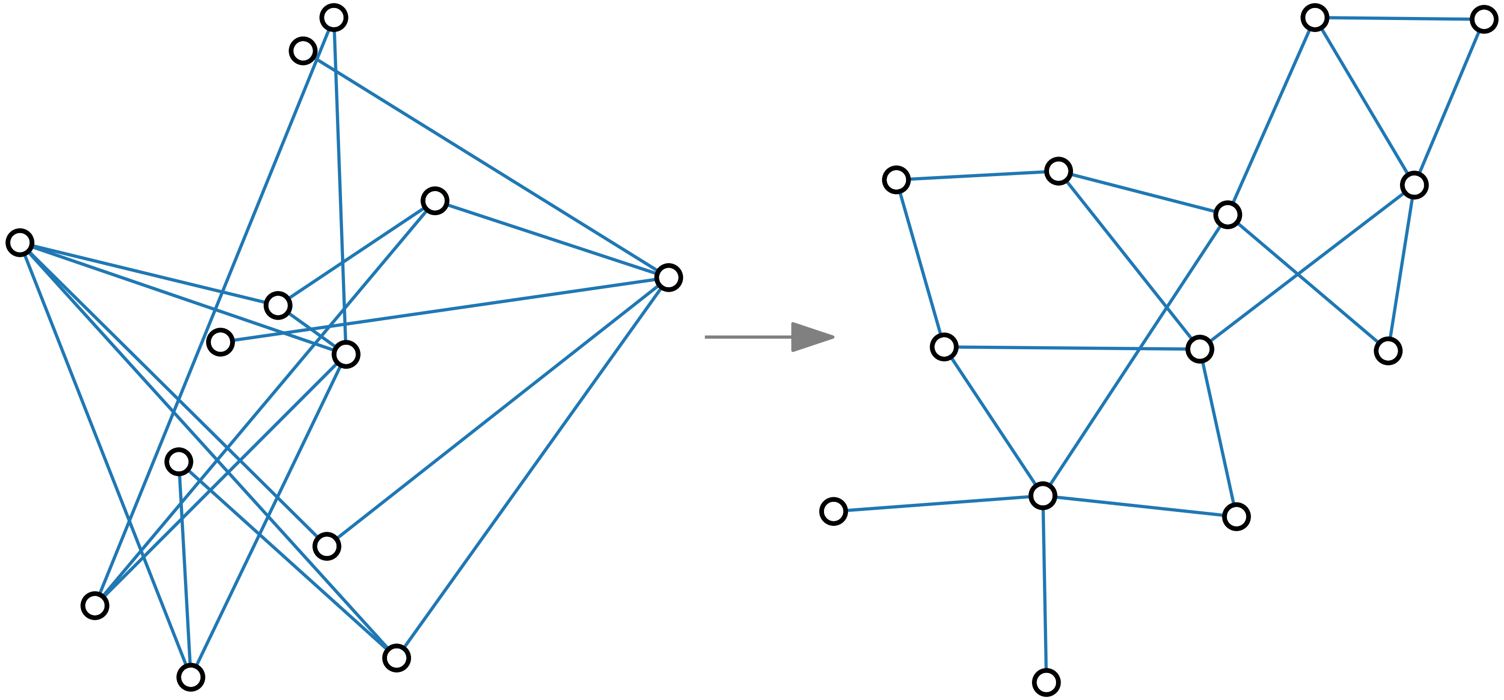
Attractive forces.



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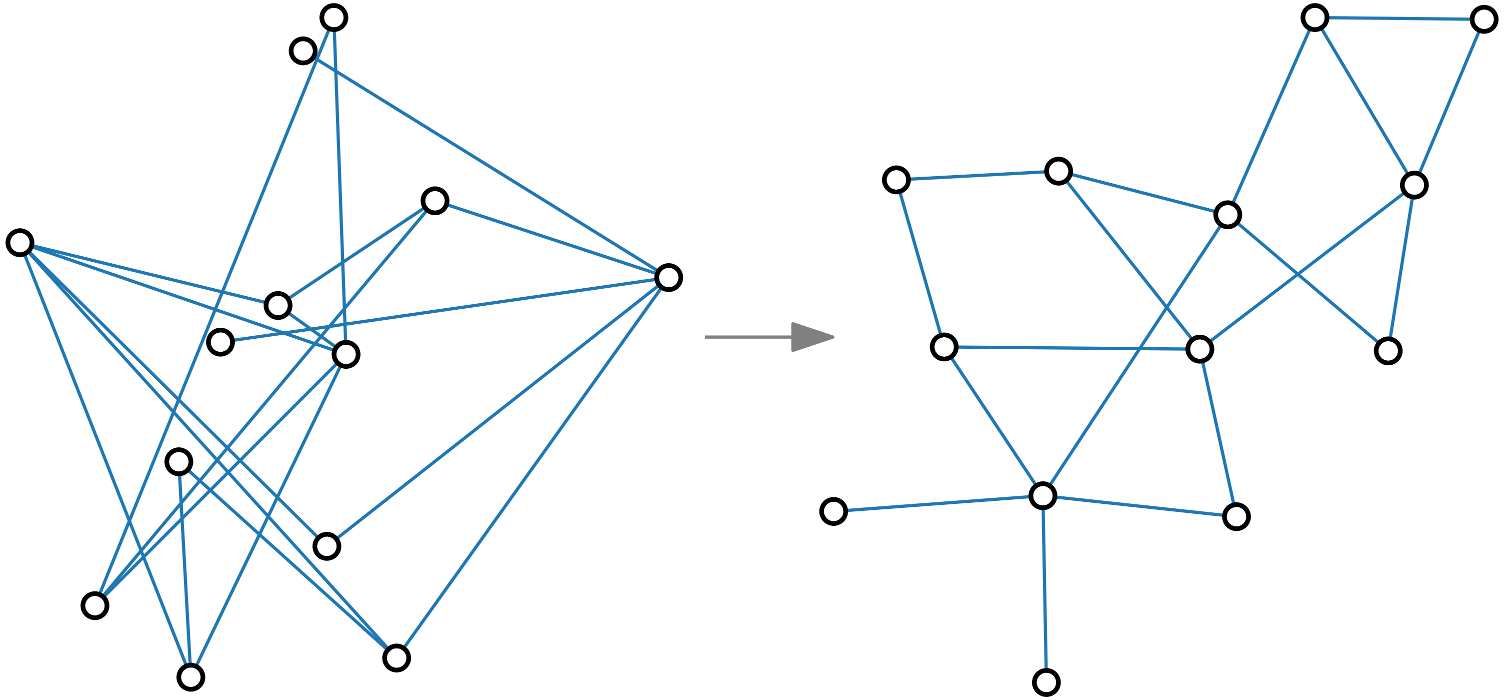

Attractive forces.

adjacent vertices u and v:

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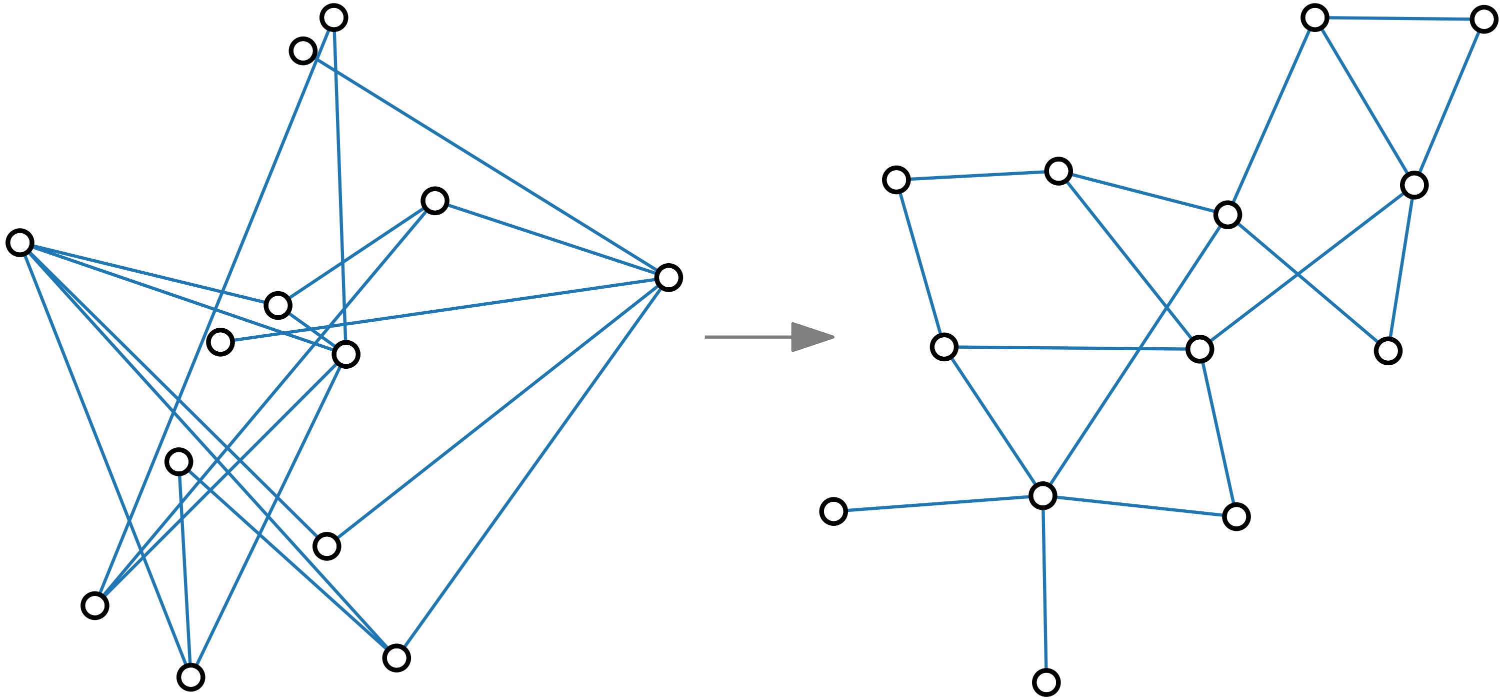

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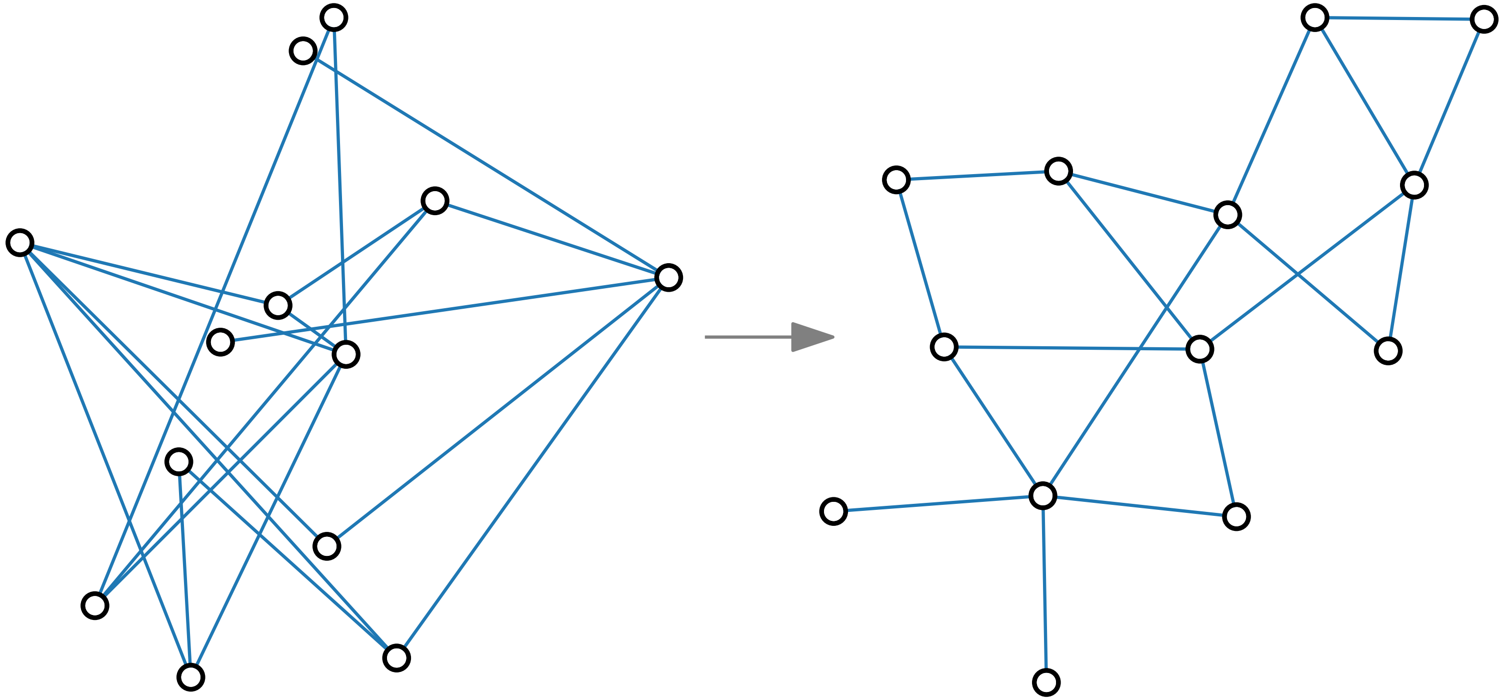

Attractive forces.

adjacent vertices u and v:

Repulsive forces.

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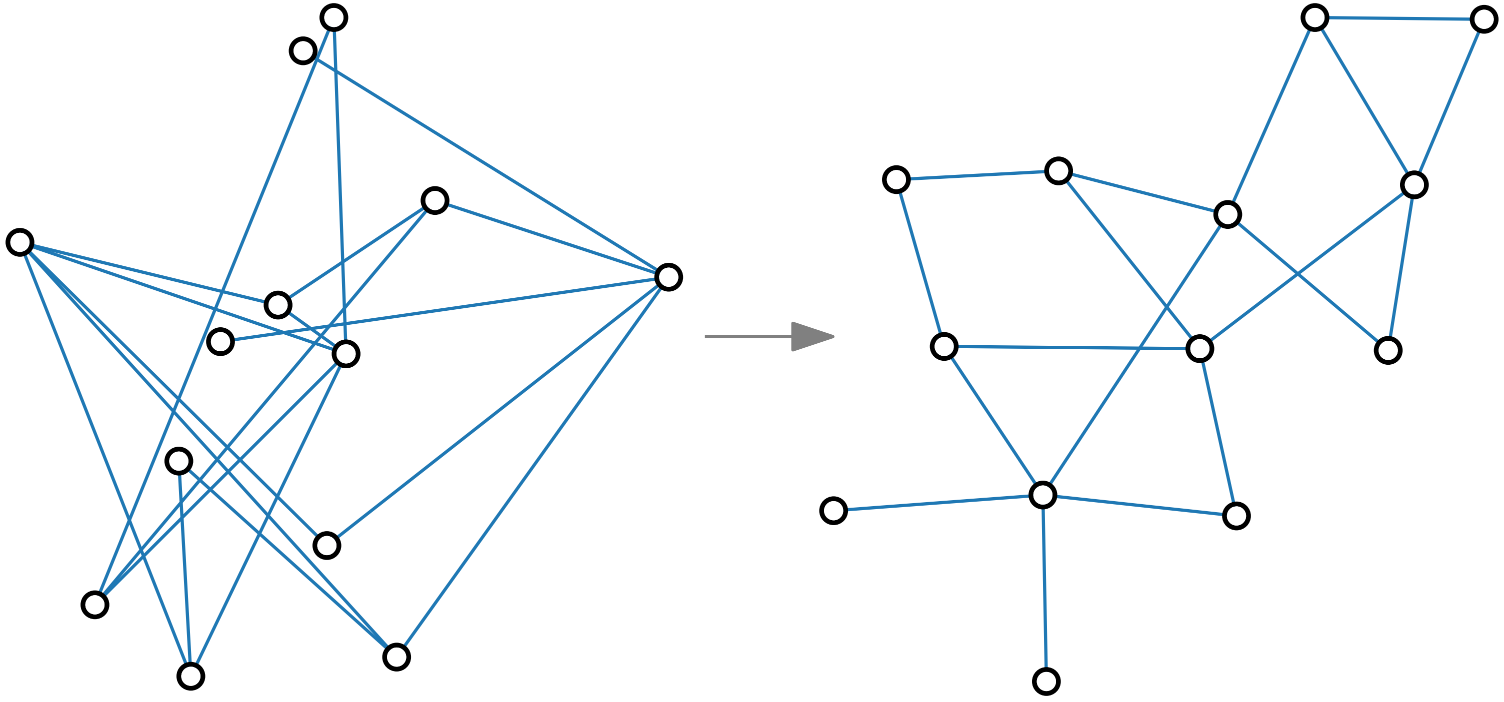

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all vertices x and y:

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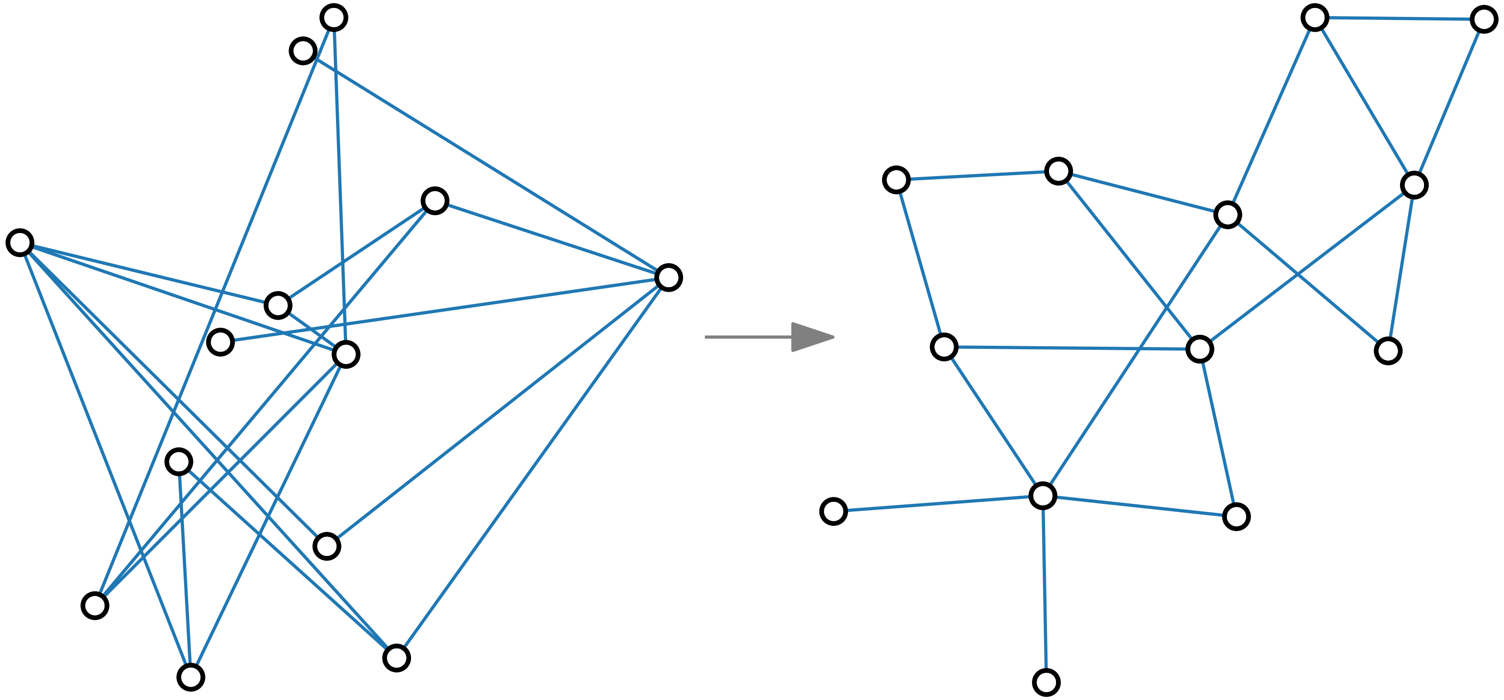

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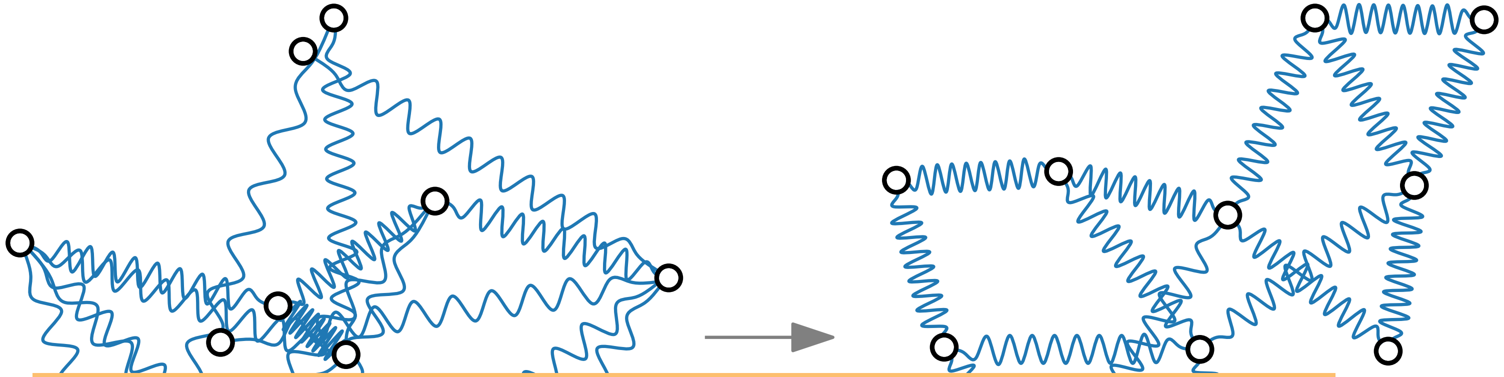

So-called **spring embedders** or **force-directed** algorithms that work according to this or similar principles are among the most frequently used graph-drawing methods in practice.

Attractive forces.

adjacent vertices u and v:

Repulsive forces.

all vertices x and y:

Force-Directed Algorithms

```
ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

    return $p$
```

Force-Directed Algorithms

initial layout

```
ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)




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Force-Directed Algorithms

initial layout

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```

end layout

Force-Directed Algorithms

initial layout

threshold

```
$\text{ForceDirected}(G = (V, E), p = (p_v)_{v \in V}, \varepsilon > 0, K \in \mathbb{N})$




    return $p$
```

end layout

Force-Directed Algorithms

initial layout

threshold

max # iterations

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Force-Directed Algorithms

initial layout

threshold

max # iterations

```
ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
    $t \leftarrow 1$
    while $t < K$ and $\max_{v \in V} \|F_v(t)\| > \varepsilon$ do
        $t \leftarrow t + 1$
    return $p$
```

end layout

Force-Directed Algorithms

initial layout

threshold

max # iterations

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ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
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Force-Directed Algorithms

initial layout

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max # iterations

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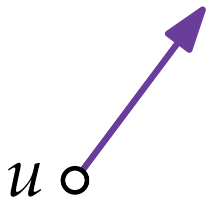

Force-Directed Algorithms

initial layout

threshold

max # iterations

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      foreach $u \in V$ do
          $F_u(t) \leftarrow \sum_{v \in V} f_{\mathrm{rep}}(u, v) +$
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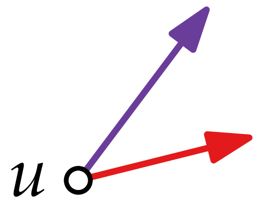
```

end layout



Force-Directed Algorithms

initial layout

threshold

max # iterations

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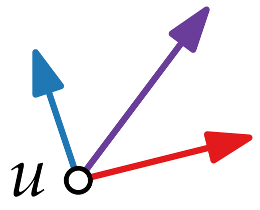
end layout



Force-Directed Algorithms

initial layout

threshold

max # iterations

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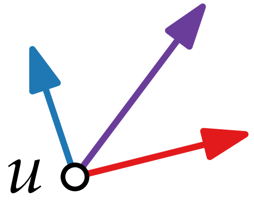
end layout



Force-Directed Algorithms

initial layout

threshold

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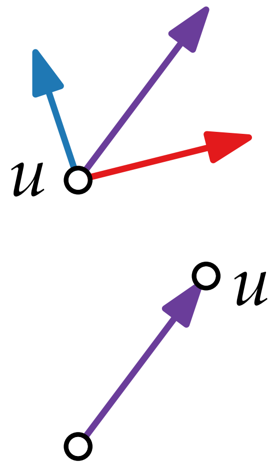
end layout



Force-Directed Algorithms

initial layout

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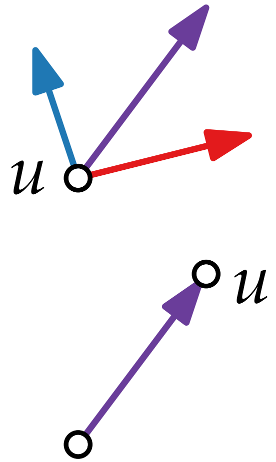
cooling factor

end layout



Force-Directed Algorithms

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          $F_u(t) \leftarrow \sum_{v \in V} f_{\text{rep}}(u, v) + \sum_{uv \in E} f_{\text{attr}}(u, v)$
      foreach $u \in V$ do
          $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$
      $t \leftarrow t + 1$
  return $p$
```
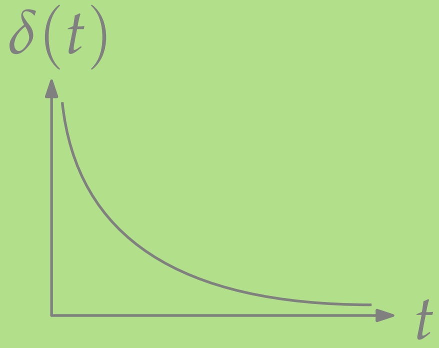
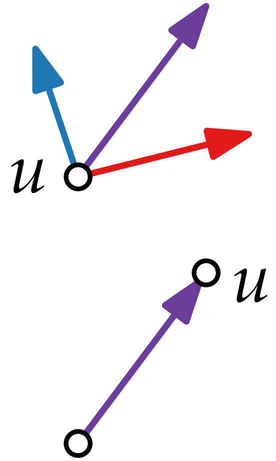

cooling factor



end layout



Force-Directed Algorithms

initial layout

threshold

max # iterations

```
ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
  $t \leftarrow 1$
  while $t < K$ and $\max_{v \in V} \|F_v(t)\| > \varepsilon$ do
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```
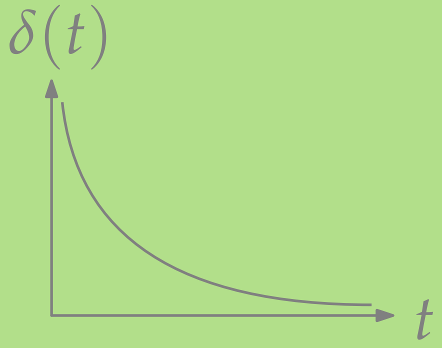
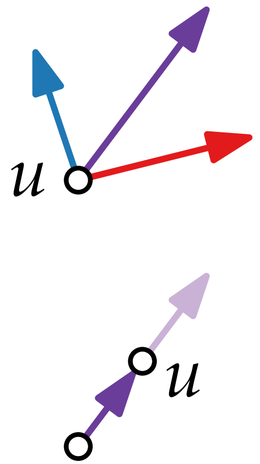

cooling factor

end layout





Visualization of Graphs

Lecture 3:
Force-Directed Drawing Algorithms

Part II:
Spring Embedder by Eades

Philipp Kindermann

Spring Embedder by Eades – Model

```
$\text{ForceDirected}(G = (V, E), p = (p_v)_{v \in V}, \varepsilon > 0, K \in \mathbb{N})$
  $t \leftarrow 1$
  while $t < K$ and $\max_{v \in V} \|F_v(t)\| > \varepsilon$ do
      foreach $u \in V$ do
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Spring Embedder by Eades – Model

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      $t \leftarrow t + 1$
  return $p$
```

- **Repulsive forces**

- **Attractive forces**

- **Resulting displacement vector**

Spring Embedder by Eades – Model

```
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    $t \leftarrow 1$
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- **Repulsive forces**

- **Attractive forces**

- **Resulting displacement vector**

$$F_u = \sum_{v \in V} f_{\mathrm{rep}}(u, v) + \sum_{uv \in E} f_{\mathrm{attr}}(u, v)$$

Spring Embedder by Eades – Model

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  return $p$
```

- **Repulsive forces**

$$f_{\text{rep}}(u,v) = \frac{c_{\text{rep}}}{||p_v - p_u||^2} \cdot \overrightarrow{p_v p_u}$$

- **Attractive forces**

- **Resulting displacement vector**

$$F_u = \sum_{v \in V} f_{\text{rep}}(u,v) + \sum_{uv \in E} f_{\text{attr}}(u,v)$$

Spring Embedder by Eades – Model

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$$F_u = \sum_{v \in V} f_{\mathrm{rep}}(u, v) + \sum_{uv \in E} f_{\mathrm{attr}}(u, v)$$

Notation.

- $||p_u - p_v||$ = Euclidean distance between u and v

Spring Embedder by Eades – Model

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Notation.

- $||p_u - p_v||$ = Euclidean distance between u and v
- $\overrightarrow{p_u p_v}$ = unit vector pointing from u to v

Spring Embedder by Eades – Model

- **Repulsive forces**

repulsion constant (e.g. 2.0)

$$f_{\text{rep}}(u,v) = \frac{c_{\text{rep}}}{||p_v - p_u||^2} \cdot \overrightarrow{p_v p_u}$$

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Notation.

- $||p_u - p_v|| =$ Euclidean distance between u and v
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Spring Embedder by Eades – Model

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repulsion constant (e.g. 2.0)

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- **Attractive forces**

$$f_{\text{spring}}(u,v) = c_{\text{spring}} \cdot \log \frac{||p_v - p_u||}{\ell} \cdot \overrightarrow{p_u p_v}$$

- **Resulting displacement vector**

$$F_u = \sum_{v \in V} f_{\text{rep}}(u,v) + \sum_{uv \in E} f_{\text{attr}}(u,v)$$

Notation.

- $||p_u - p_v||$ = Euclidean distance between u and v
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Spring Embedder by Eades – Model

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Spring Embedder by Eades – Model

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repulsion constant (e.g. 2.0)

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spring constant (e.g. 1.0)

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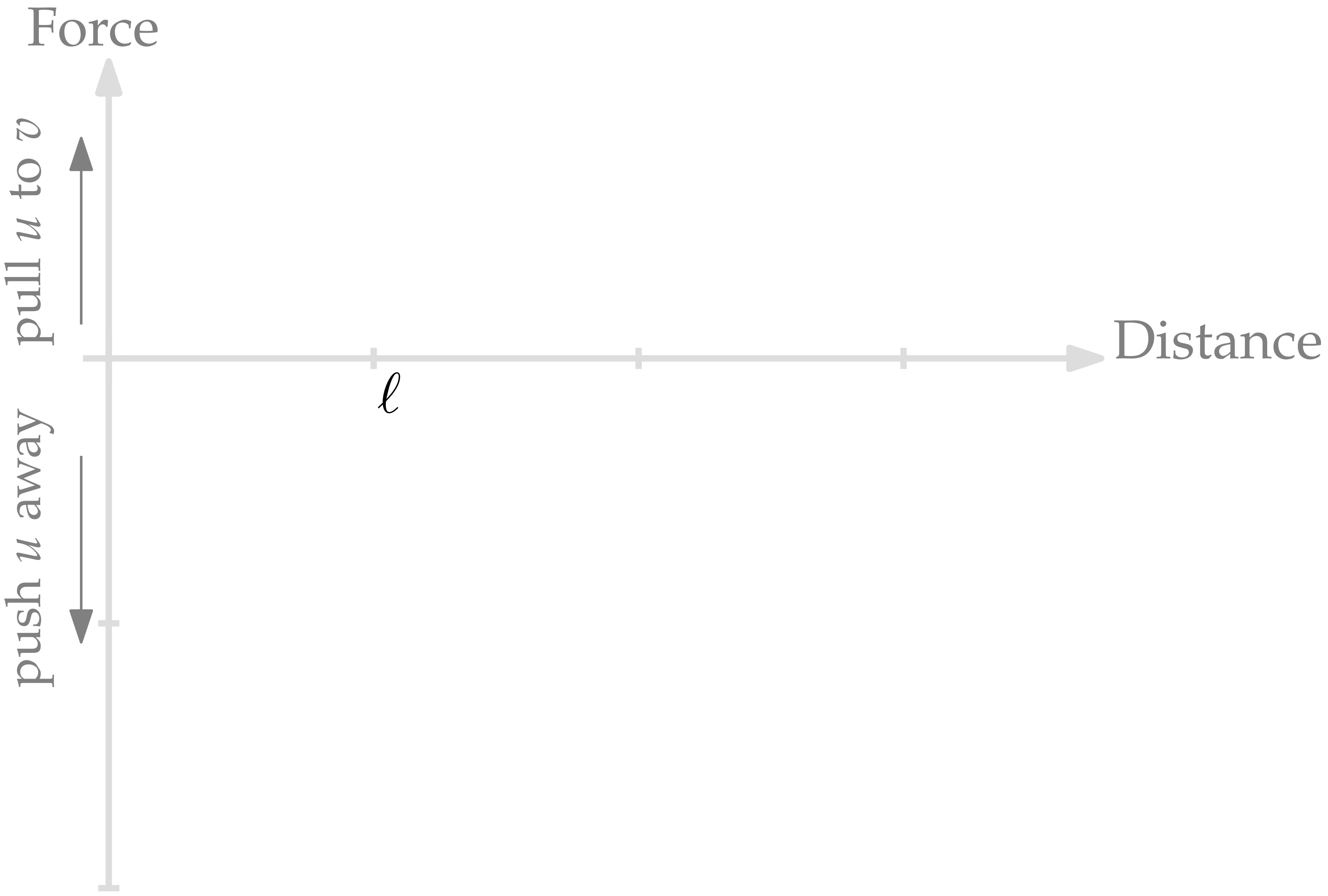
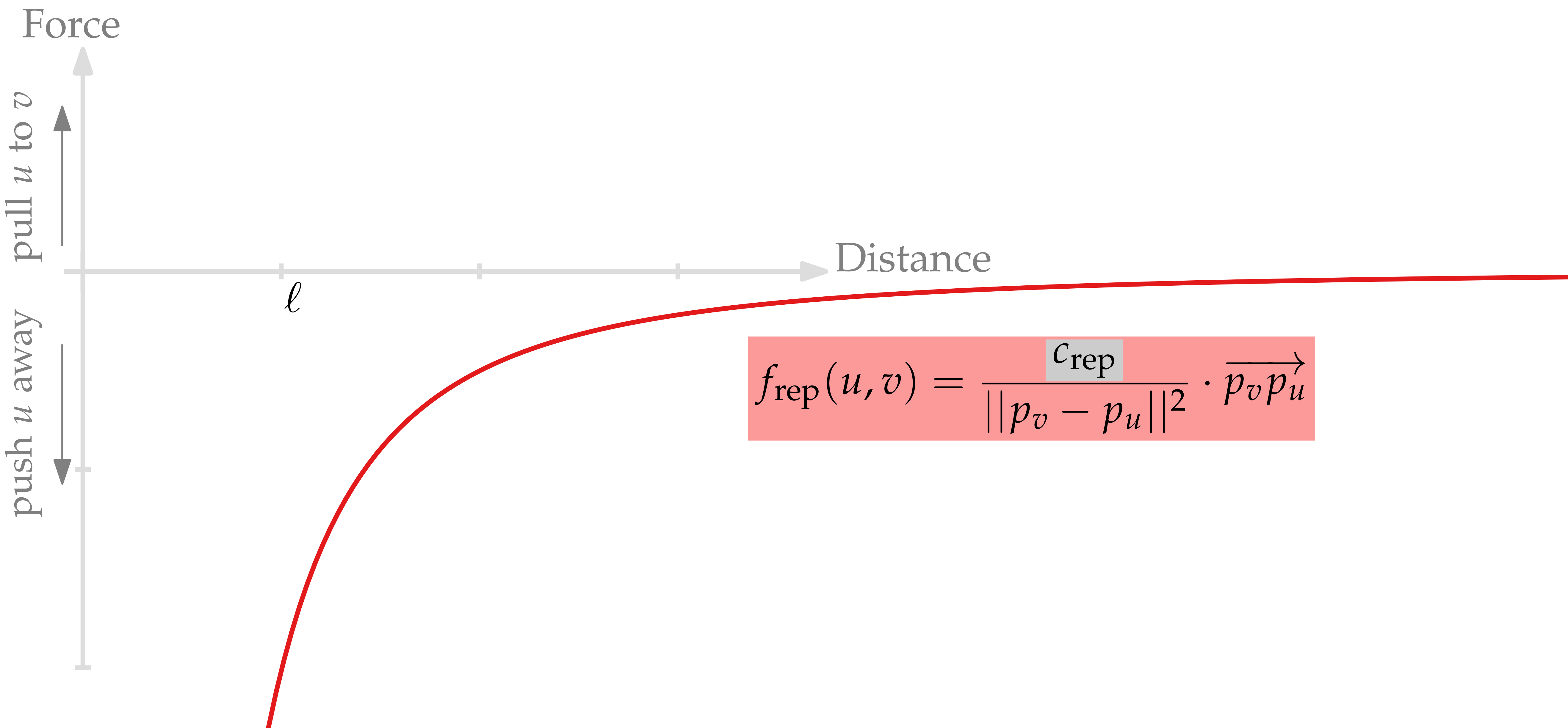
Spring Embedder by Eades – Force Diagram



Spring Embedder by Eades – Force Diagram



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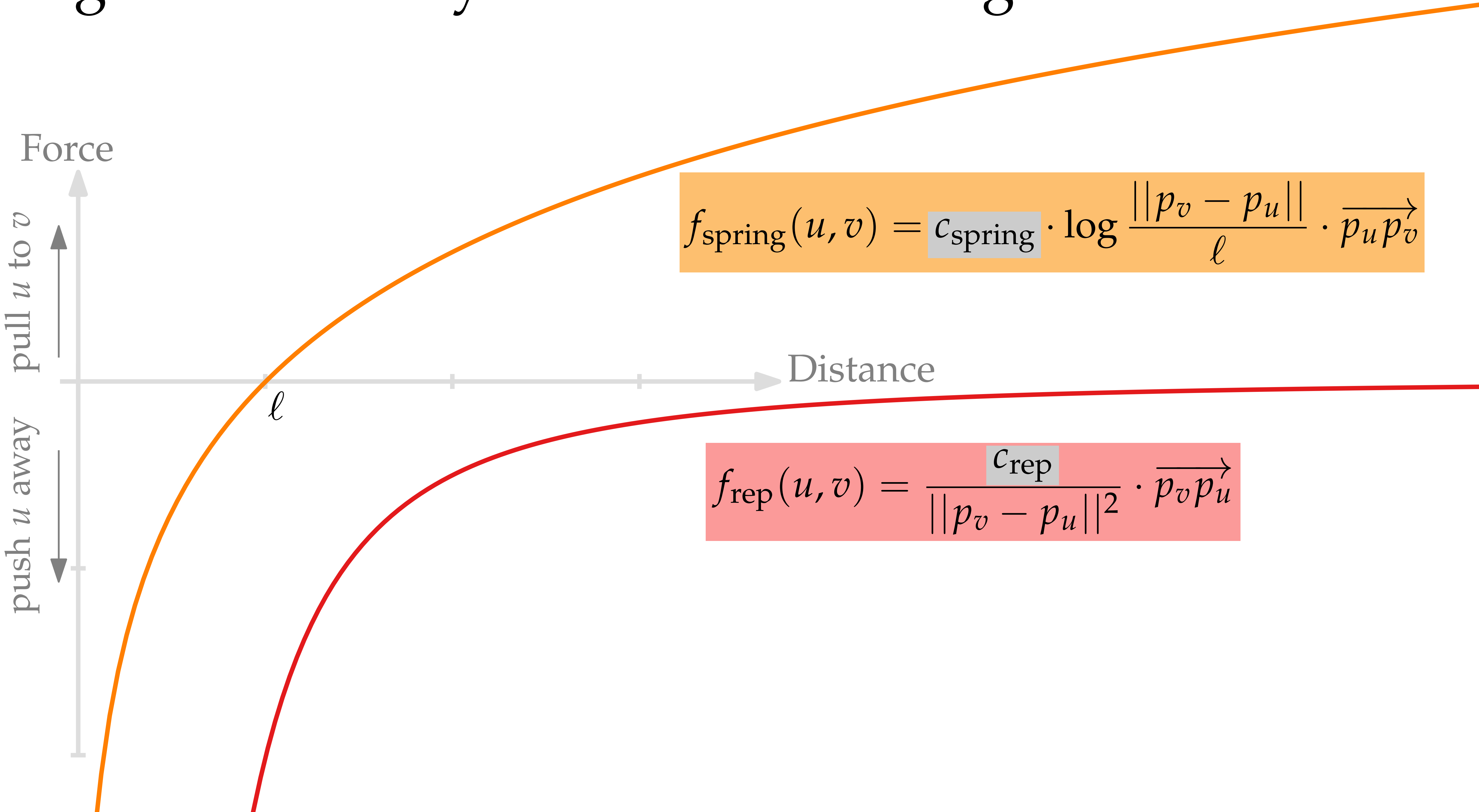

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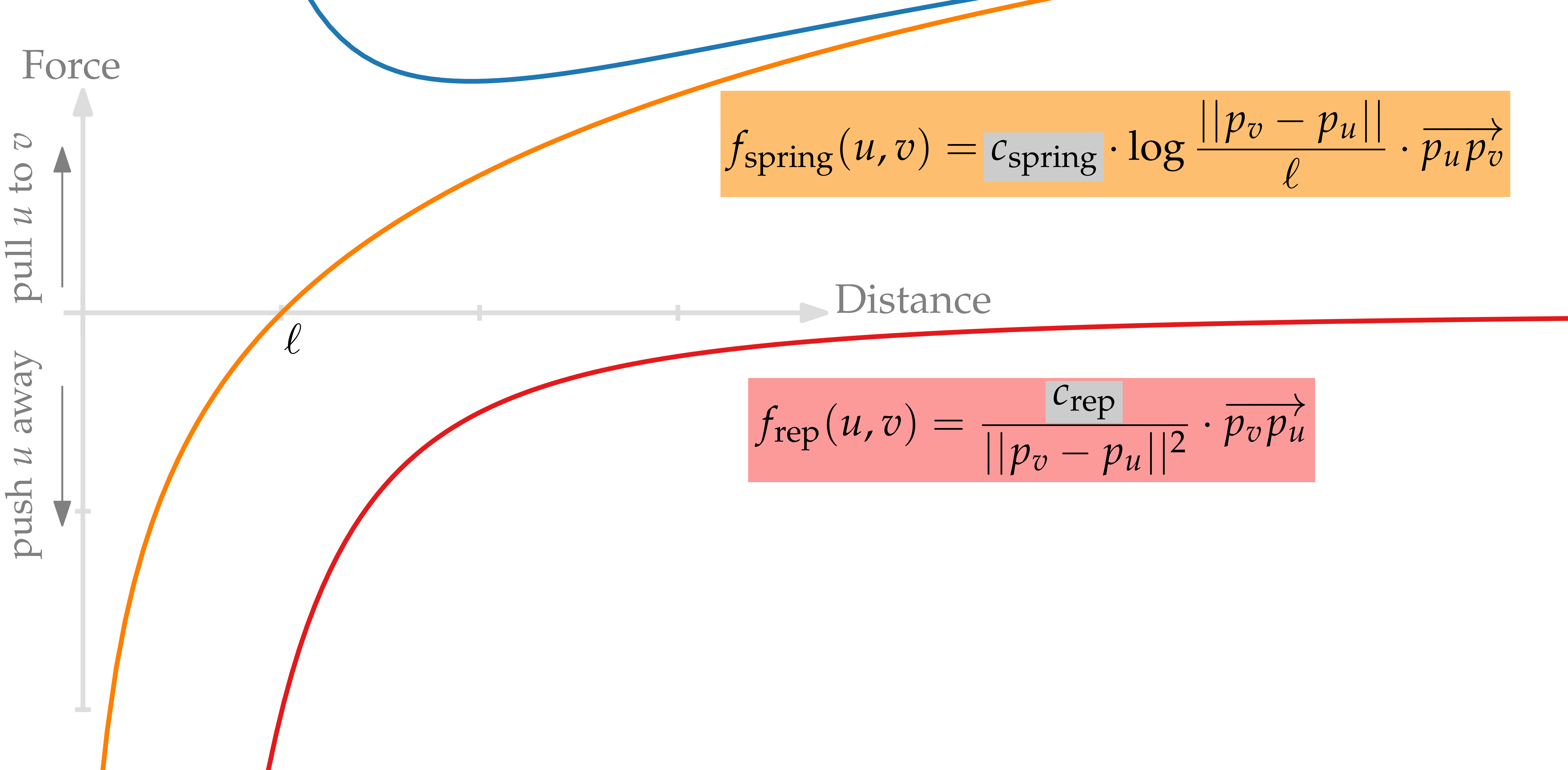
Spring Embedder by Eades – Force Diagram

$$f_{\text{attr}}(u,v) = f_{\text{spring}}(u,v) - f_{\text{rep}}(u,v)$$



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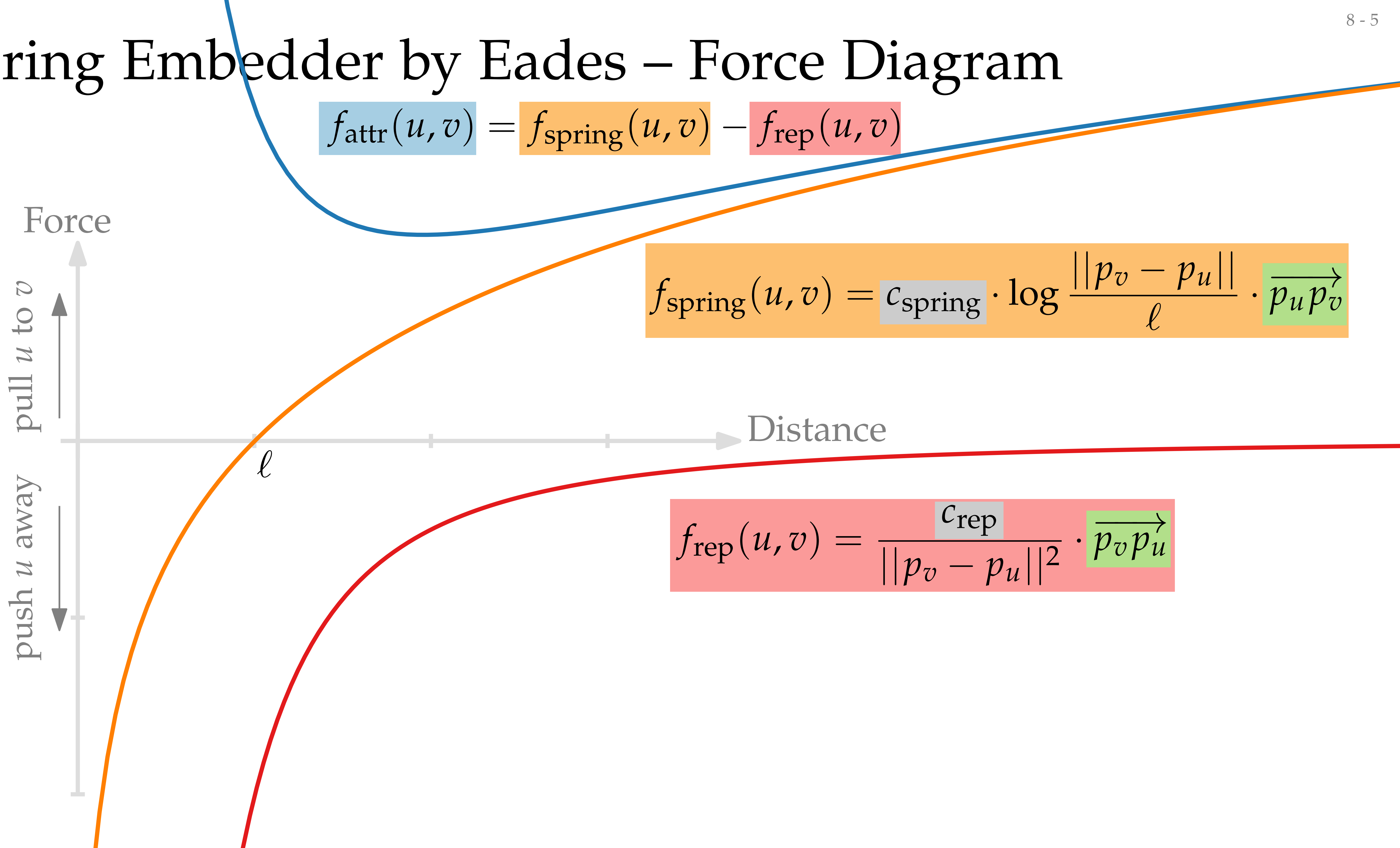

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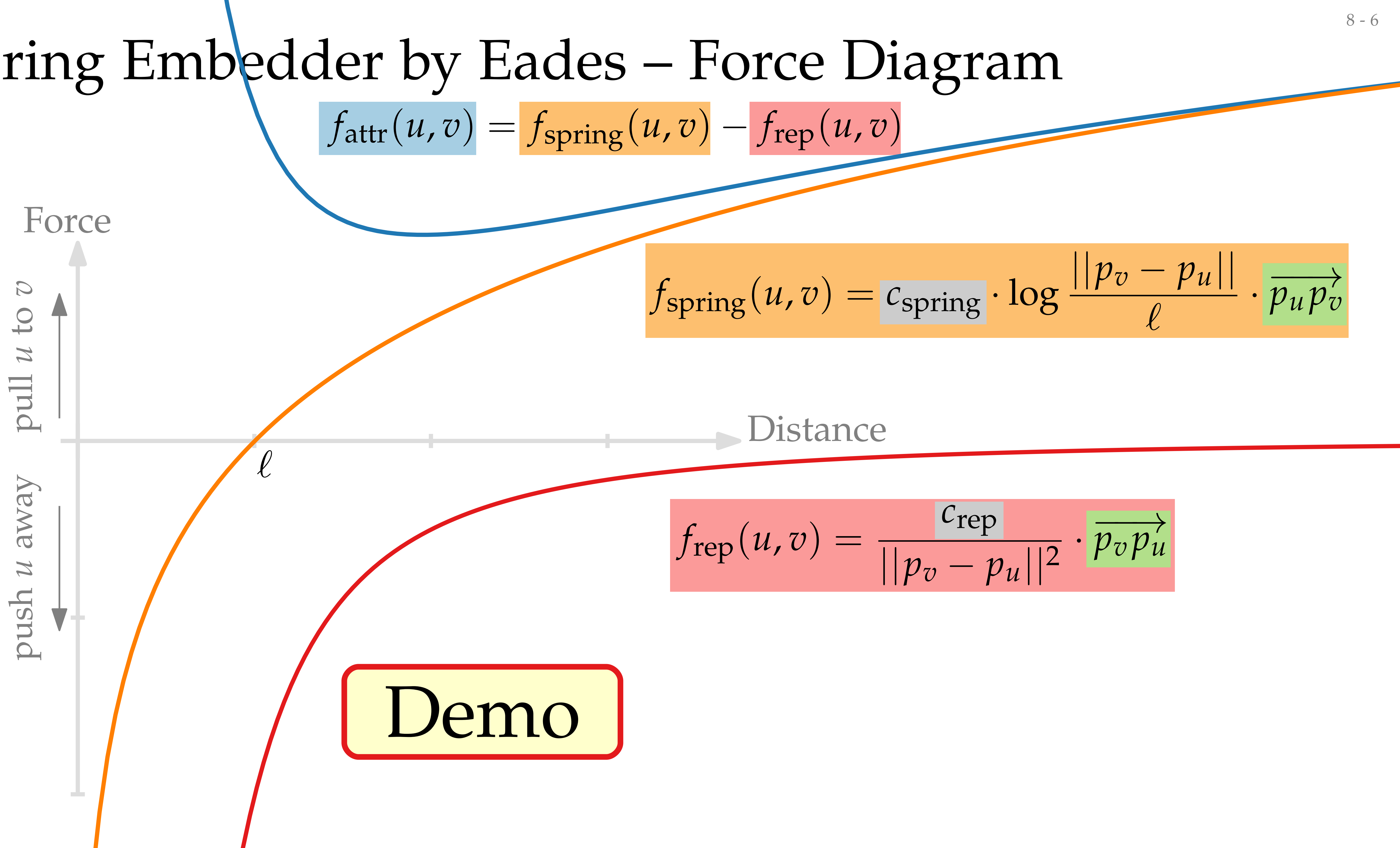

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Demo

Spring Embedder by Eades – Discussion

Advantages.

Spring Embedder by Eades – Discussion

Advantages.

- very simple algorithm

Spring Embedder by Eades – Discussion

Advantages.

- very simple algorithm
- good results for small and medium-sized graphs

Spring Embedder by Eades – Discussion

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- system is not stable at the end
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- very simple algorithm
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- system is not stable at the end
- converging to local minima
- timewise f_{spring} in $\mathcal{O}(|E|)$ and f_{rep} in $\mathcal{O}(|V|^2)$

Spring Embedder by Eades – Discussion

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- system is not stable at the end
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Influence.

Spring Embedder by Eades – Discussion

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- system is not stable at the end
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Influence.

- original paper by Peter Eades [Eades '84] got $\sim$ 2000 citations

Spring Embedder by Eades – Discussion

Advantages.

- very simple algorithm
- good results for small and medium-sized graphs
- empirically good representation of symmetry and structure

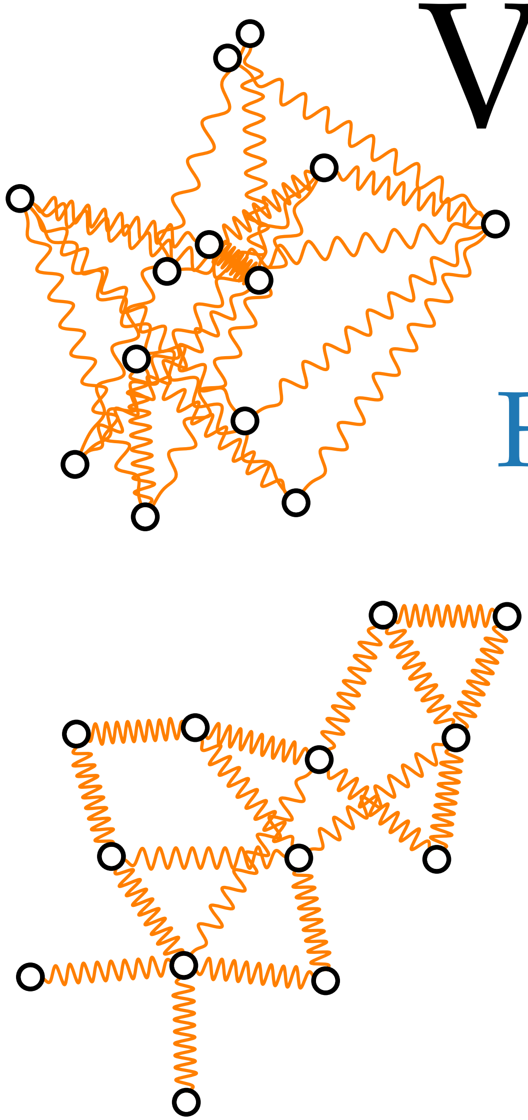
Disadvantages.

- system is not stable at the end
- converging to local minima
- timewise f_{spring} in $\mathcal{O}(|E|)$ and f_{rep} in $\mathcal{O}(|V|^2)$

Influence.

- original paper by Peter Eades [Eades '84] got $\sim$ 2000 citations
- basis for many further ideas

Universität Trier



Visualization of Graphs

Lecture 3:
Force-Directed Drawing Algorithms

Part III:
Variant by Fruchterman & Reingold

Philipp Kindermann

Variant by Fruchterman & Reingold

```
ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
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```

- **Repulsive forces**

repulsion constant (e.g. 2.0)

$$f_{\text{rep}}(u, v) = \frac{c_{\text{rep}}}{||p_v - p_u||^2} \cdot \overrightarrow{p_v p_u}$$

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spring constant (e.g. 1.0)

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- **Resulting displacement vector**

$$F_u = \sum_{v \in V} f_{\text{rep}}(u, v) + \sum_{uv \in E} f_{\text{attr}}(u, v)$$

Notation.

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```

- **Repulsive forces**

$$f_{\text{rep}}(u, v) = \frac{\ell^2}{||p_v - p_u||} \cdot \overrightarrow{p_v p_u}$$

- **Attractive forces**

spring constant (e.g. 1.0)

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    while $t < K$ and $\max_{v \in V} \|F_v(t)\| > \varepsilon$ do
        foreach $u \in V$ do
            $F_u(t) \leftarrow \sum_{v \in V} f_{\mathrm{rep}}(u,v) + \sum_{uv \in E} f_{\mathrm{attr}}(u,v)$
        foreach $u \in V$ do
            $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$
        $t \leftarrow t + 1$
    return $p$
```

- **Repulsive forces**

$$f_{\mathrm{rep}}(u,v) = \frac{\ell^2}{||p_v - p_u||} \cdot \overrightarrow{p_v p_u}$$

- **Attractive forces**

$$f_{\mathrm{attr}}(u,v) = \frac{||p_v - p_u||^2}{\ell} \cdot \overrightarrow{p_u p_v}$$

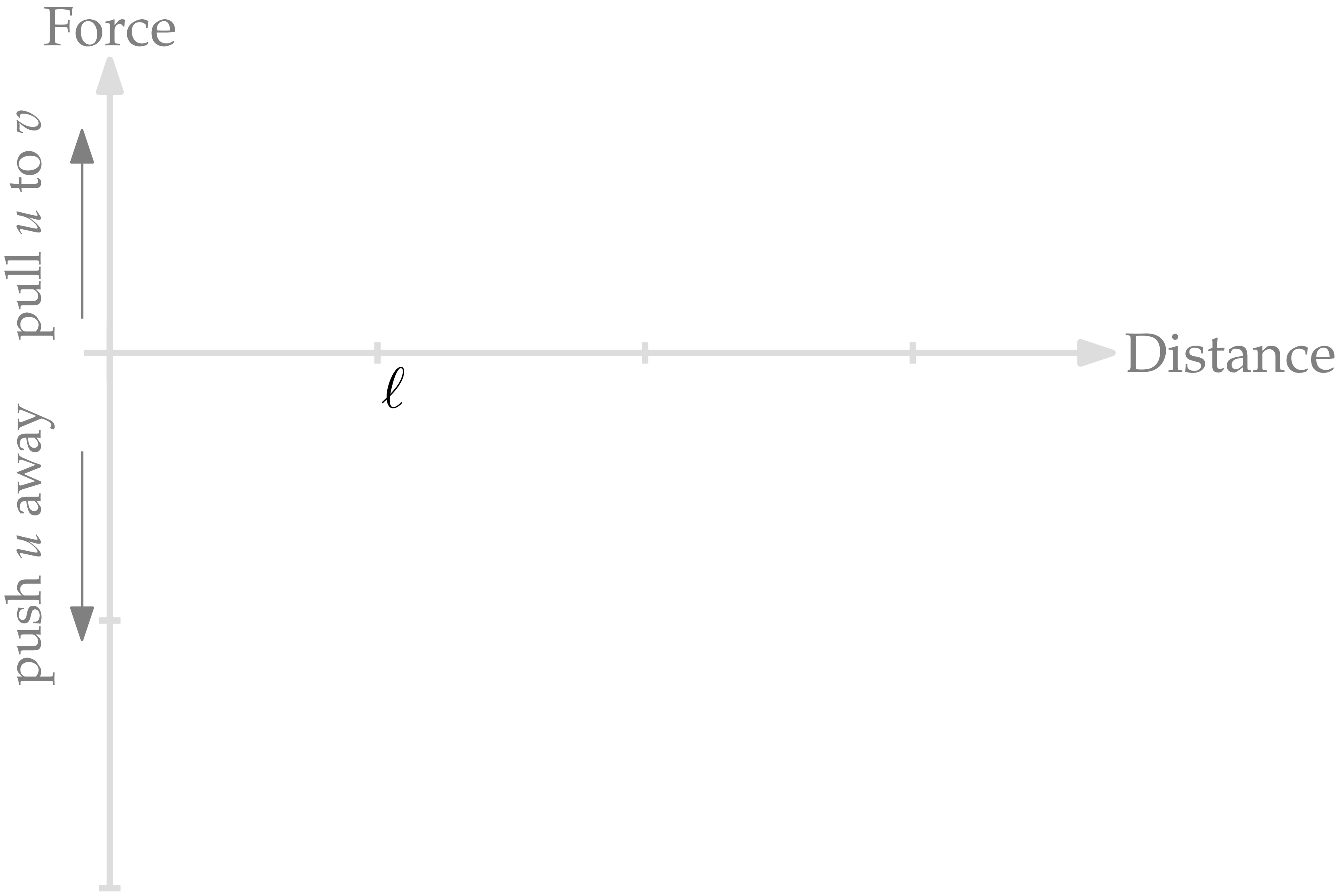
- **Resulting displacement vector**

$$F_u = \sum_{v \in V} f_{\mathrm{rep}}(u,v) + \sum_{uv \in E} f_{\mathrm{attr}}(u,v)$$

Notation.

- $||p_u - p_v||$ = Euclidean distance between u and v
- $\overrightarrow{p_u p_v}$ = unit vector pointing from u to v
- ℓ = ideal spring length for edges

Fruchterman & Reingold – Force Diagram



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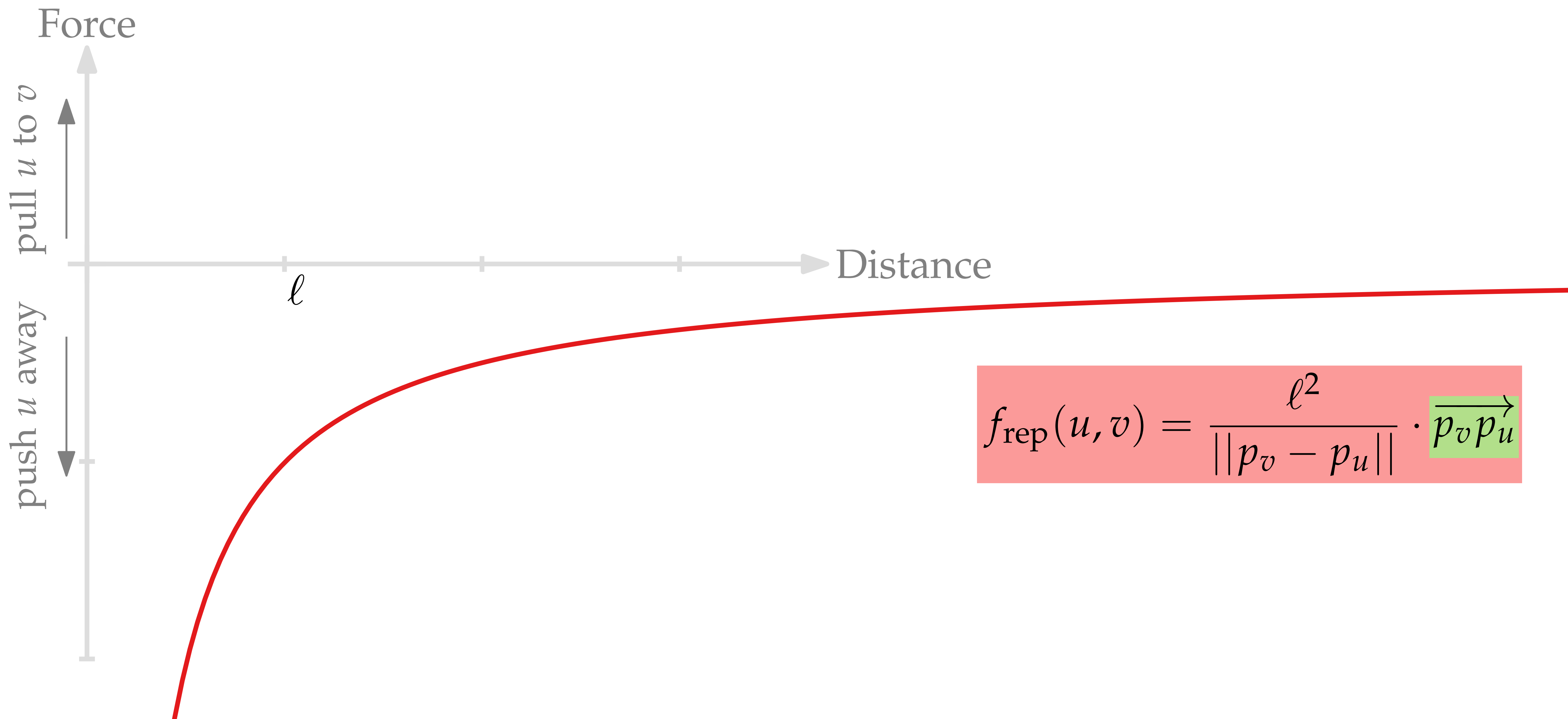
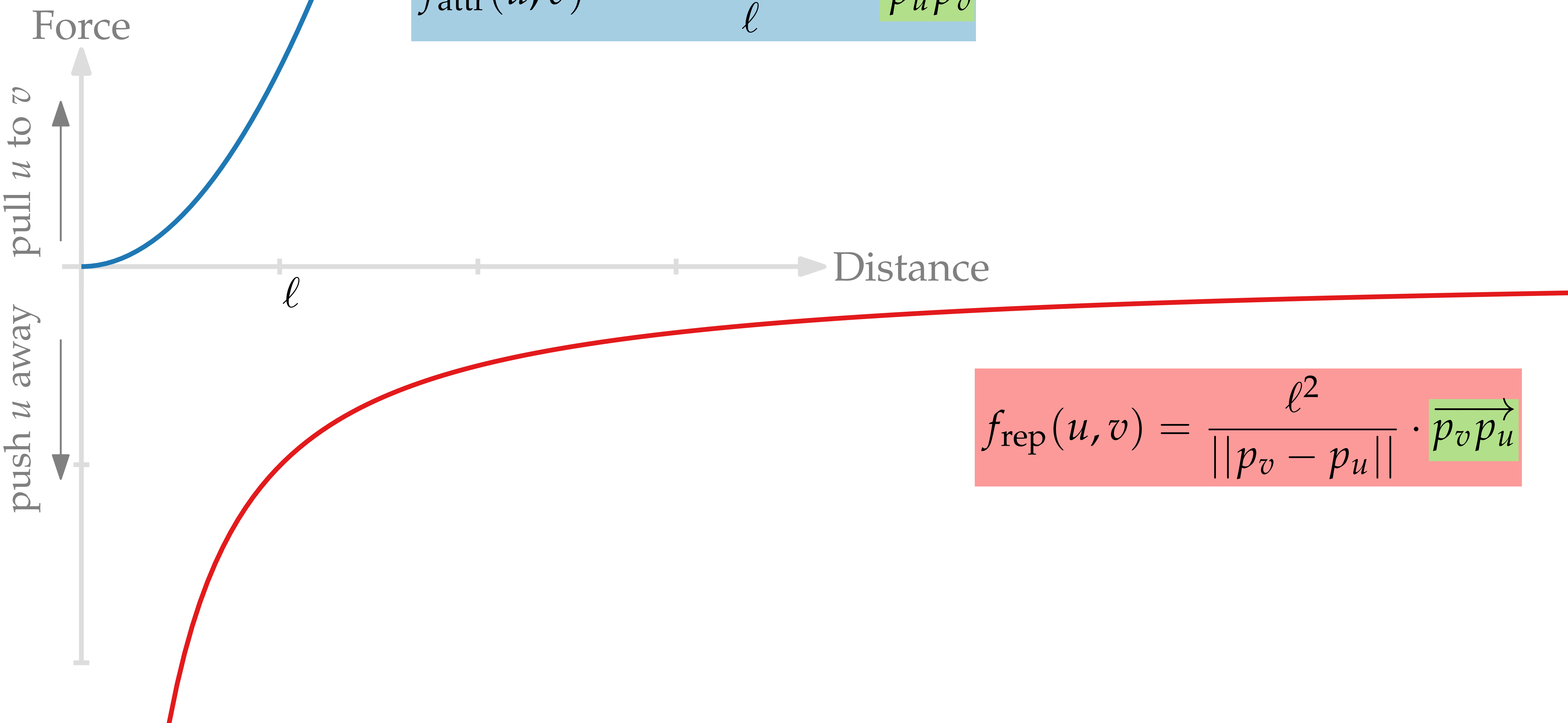


$$f_{\text{rep}}(u,v) = \frac{\ell^2}{||p_v - p_u||} \cdot \overrightarrow{p_v p_u}$$

Fruchterman & Reingold – Force Diagram

$$f_{\mathrm{attr}}(u,v) = \frac{||p_v - p_u||^2}{\ell} \cdot \overrightarrow{p_u p_v}$$



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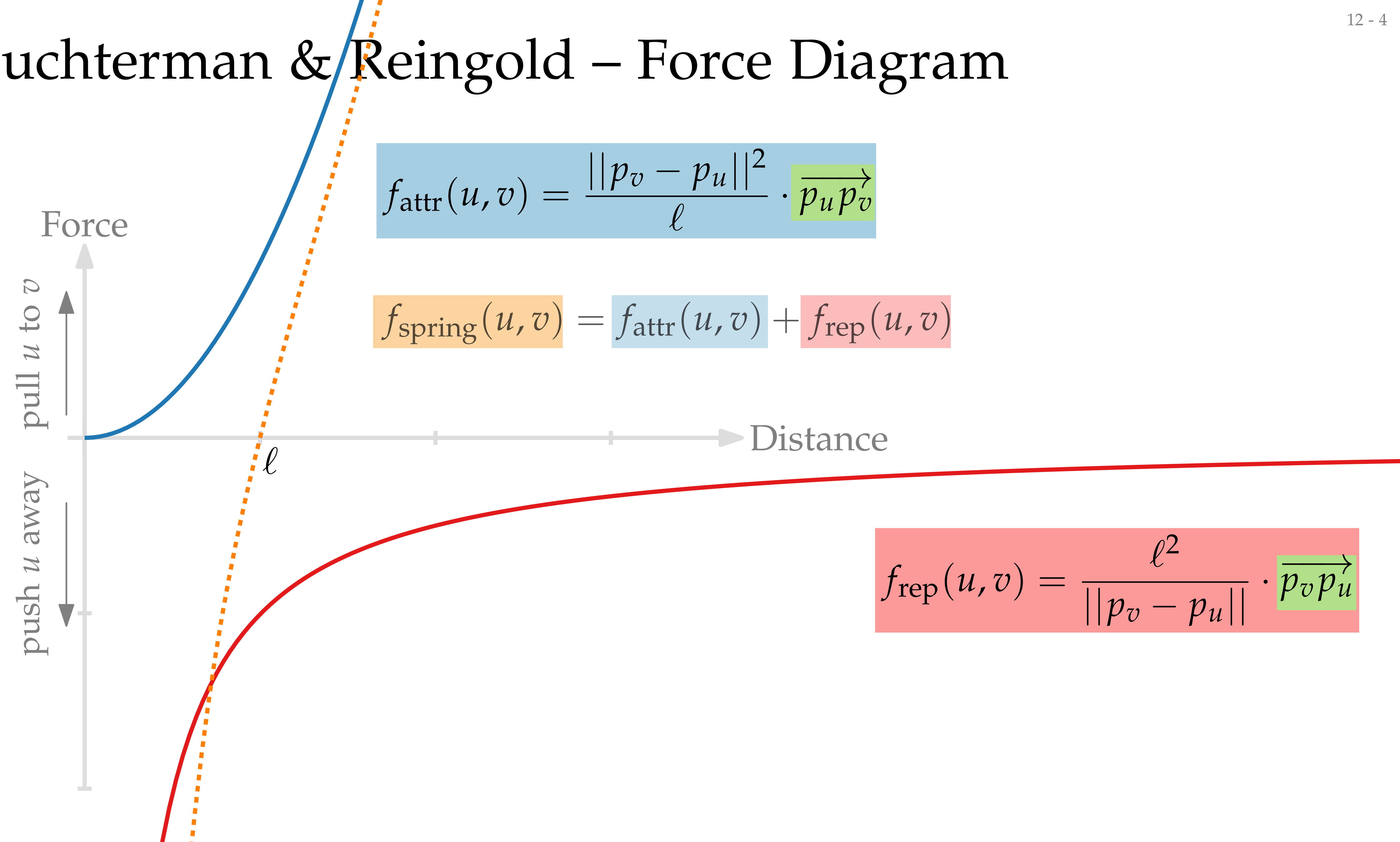

$$f_{\text{attr}}(u,v) = \frac{||p_v - p_u||^2}{\ell} \cdot \overrightarrow{p_u p_v}$$

$$f_{\text{spring}}(u,v) = f_{\text{attr}}(u,v) + f_{\text{rep}}(u,v)$$

$$f_{\text{rep}}(u,v) = \frac{\ell^2}{||p_v - p_u||} \cdot \overrightarrow{p_v p_u}$$

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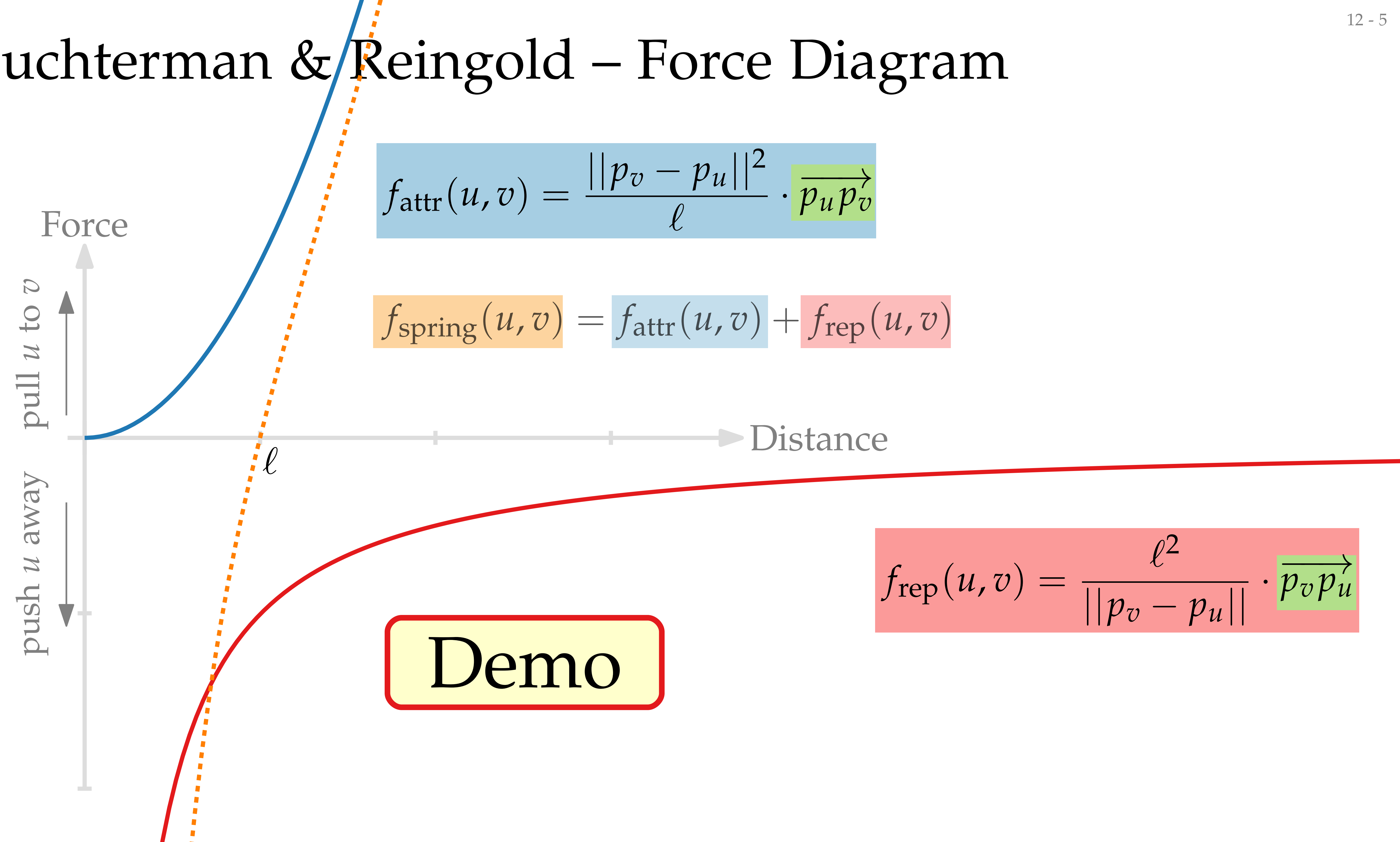
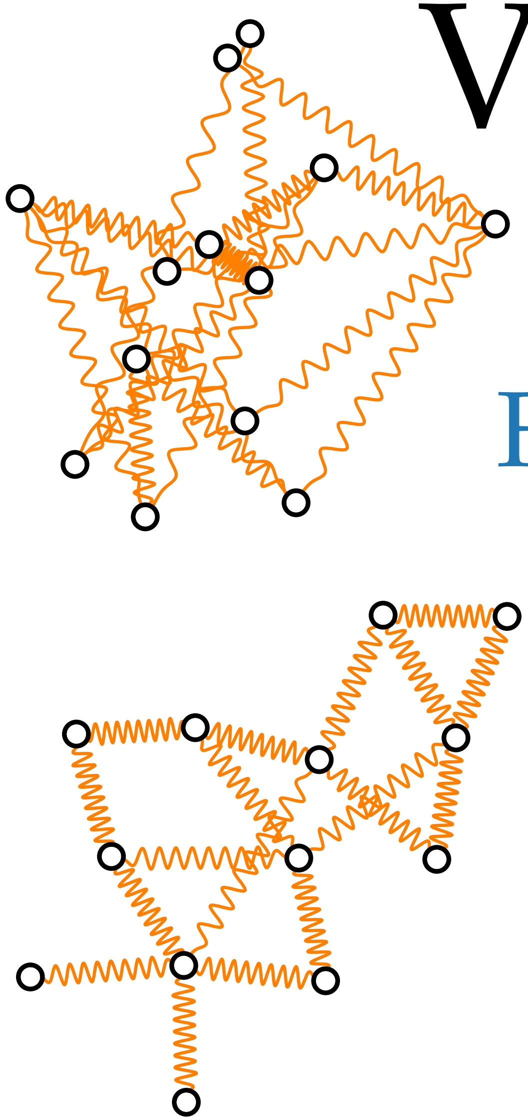


$$f_{\text{rep}}(u,v) = \frac{\ell^2}{||p_v - p_u||} \cdot \overrightarrow{p_v p_u}$$

Demo

Universität Trier



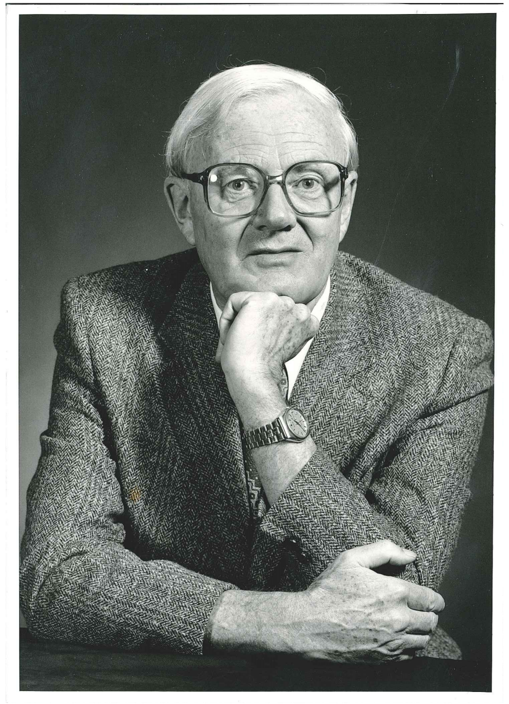
Visualization of Graphs

Lecture 3:
Force-Directed Drawing Algorithms

Part IV:
Tutte Drawing

Philipp Kindermann

Idea



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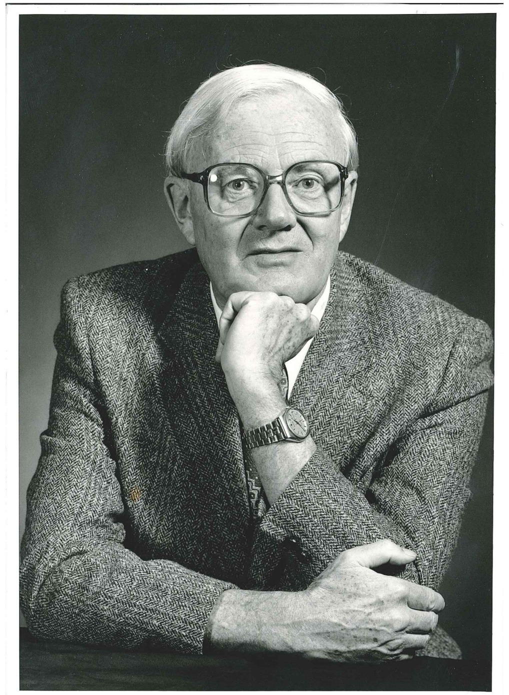
1917 – 2002

Idea



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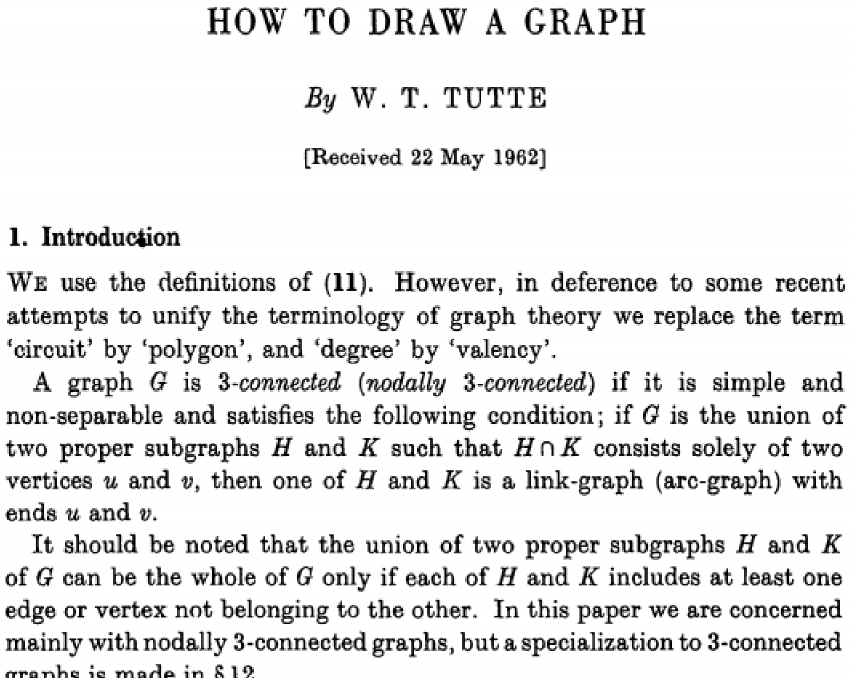

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By W. T. TUTTE

[Received 22 May 1962]

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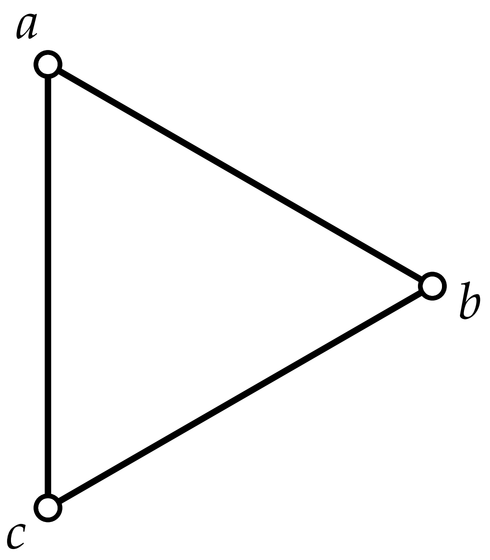

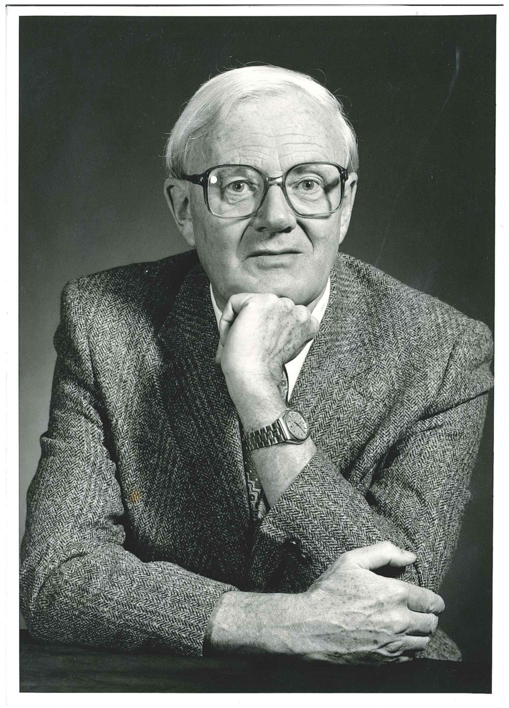

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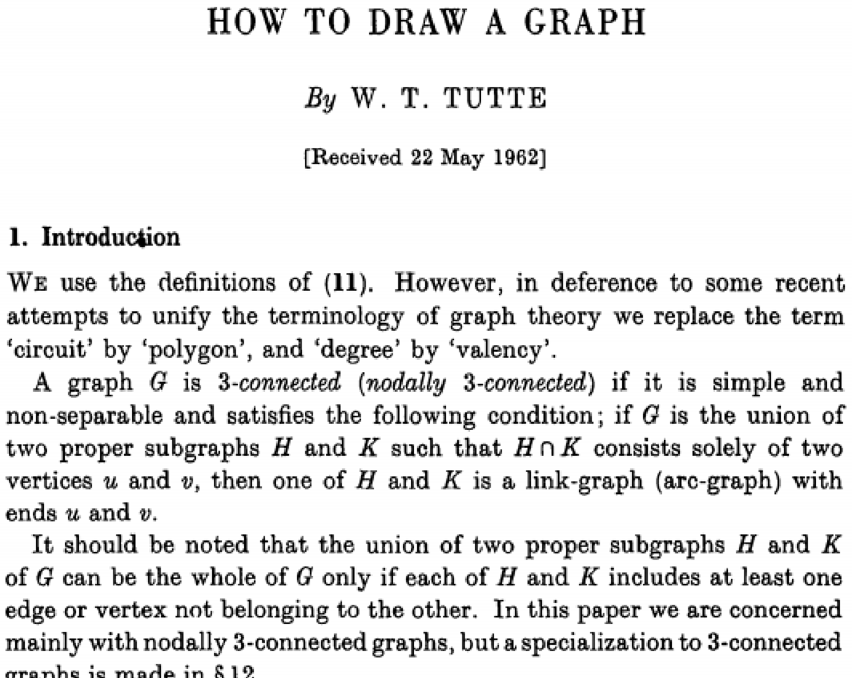

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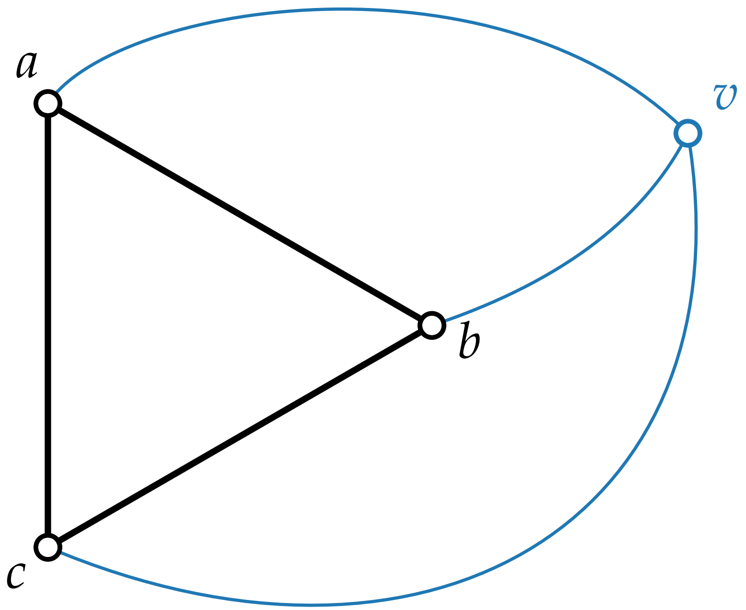

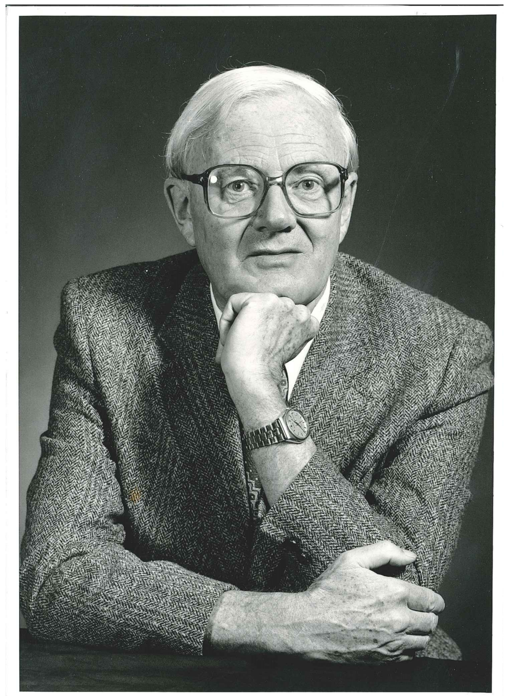

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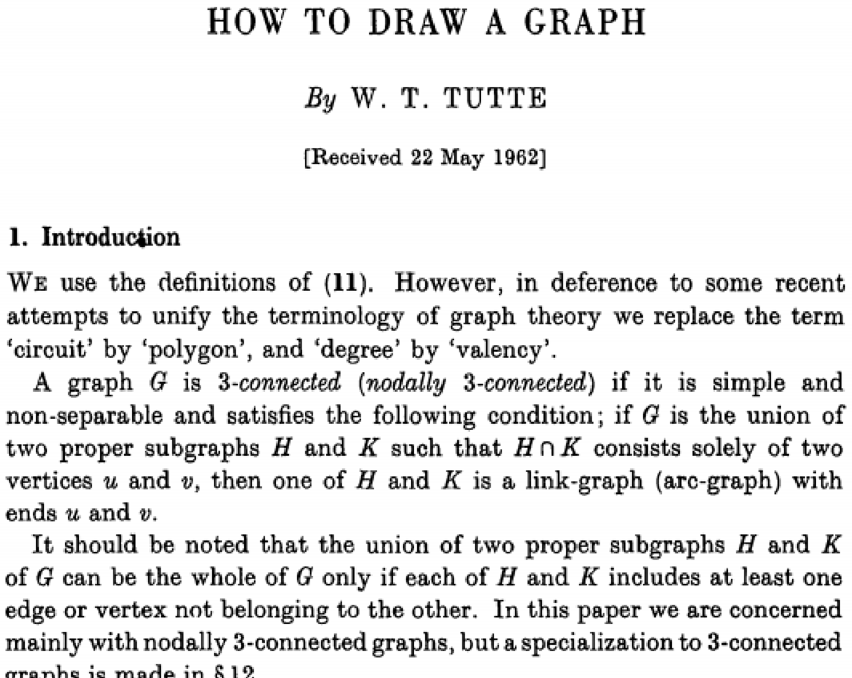

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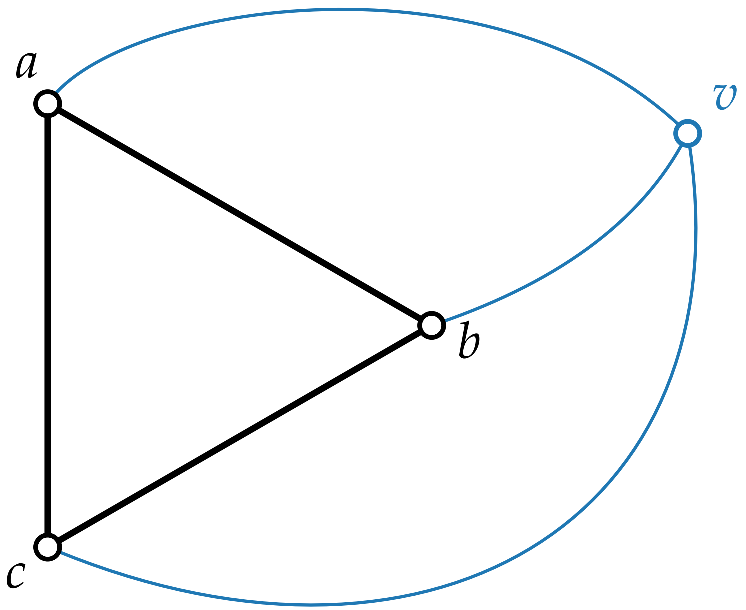

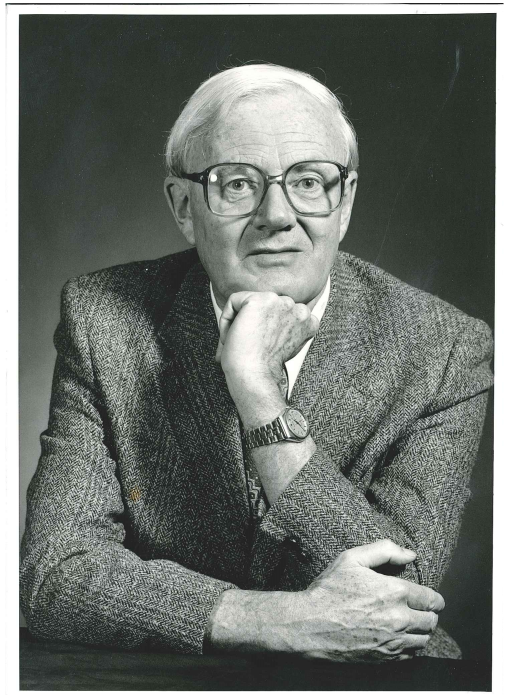

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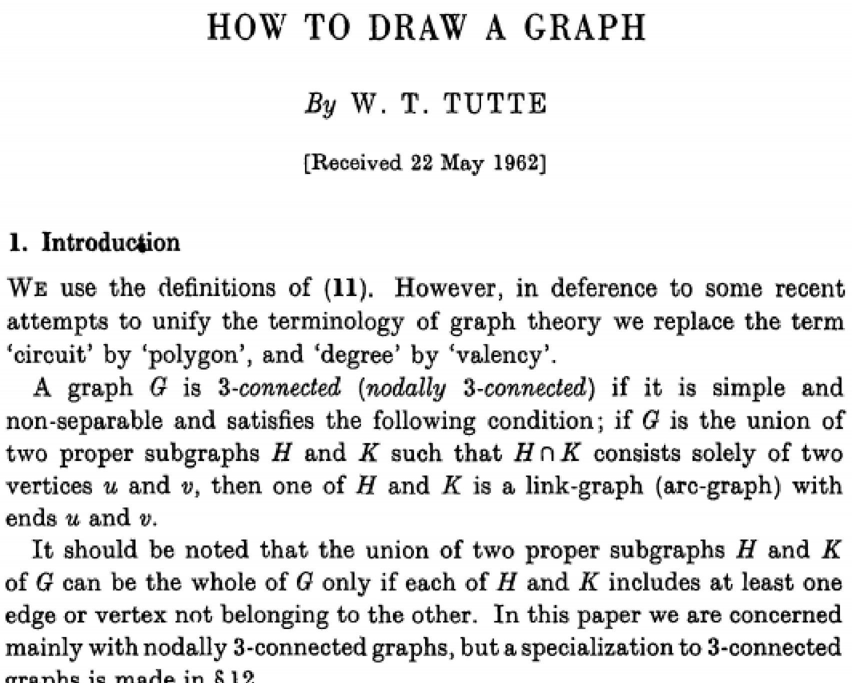

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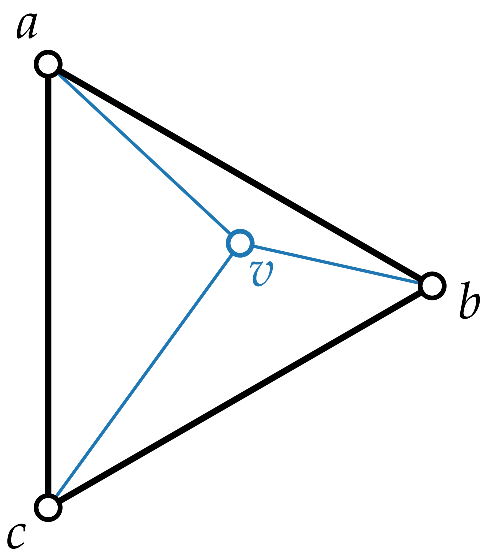

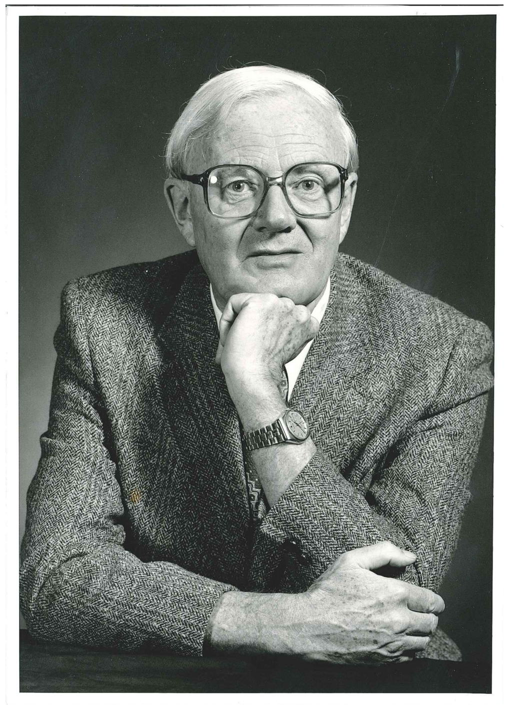

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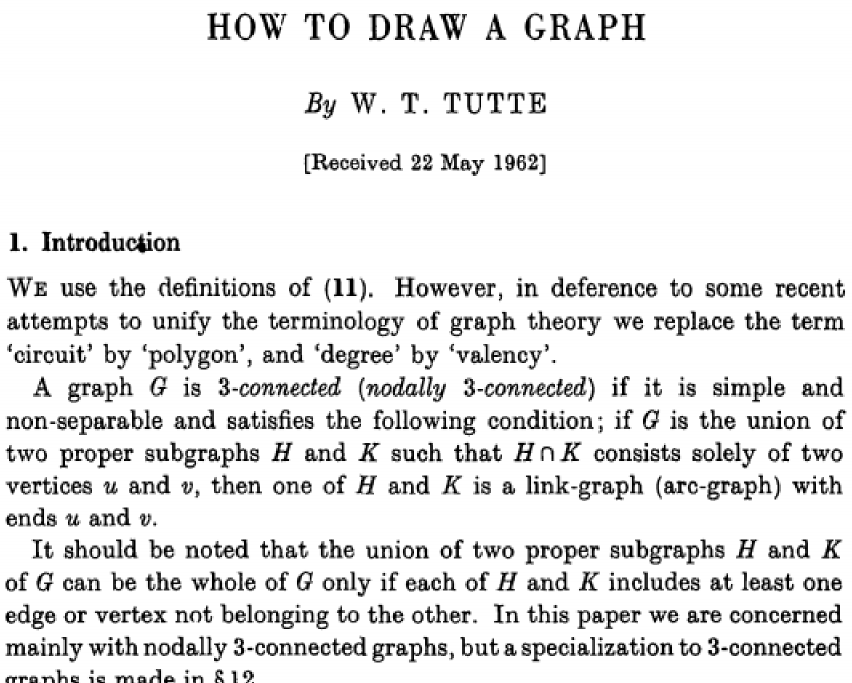

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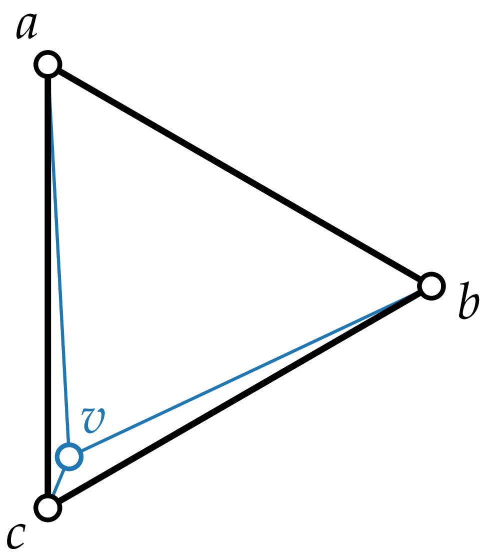

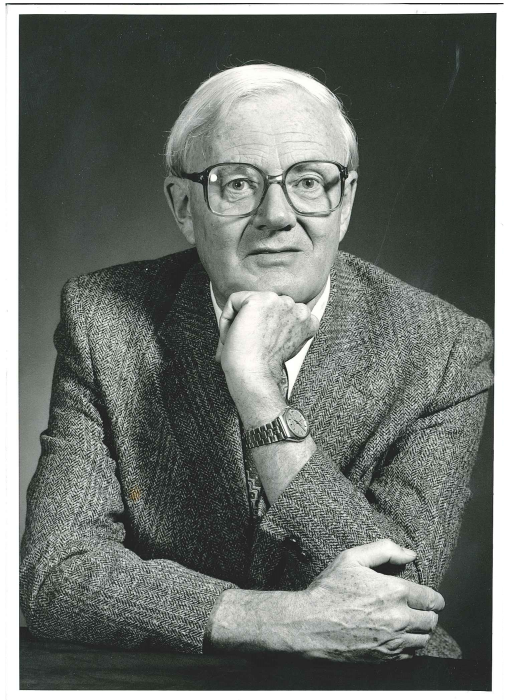

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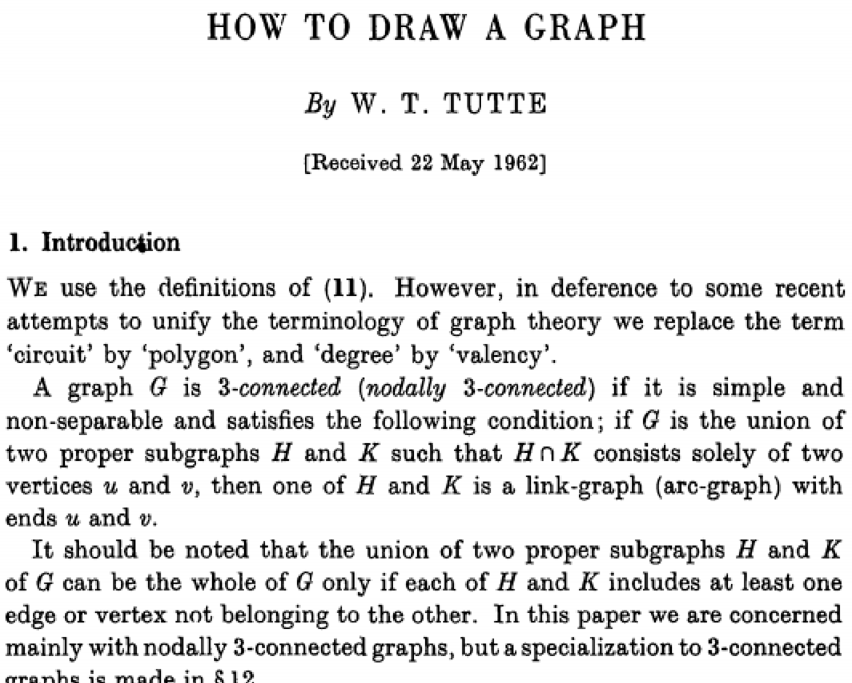

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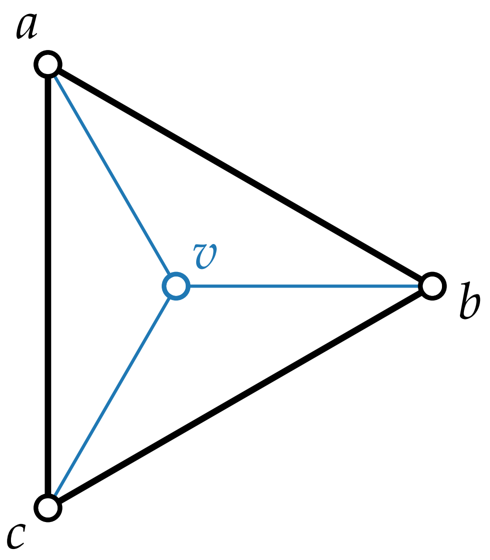

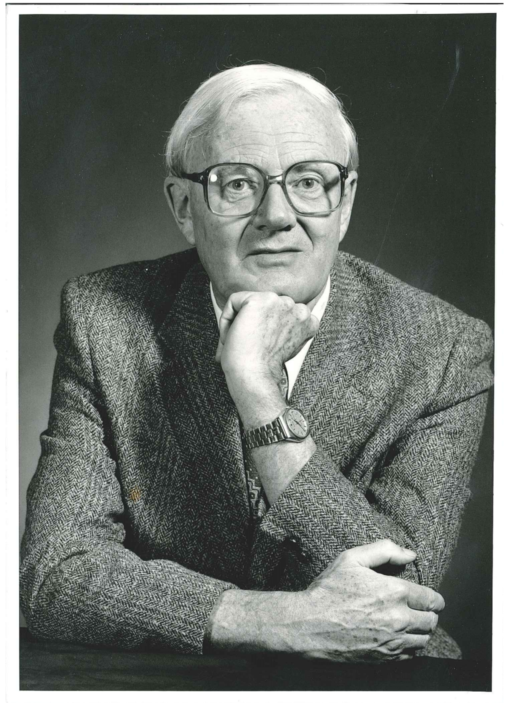

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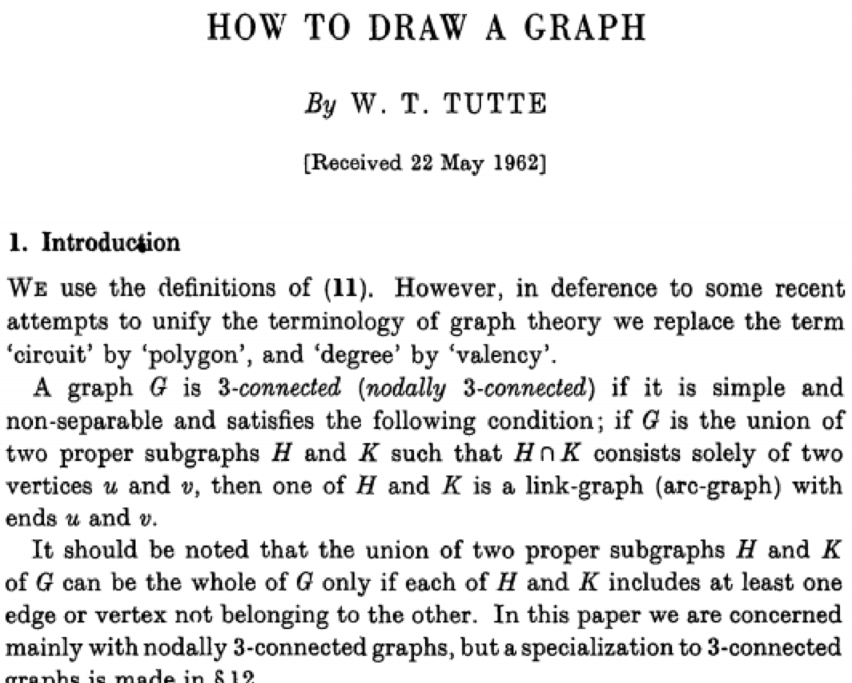

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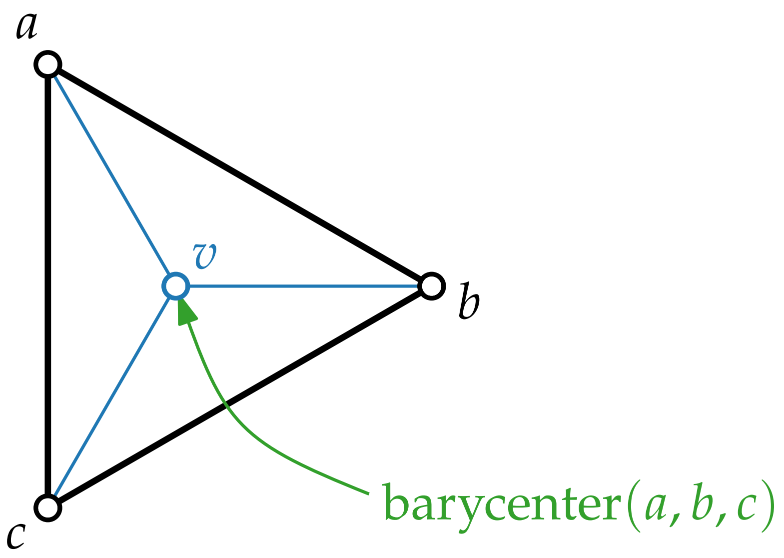

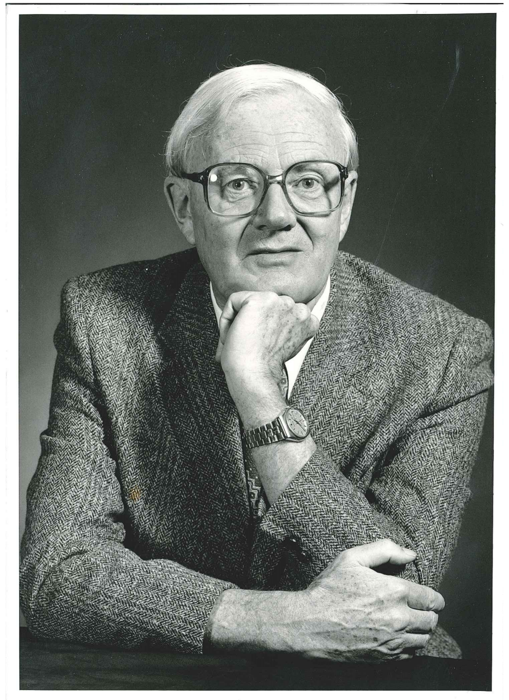

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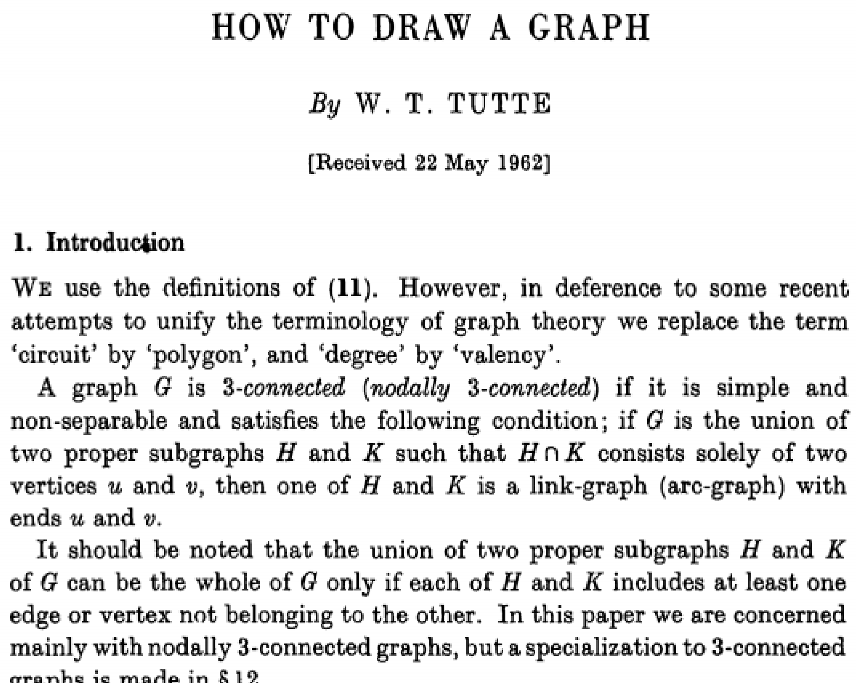

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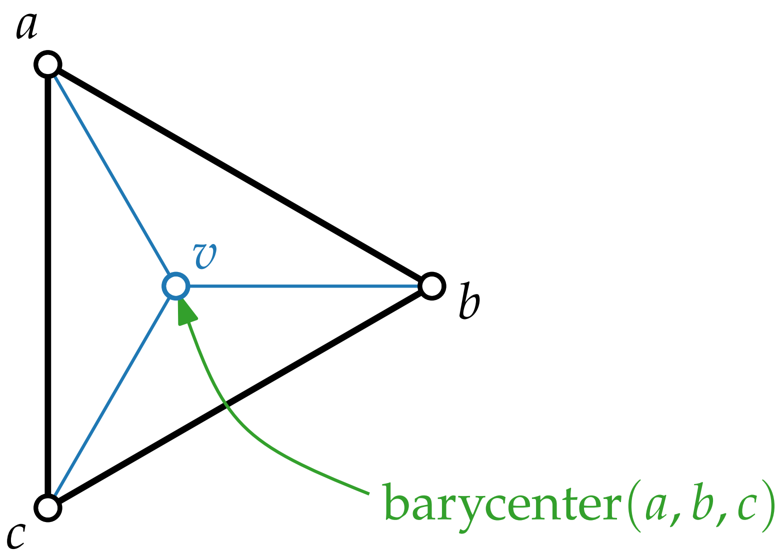
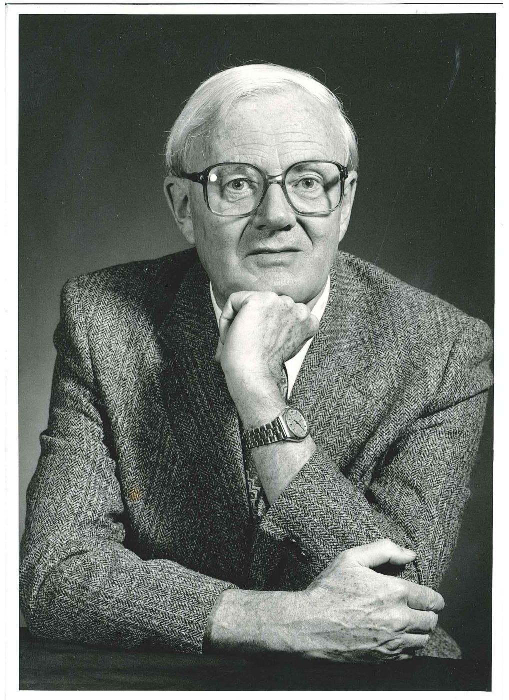


$$\text{barycenter}(x_1, \ldots, x_k) = \quad ?$$



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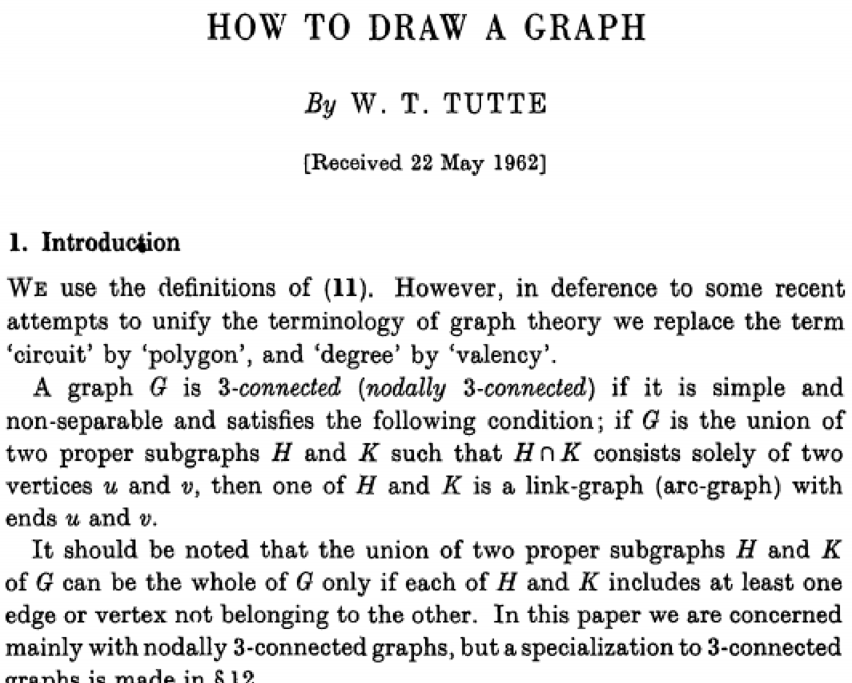

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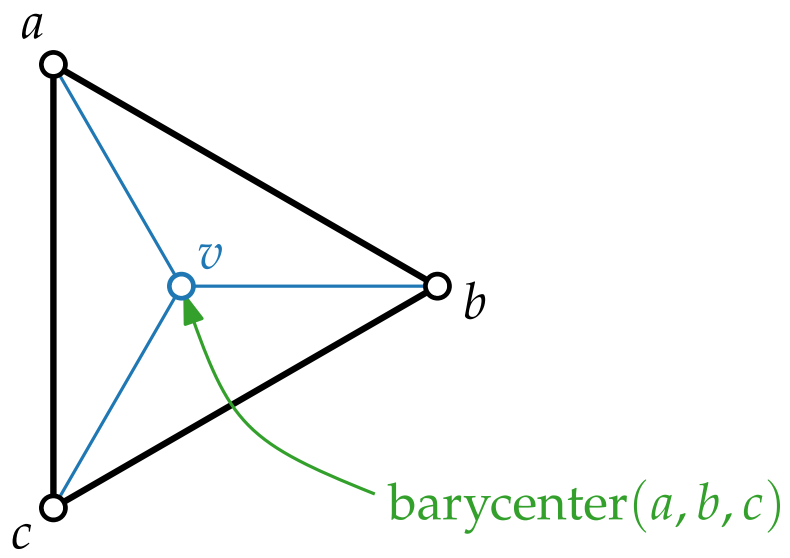
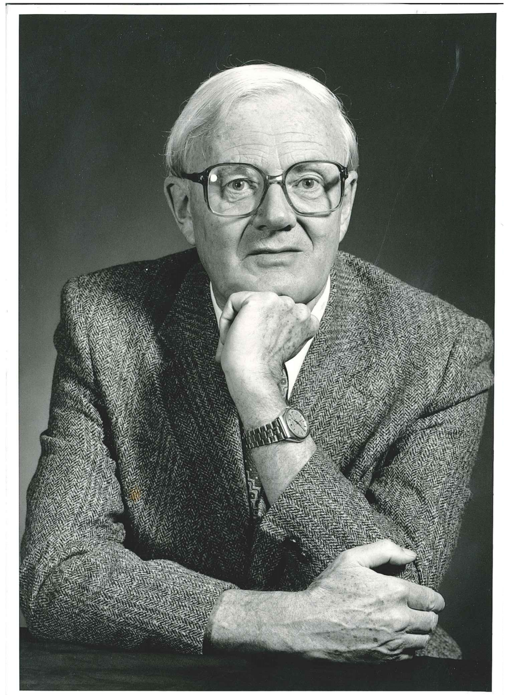


$$\text{barycenter}(x_1, \ldots, x_k) = \textstyle\sum_{i=1}^{k} x_i / k$$



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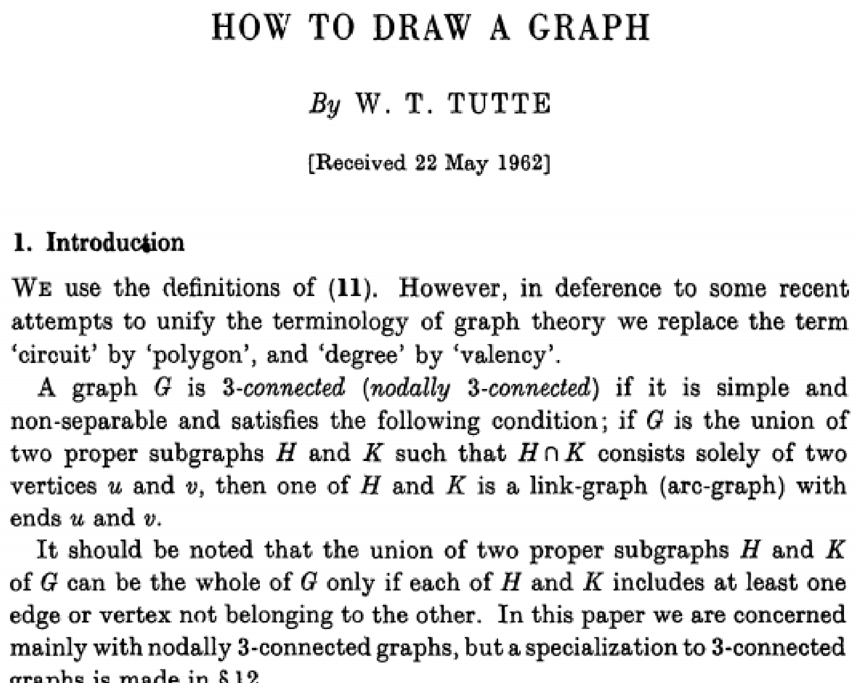

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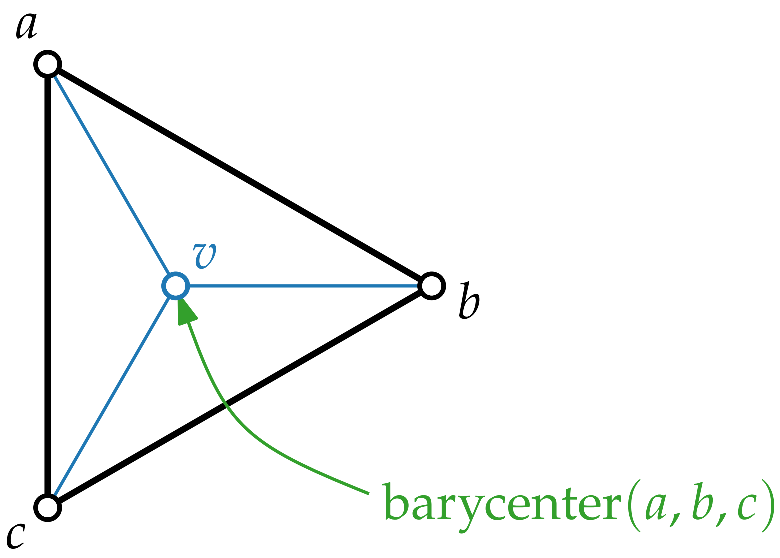
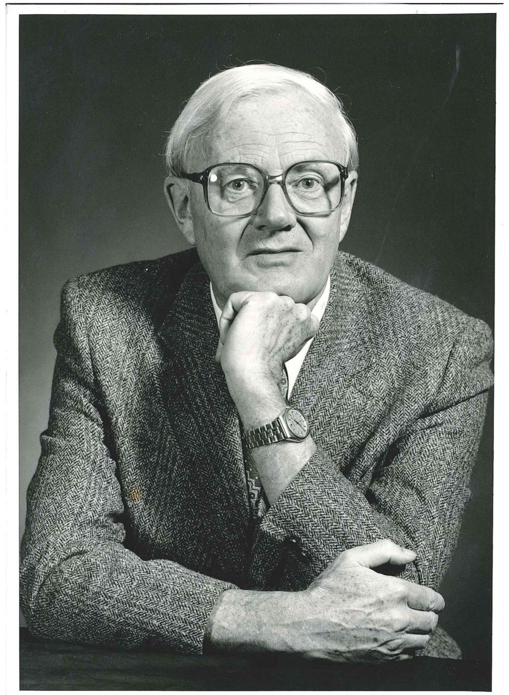


$$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$$

Idea.
Repeatedly place every vertex at barycenter of neighbors.



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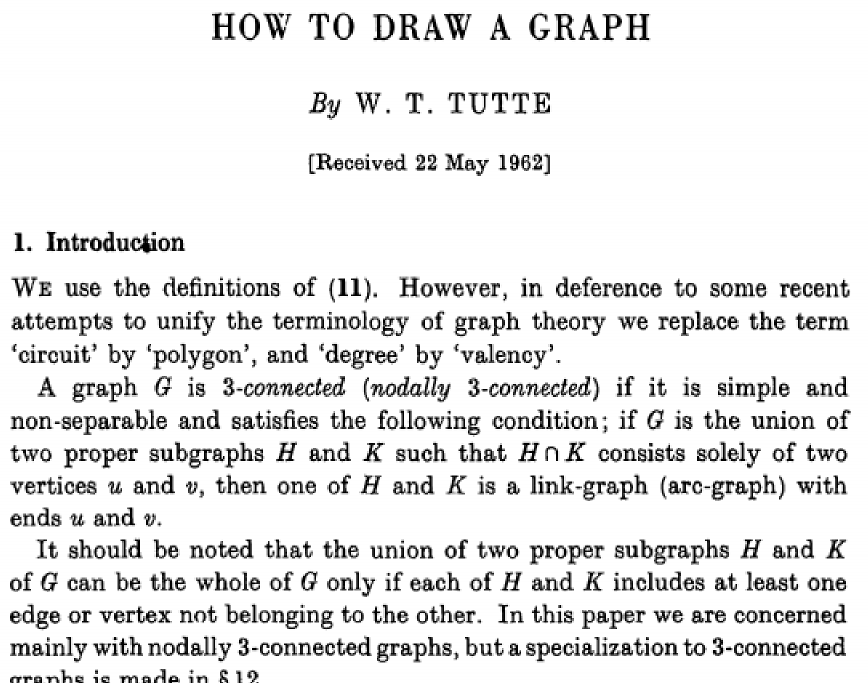

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Tutte's Forces

```
ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
    $t \leftarrow 1$
    while $t < K$ and $\max_{v \in V} \|F_v(t)\| > \varepsilon$ do
        foreach $u \in V$ do
            $F_u(t) \leftarrow \sum_{v \in V} f_{\text{rep}}(u, v) + \sum_{uv \in E} f_{\text{attr}}(u, v)$
        foreach $u \in V$ do
            $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$
        $t \leftarrow t + 1$
    return $p$
```

Tutte's Forces

```
ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
  $t \leftarrow 1$
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  return $p$
```

Tutte's Forces

Goal.

$p_u = \text{barycenter}(\bigcup_{uv \in E} v)$

```
ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
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      foreach $u \in V$ do
          $p_u \leftarrow p_u + \cancel{\delta(t)}\ 1 \cdot F_u(t)$
      $t \leftarrow t + 1$
  return $p$
```

$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\textstyle\bigcup_{uv \in E} v)$$
$$= \textstyle\sum_{uv \in E} p_v /$$

```
ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
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```

$$\text{barycenter}(x_1, \dots, x_k) = \textstyle\sum_{i=1}^{k} x_i / k$$

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\textstyle\bigcup_{uv \in E} v)$$
$$= \textstyle\sum_{uv \in E} p_v / \deg(u)$$

```
ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
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    while $t < K$ and $\max_{v \in V} \|F_v(t)\| > \varepsilon$ do
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Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\bigcup_{uv \in E} v)$$
$$= \sum_{uv \in E} p_v / \deg(u)$$

$$\boxed{F_u(t)} = \sum_{uv \in E} p_v / \deg(u) - p_u$$

```
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Tutte's Forces

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$$F_u(t) = \sum_{uv \in E} p_v / \deg(u) - p_u$$
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Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\textstyle\bigcup_{uv\in E} v)$$
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Tutte's Forces

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$$F_u(t) = \sum_{uv \in E} p_v / \deg(u) - p_u$$
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Tutte's Forces

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```

$$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$$

- **Repulsive forces**

$$f_{\text{rep}}(u, v) = 0$$

Tutte's Forces

Goal.

$$p_u = \text{barycenter}\left(\bigcup\nolimits_{uv\in E} v\right) = \sum\nolimits_{uv\in E} p_v / \deg(u)$$

$$F_u(t) = \sum\nolimits_{uv\in E} p_v / \deg(u) - p_u = \sum\nolimits_{uv\in E} (p_v - p_u) / \deg(u) = \sum\nolimits_{uv\in E} ||p_u - p_v|| / \deg(u)$$

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$$\text{barycenter}(x_1, \ldots, x_k) = \sum\nolimits_{i=1}^{k} x_i / k$$

- **Repulsive forces**

$$f_{\text{rep}}(u, v) = 0$$

- **Attractive forces**

$$f_{\text{attr}}(u, v) = \frac{1}{\deg(u)} \cdot ||p_u - p_v||$$

Tutte's Forces

Goal.

$$p_u = \text{barycenter}\left(\bigcup_{uv\in E} v\right) = \textstyle\sum_{uv\in E} p_v / \deg(u)$$

$$F_u(t) = \textstyle\sum_{uv\in E} p_v/\deg(u) - p_u = \sum_{uv\in E}(p_v - p_u)/\deg(u) = \sum_{uv\in E} ||p_u - p_v|| / \deg(u)$$

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$$f_{\mathrm{attr}}(u,v) = \frac{1}{\deg(u)} \cdot ||p_u - p_v||$$

Demo

Tutte’s Forces

Goal.

$$p_u = \text{barycenter}(\textstyle\bigcup_{uv\in E} v)$$
$$= \textstyle\sum_{uv\in E} p_v / \deg(u)$$

$$F_u(t) = \textstyle\sum_{uv\in E} p_v / \deg(u) - p_u$$
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```

$$\text{barycenter}(x_1, \ldots, x_k) = \textstyle\sum_{i=1}^{k} x_i / k$$

- **Repulsive forces**

$$f_{\text{rep}}(u, v) = 0$$

Solution: $p_u = (0, 0)\ \forall u \in V$

- **Attractive forces**

$$f_{\text{attr}}(u, v) = \frac{1}{\deg(u)} \cdot ||p_u - p_v||$$

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$$

$$\boxed{F_u(t)} = \sum_{uv \in E} p_v / \deg(u) - p_u = \sum_{uv \in E} (p_v - p_u) / \deg(u) = \boxed{\sum_{uv \in E} ||p_u - p_v|| / \deg(u)}$$

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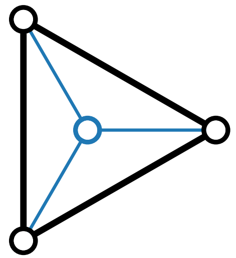

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\textstyle\bigcup_{uv \in E} v)$$
$$= \textstyle\sum_{uv \in E} p_v / \deg(u)$$

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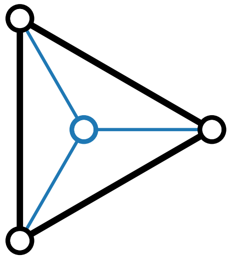

Fix coordinates of outer face!

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\textstyle\bigcup_{uv \in E} v)$$
$$= \textstyle\sum_{uv \in E} p_v / \deg(u)$$

$$F_u(t) = \textstyle\sum_{uv \in E} p_v / \deg(u) - p_u$$
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```

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- **Repulsive forces**

$$f_{\text{rep}}(u, v) = 0$$

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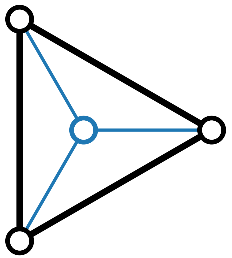
$$f_{\text{attr}}(u, v) = \begin{cases} 0 & u \text{ fixed} \\ \frac{1}{\deg(u)} \cdot ||p_u - p_v|| & \text{else} \end{cases}$$

Solution: $p_u = (0, 0)\ \forall u \in V$



Fix coordinates of outer face!

Tutte's Forces

Goal.

$$\begin{aligned} p_u &= \text{barycenter}(\textstyle\bigcup_{uv \in E} v) \\ &= \textstyle\sum_{uv \in E} p_v / \deg(u) \end{aligned}$$

$$\begin{aligned} F_u(t) &= \textstyle\sum_{uv \in E} p_v / \deg(u) - p_u \\ &= \textstyle\sum_{uv \in E} (p_v - p_u) / \deg(u) \\ &= \textstyle\sum_{uv \in E} \|p_u - p_v\| / \deg(u) \end{aligned}$$

```
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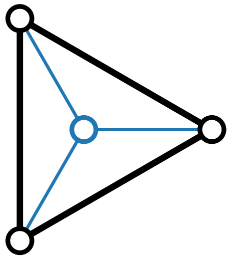

Fix coordinates of outer face!

Demo

Linear System of Equations

Goal.

$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\bigcup_{uv \in E} v) \ = \sum_{uv \in E} p_v / \deg(u)$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

$x_u = \sum_{uv \in E} x_v / \deg(u)$

$y_u = \sum_{uv \in E} y_v / \deg(u)$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

$x_u = \sum_{uv \in E} x_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v$

$y_u = \sum_{uv \in E} y_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot y_u = \sum_{uv \in E} y_v$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

$$x_u = \sum_{uv \in E} x_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \qquad \Leftrightarrow \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$$

$$y_u = \sum_{uv \in E} y_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot y_u = \sum_{uv \in E} y_v \qquad \Leftrightarrow \deg(u) \cdot y_u - \sum_{uv \in E} y_v = 0$$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

2 Systems of linear equations

$$x_u = \sum_{uv \in E} x_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \quad \Leftrightarrow \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$$

$$y_u = \sum_{uv \in E} y_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot y_u = \sum_{uv \in E} y_v \quad \Leftrightarrow \deg(u) \cdot y_u - \sum_{uv \in E} y_v = 0$$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

$Ax = b$

2 Systems of linear equations

$$x_u = \sum_{uv \in E} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$$

$$y_u = \sum_{uv \in E} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{uv \in E} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{uv \in E} y_v = 0$$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

$Ax = b \qquad Ay = b$

2 Systems of linear equations

$$x_u = \sum_{uv \in E} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$$

$$y_u = \sum_{uv \in E} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{uv \in E} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{uv \in E} y_v = 0$$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

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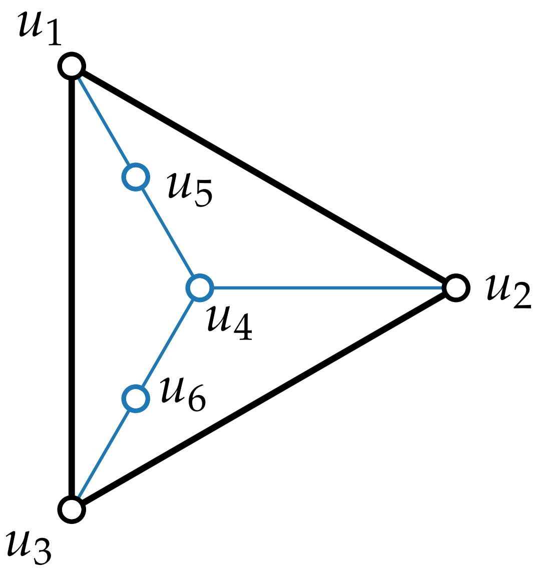

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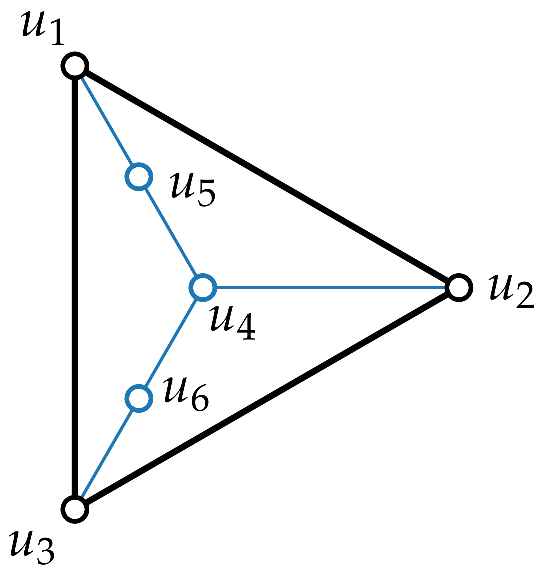
A



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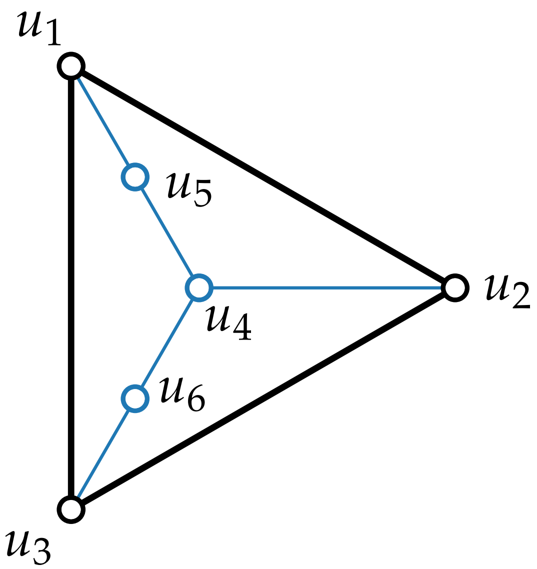

A

$$\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{pmatrix} \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

Linear System of Equations

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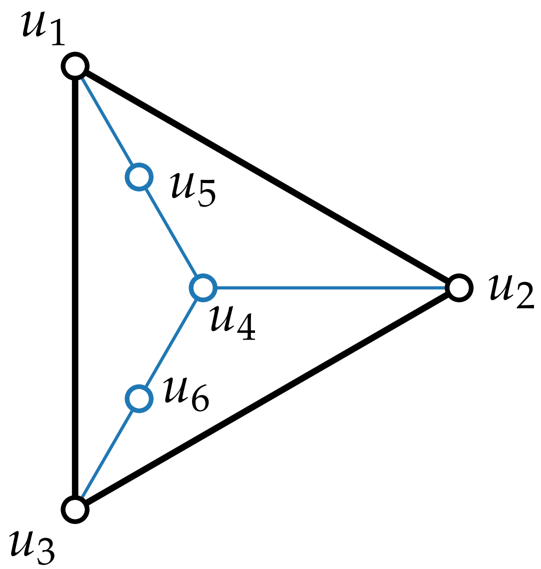

$$\begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & & & & & & \\ u_2 & & & & & & \\ u_3 & & & & & & \\ u_4 & & & & & & \\ u_5 & & & & & & \\ u_6 & & & & & & \end{array} \quad A$$

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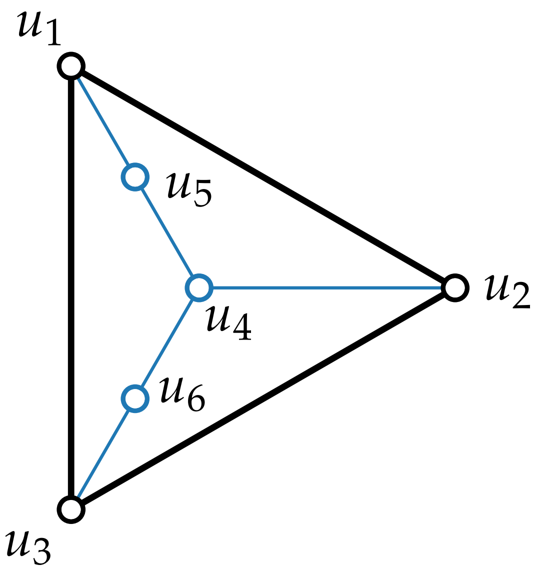

$$A \quad \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 3 & & & & & \\ u_2 & & & & & & \\ u_3 & & & & & & \\ u_4 & & & & & & \\ u_5 & & & & & & \\ u_6 & & & & & & \end{array}$$

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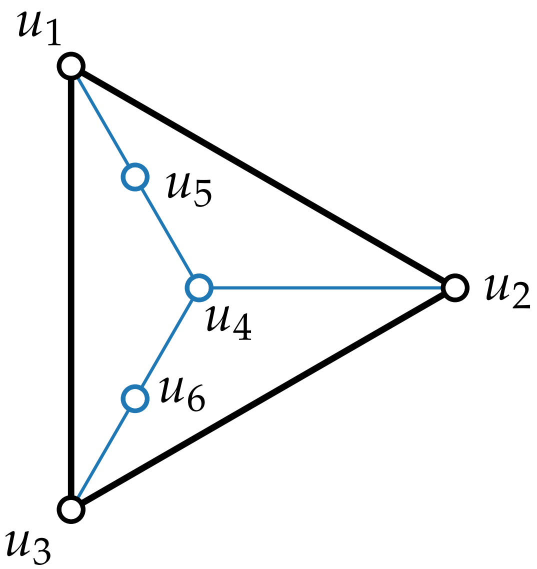

$$\begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 3 & -1 & & & & \\ u_2 & & & & & & \\ u_3 & & & & & & \\ u_4 & & & & & & \\ u_5 & & & & & & \\ u_6 & & & & & & \end{array} \quad A$$

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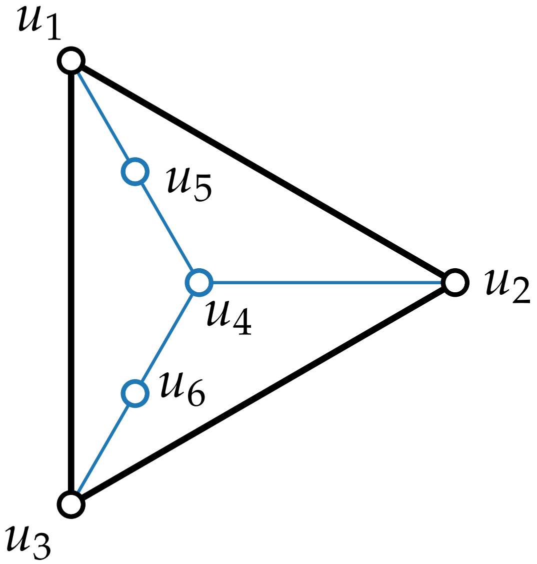

$$A \quad \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 3 & -1 & -1 & & & \\ u_2 & & & & & & \\ u_3 & & & & & & \\ u_4 & & & & & & \\ u_5 & & & & & & \\ u_6 & & & & & & \end{array}$$

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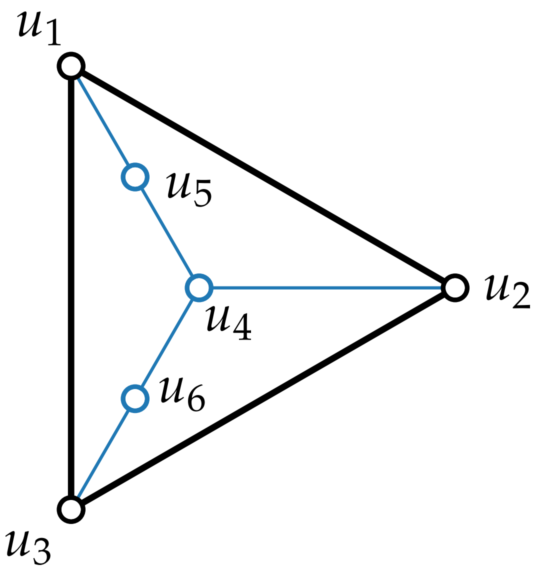

$$\begin{array}{c|c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \left(\begin{array}{cccccc} 3 & -1 & -1 & 0 & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{array}\right) \end{array} \quad A$$

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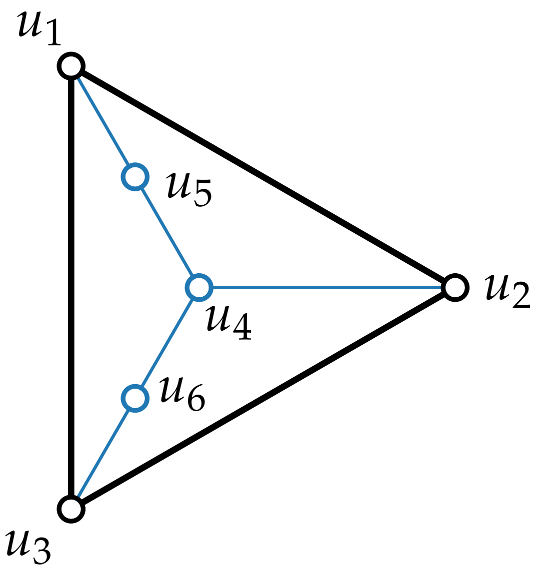

$$\begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 3 & -1 & -1 & 0 & -1 & \\ u_2 & & & & & & \\ u_3 & & & & & & \\ u_4 & & & & & & \\ u_5 & & & & & & \\ u_6 & & & & & & \end{array} \quad A$$

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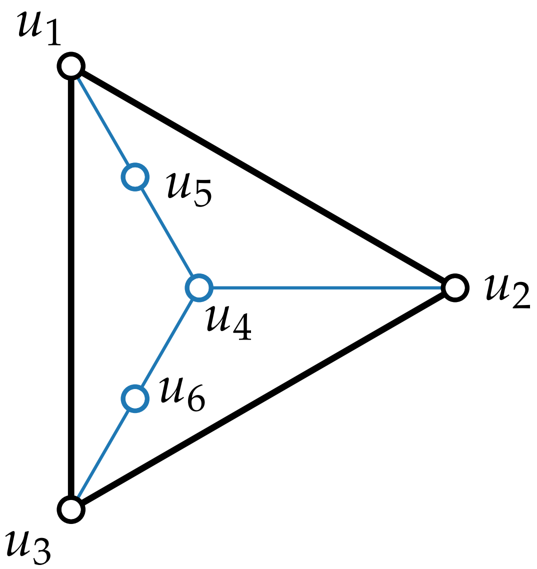

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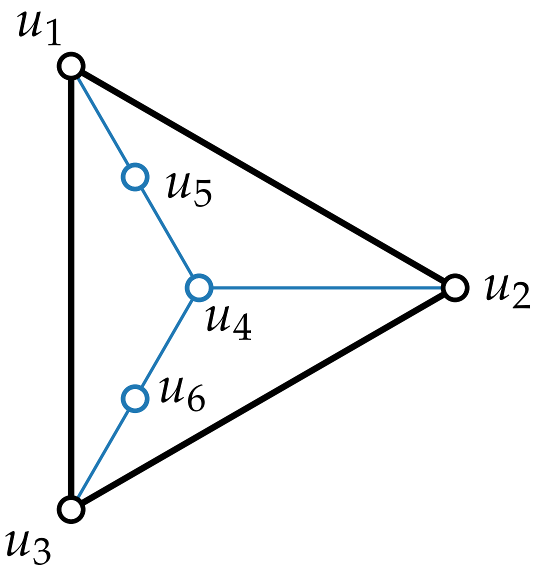

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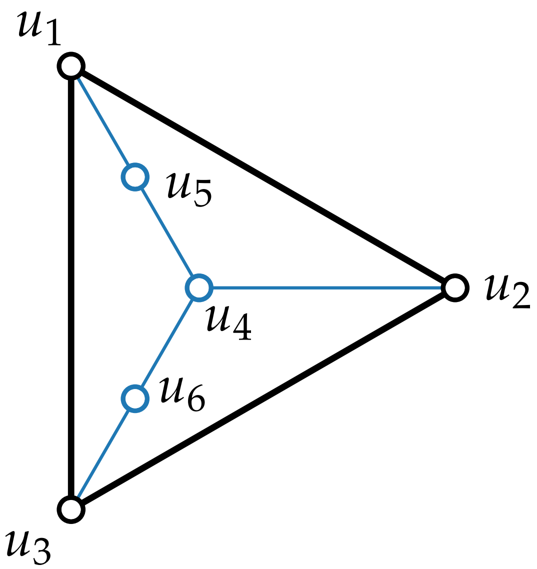

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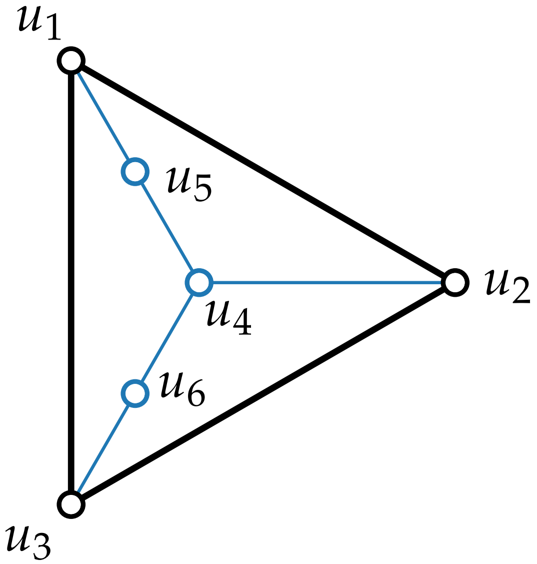

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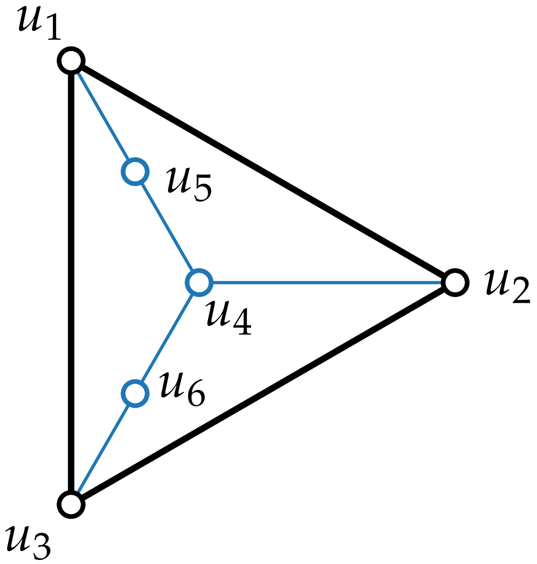

$$\begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 3 & -1 & -1 & 0 & -1 & 0 \\ u_2 & -1 & 3 & -1 & -1 & 0 & 0 \\ u_3 & -1 & -1 & 3 & 0 & 0 & -1 \\ u_4 & 0 & -1 & 0 & 3 & -1 & -1 \\ u_5 & & & & & & \\ u_6 & & & & & & \end{array} \quad A$$

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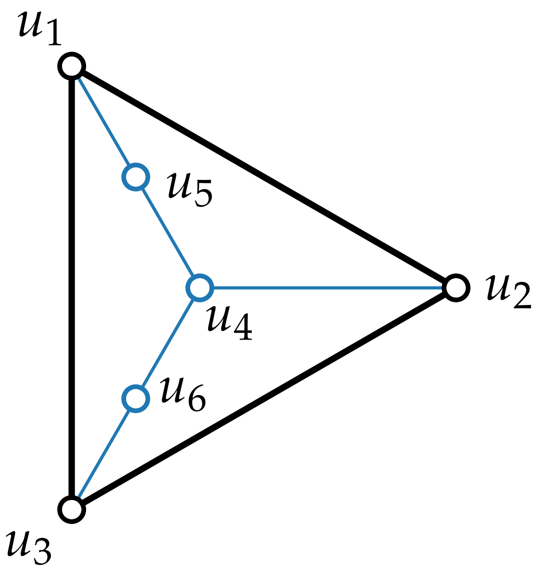

$$\begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 3 & -1 & -1 & 0 & -1 & 0 \\ u_2 & -1 & 3 & -1 & -1 & 0 & 0 \\ u_3 & -1 & -1 & 3 & 0 & 0 & -1 \\ u_4 & 0 & -1 & 0 & 3 & -1 & -1 \\ u_5 & -1 & 0 & 0 & -1 & 2 & 0 \\ u_6 & & & & & & \end{array} \quad A$$

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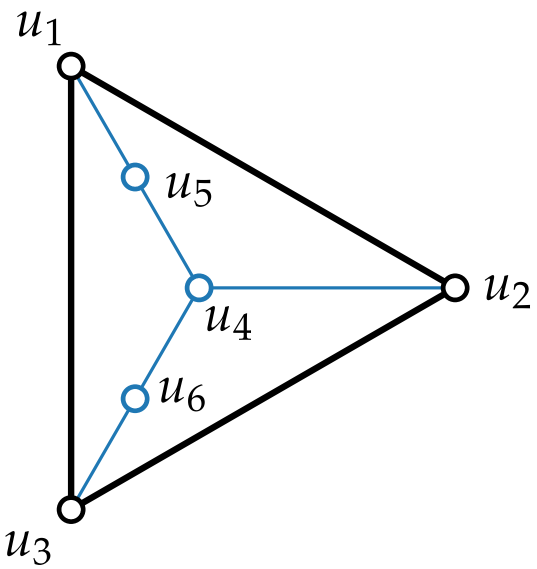

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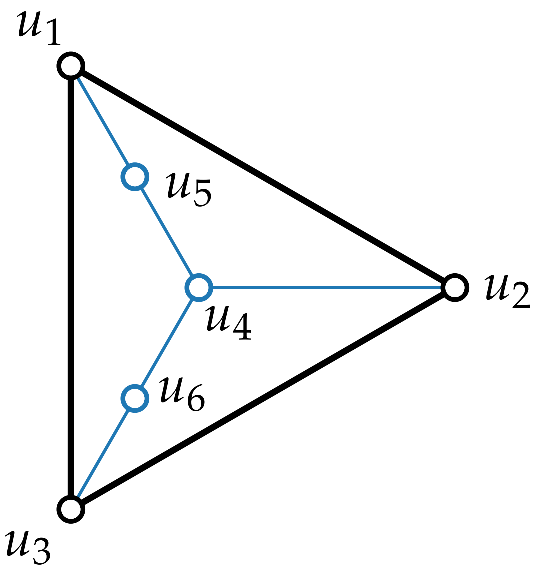

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$A_{ii} = \deg(u_i)$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

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2 Systems of linear equations

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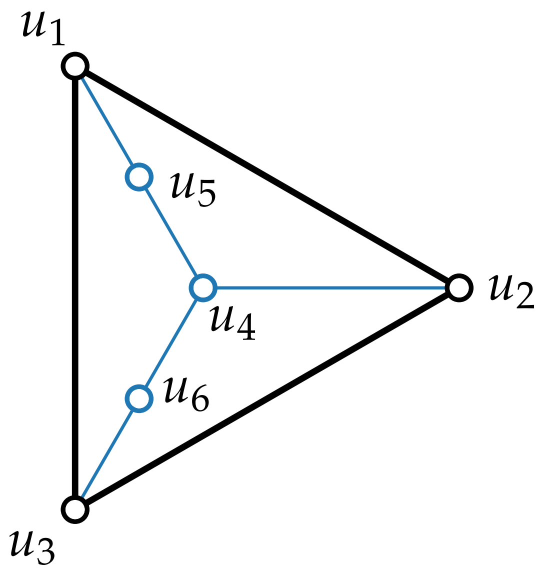

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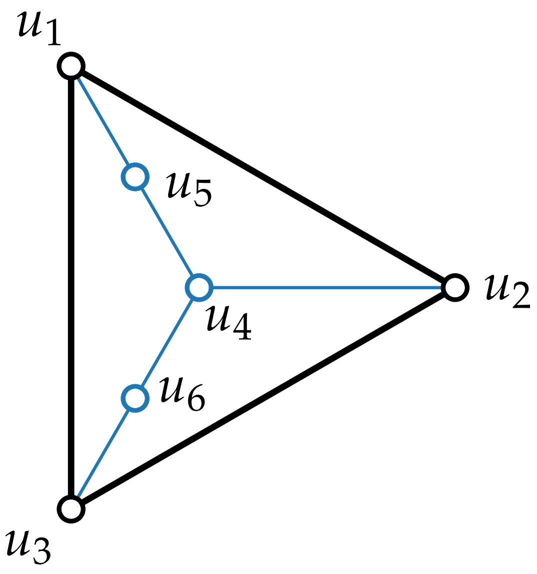

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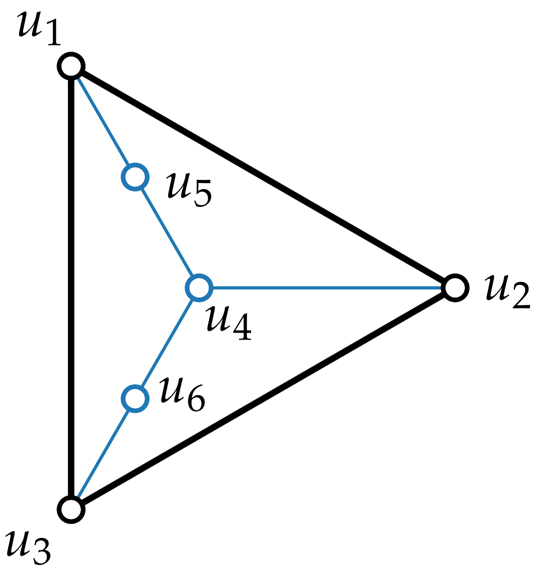

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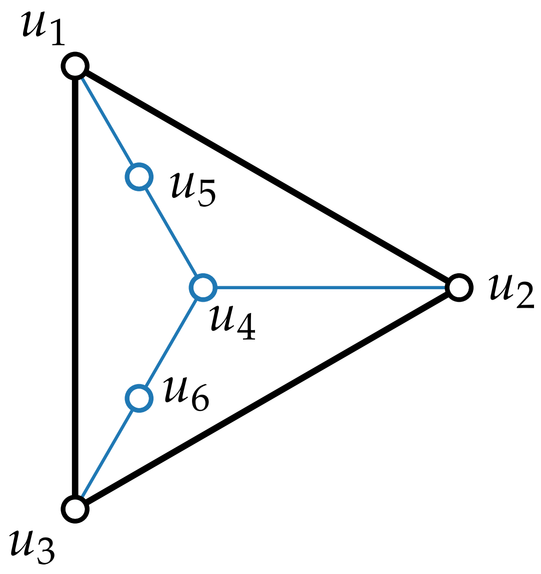

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variables, constraints, $\det(A) =$

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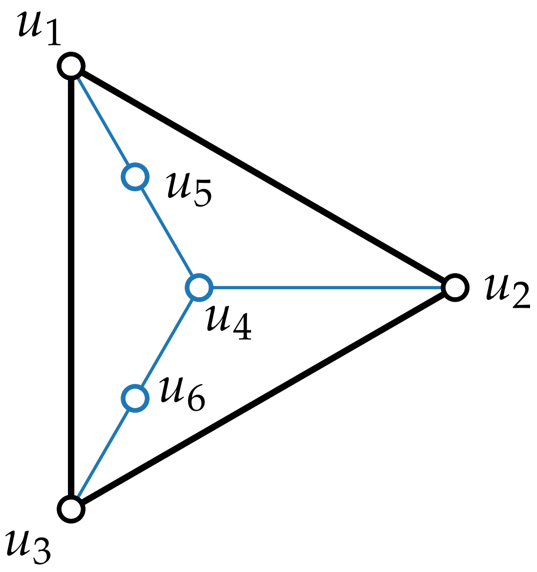

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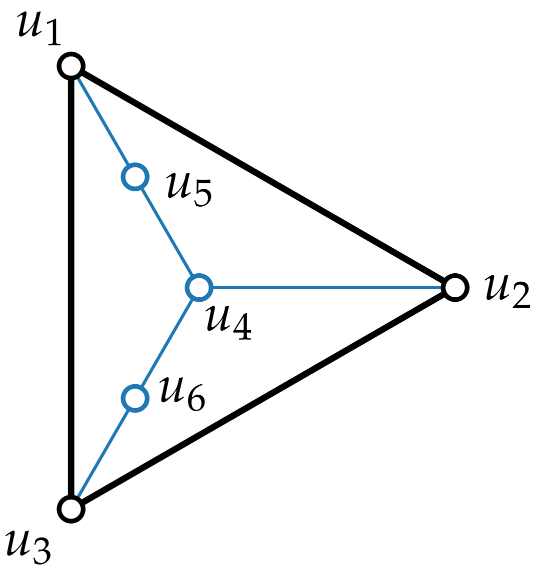

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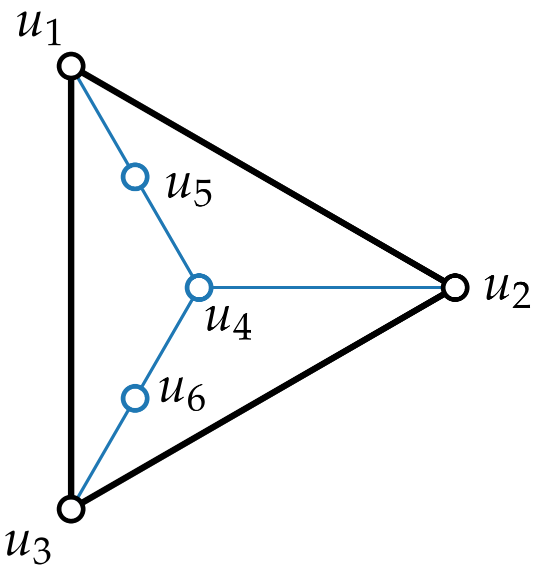

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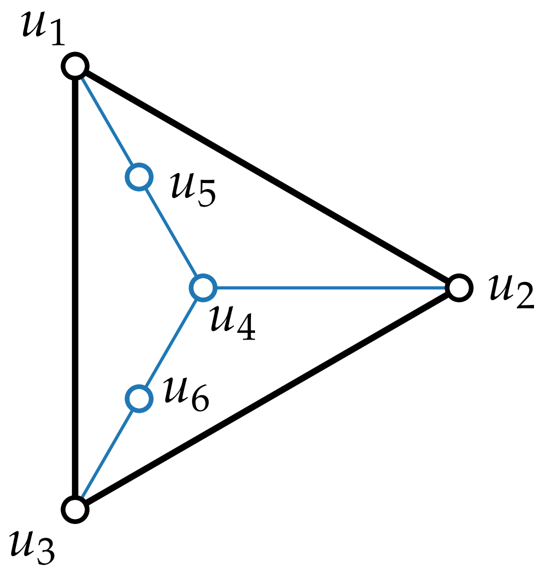

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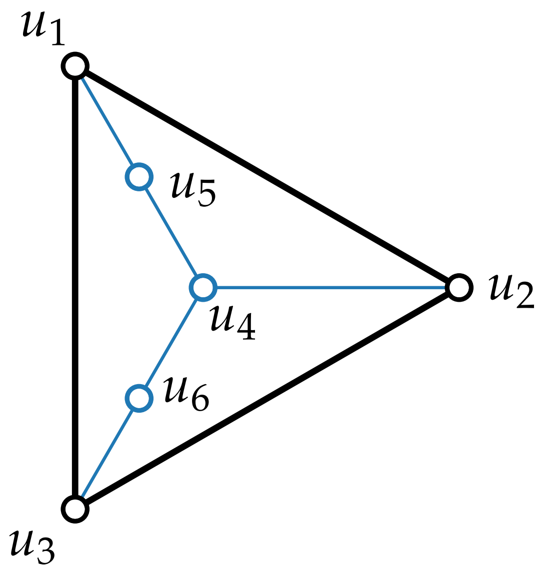

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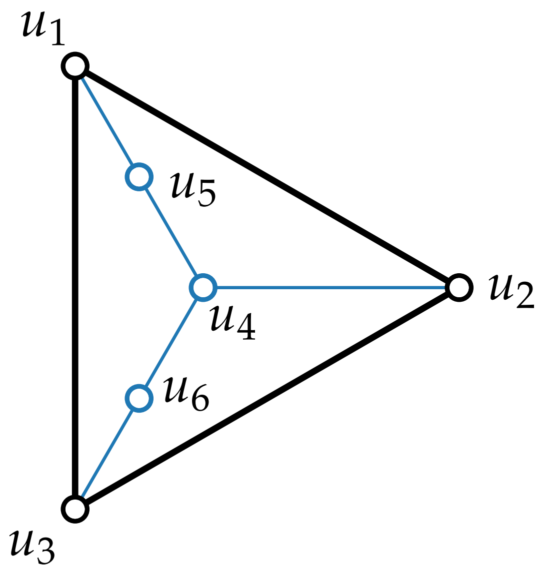

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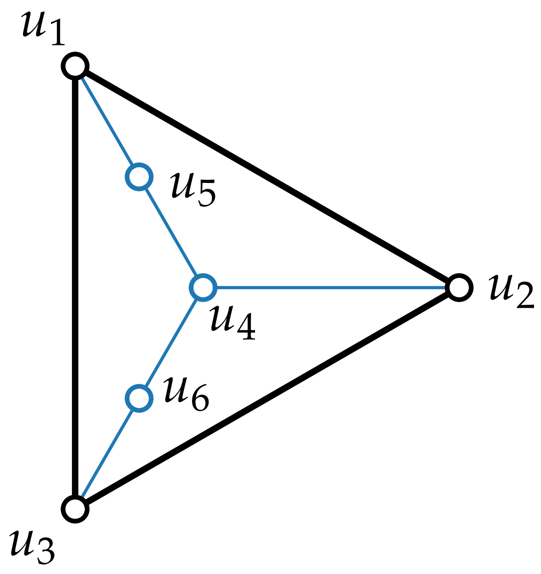

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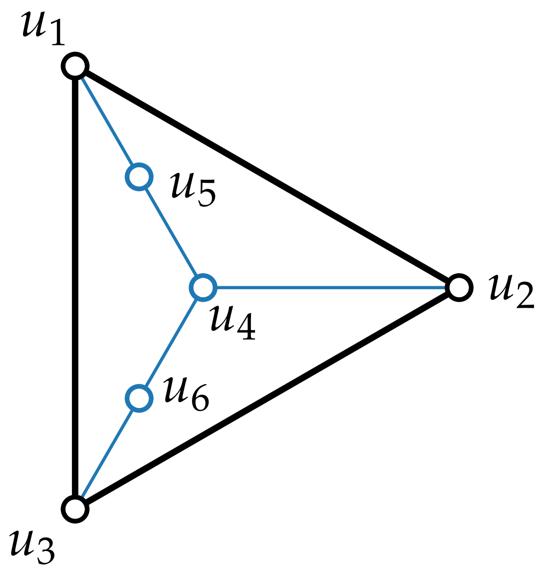

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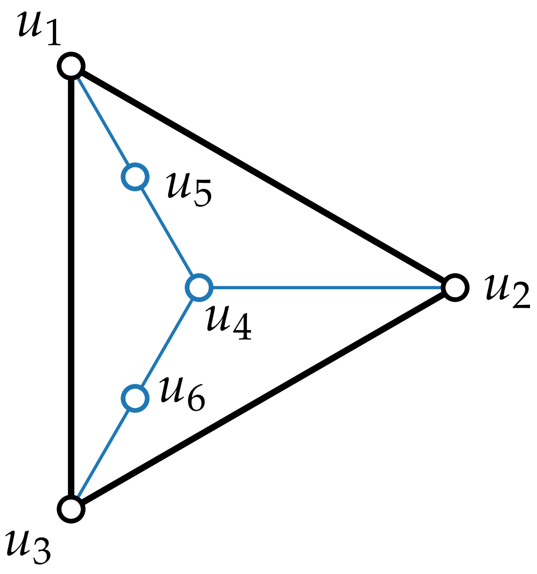

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$$A_{ij, i \neq j} = \begin{cases} -1 & u_i u_j \in E \\ 0 & u_i u_j \notin E \end{cases}$$

k variables, k constraints, $\det(A) > 0$

$k = \#\text{free vertices}$

$\Rightarrow$ no unique solution ☹

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

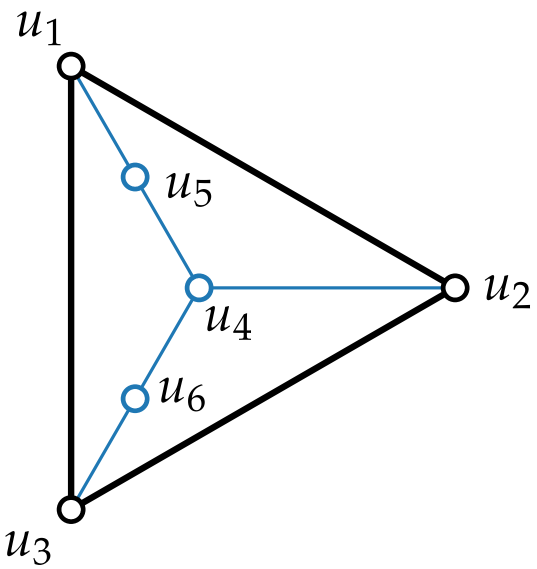
$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

$Ax = b \quad Ay = b \quad b = (0)_n$

2 Systems of linear equations

$$x_u = \sum_{uv \in E} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$$

$$y_u = \sum_{uv \in E} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{uv \in E} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{uv \in E} y_v = 0$$



$$A = \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 3 & -1 & -1 & 0 & -1 & 0 \\ u_2 & -1 & 3 & -1 & -1 & 0 & 0 \\ u_3 & -1 & -1 & 3 & 0 & 0 & -1 \\ u_4 & 0 & -1 & 0 & 3 & -1 & -1 \\ u_5 & -1 & 0 & 0 & -1 & 2 & 0 \\ u_6 & 0 & 0 & -1 & -1 & 0 & 2 \end{array}$$

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Linear System of Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\bigcup_{uv \in E} v)$

Theorem.
Tutte's barycentric algorithm admits a unique solution.
It can be computed in polynomial time.

$$x_u = \sum_{uv \in E} x_v / \deg(u) \;\Leftrightarrow\; \deg(u) \cdot x_u = \sum_{uv \in E} x_v \;\Leftrightarrow\; \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$$

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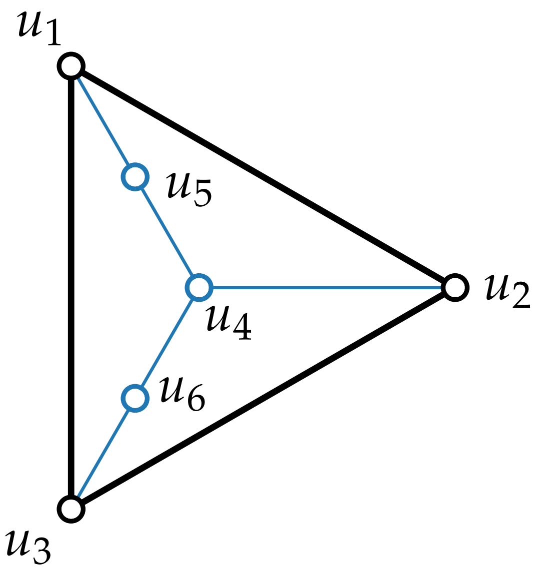

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k variables, k constraints, $\det(A) > 0$

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$\Rightarrow$ unique solution 😃

Linear System of Equations

Theorem. **Tutte drawing**
Tutte's barycentric algorithm admits a unique solution.
It can be computed in polynomial time.

Goal. $p_u = (x_u, y_u)$
$p_u = \text{barycenter}(\bigcup_{uv \in E} v)$

$$x_u = \sum_{uv \in E} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$$
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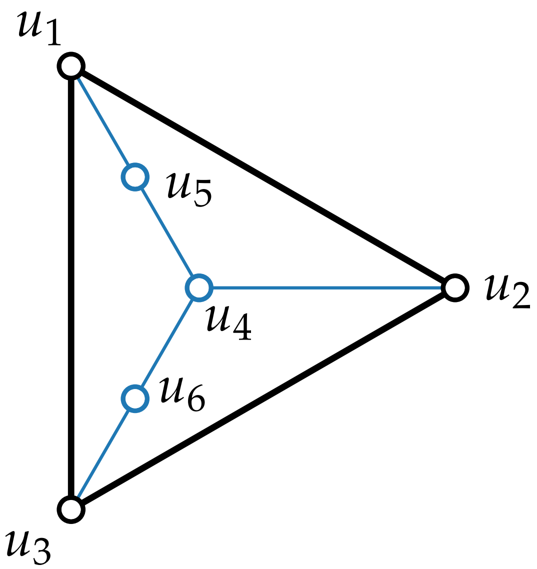

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array} A$$

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Universität Trier

Visualization of Graphs

Lecture 3:
Force-Directed Drawing Algorithms

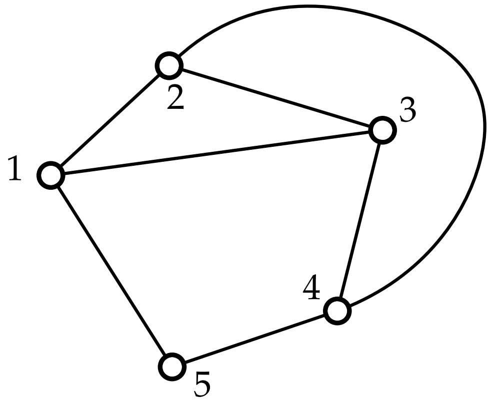
Part V:
Tutte's Theorem

Philipp Kindermann

3-Connected Planar Graphs

planar: G can be drawn in such a way that no edges cross each other

connected: There is a u-v-path for every $u, v \in V$



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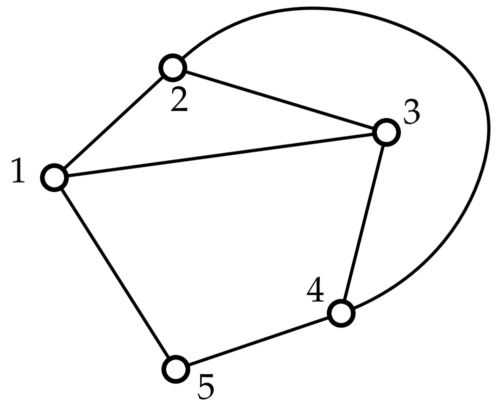

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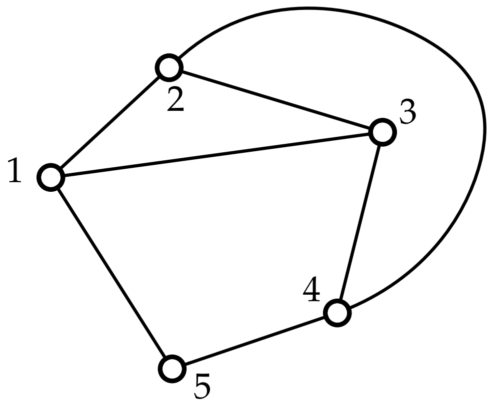

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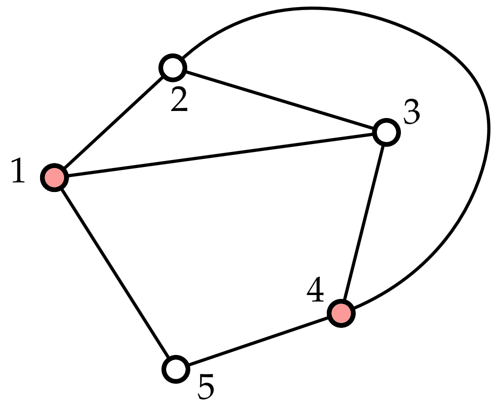

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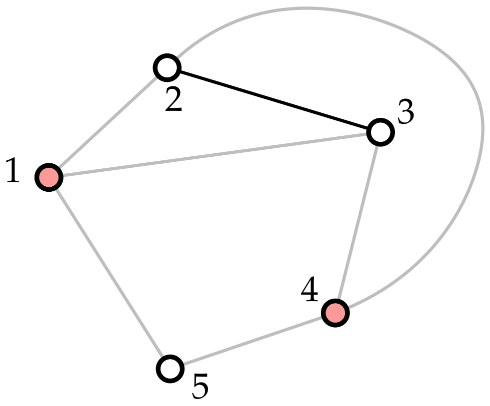

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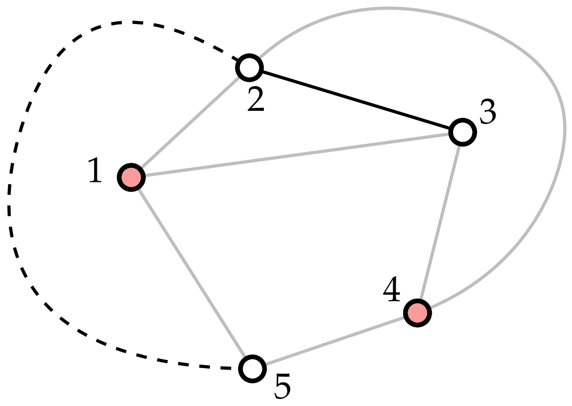

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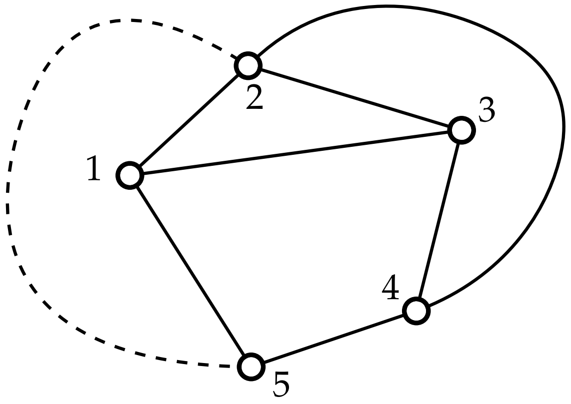

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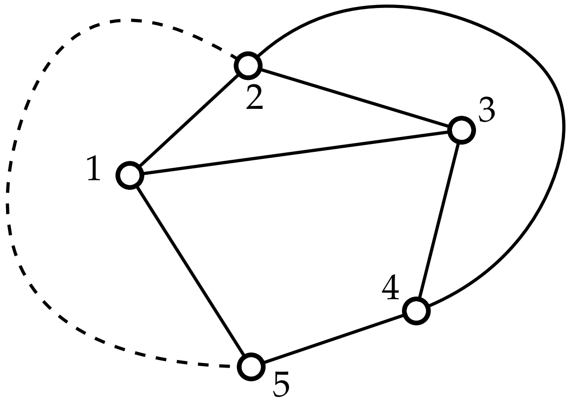

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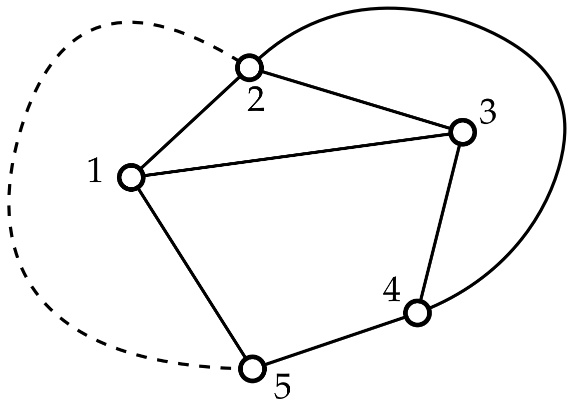

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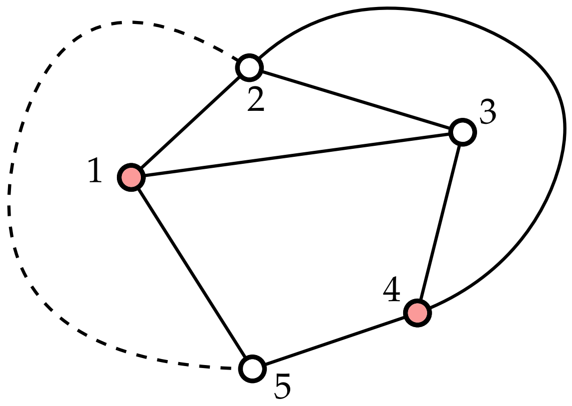

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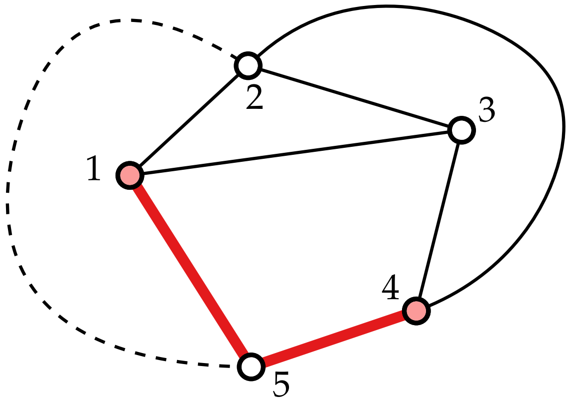

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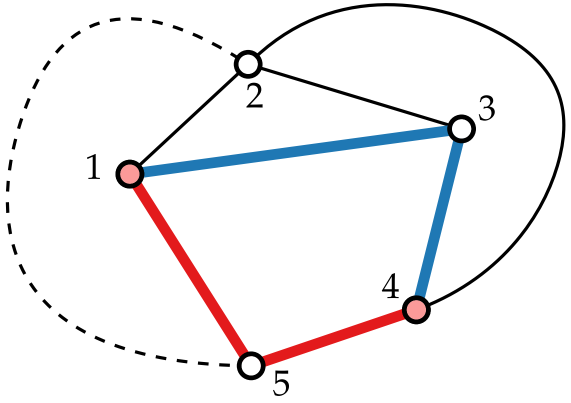

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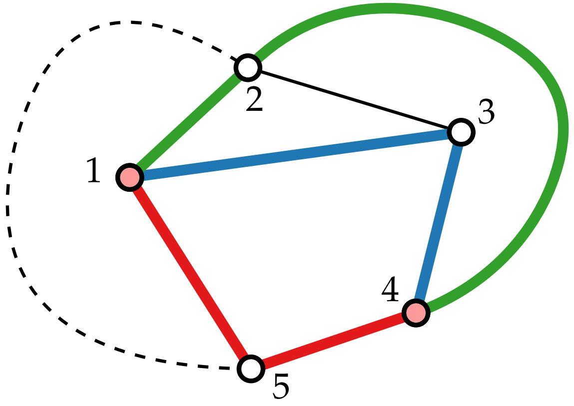

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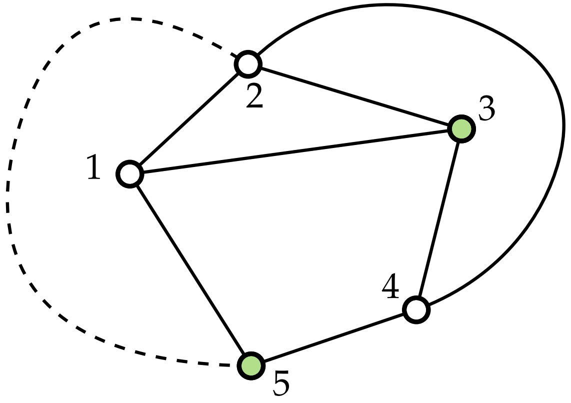

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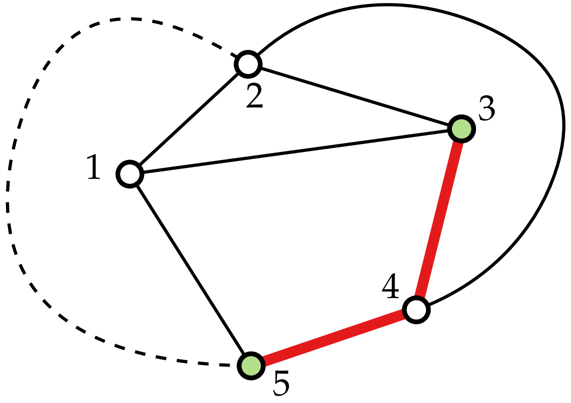

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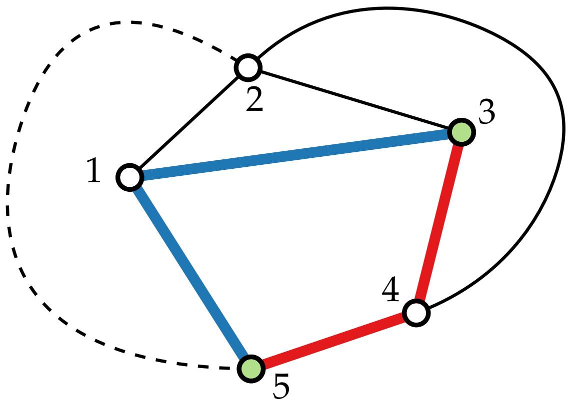

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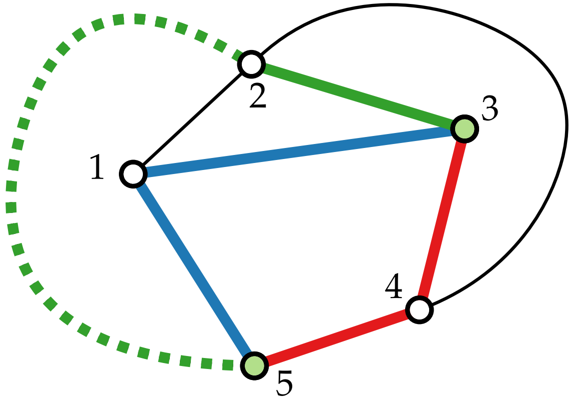

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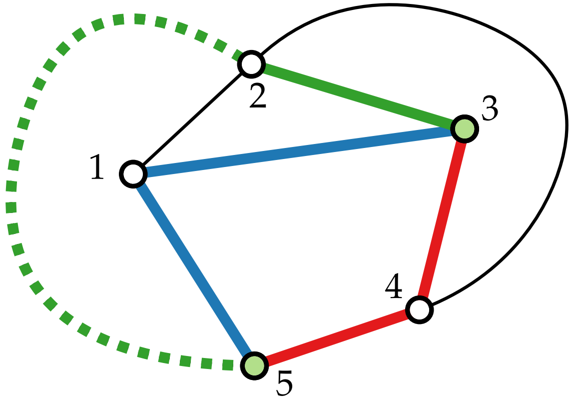
Theorem. [Whitney 1933]
Every 3-connected planar graph has a unique planar embedding.



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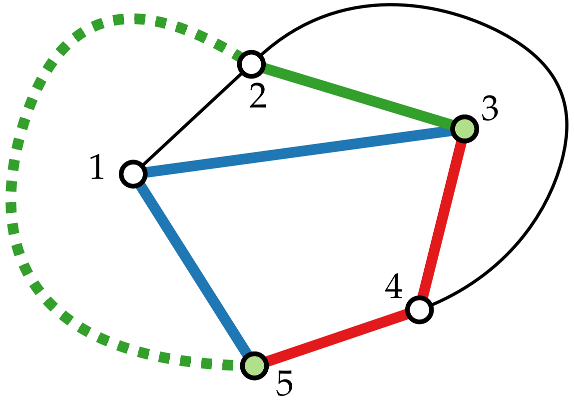

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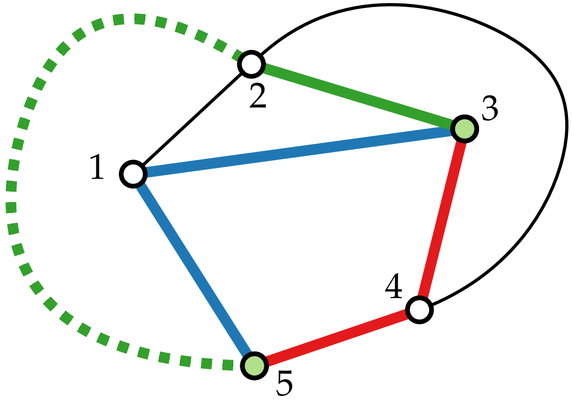
Γ_1, Γ_2 embeddings of G



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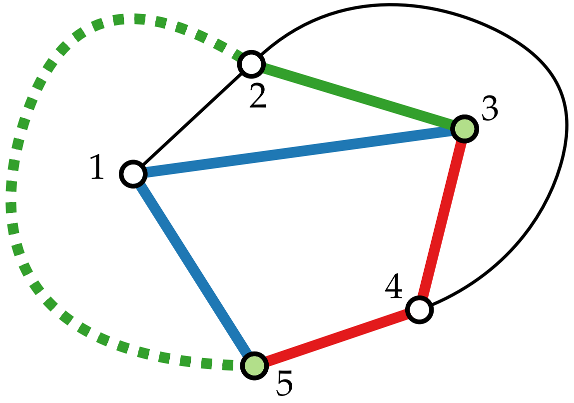
C face of Γ_2, but not Γ_1



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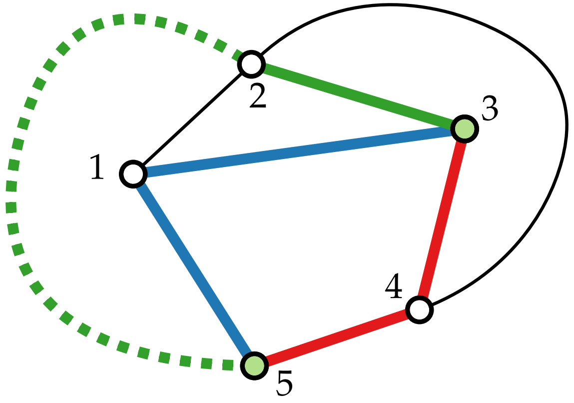

Γ_1

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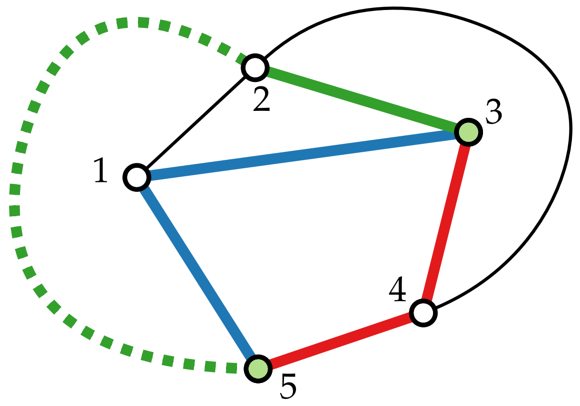

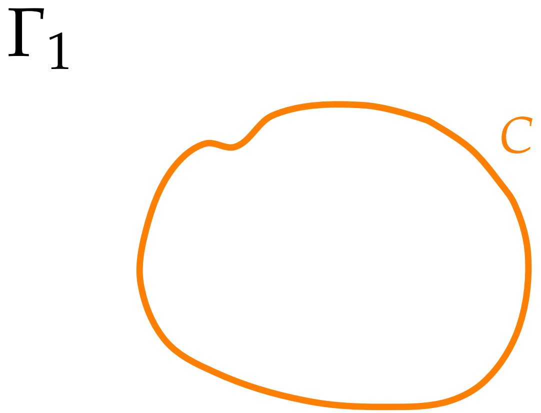

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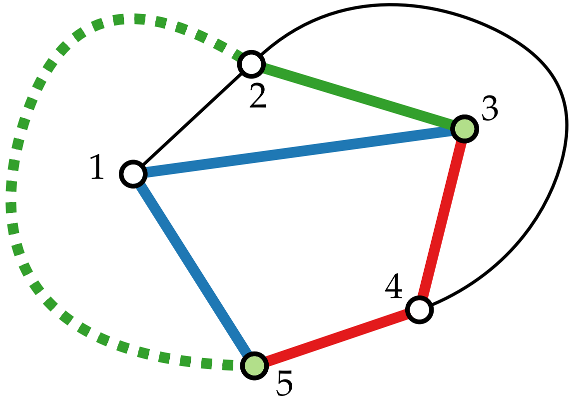

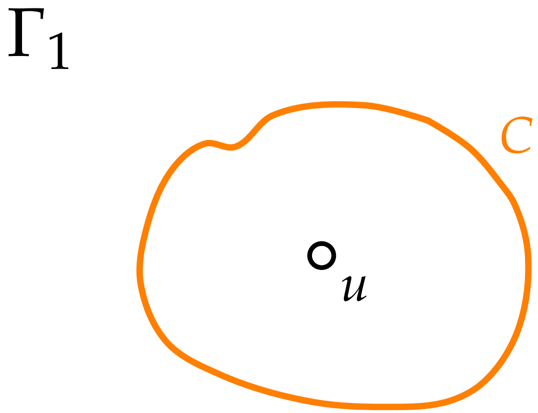

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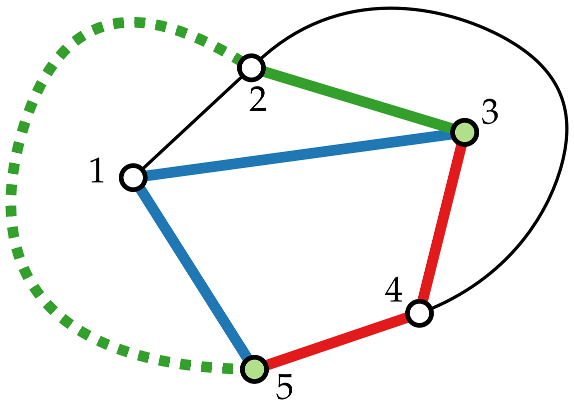

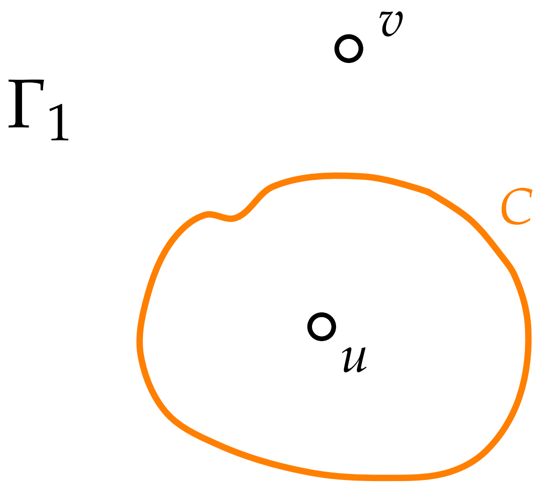

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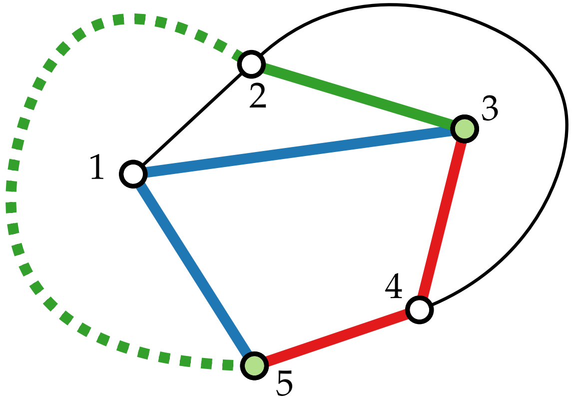

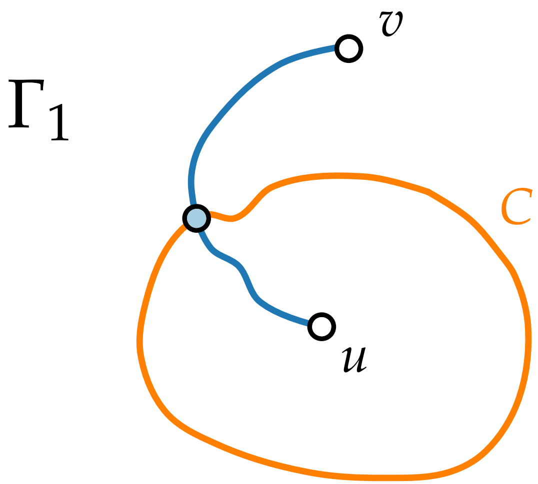

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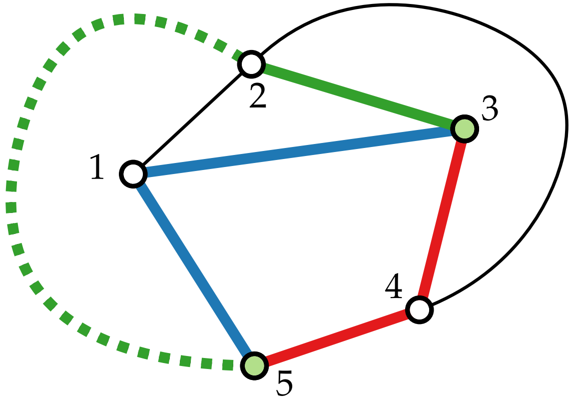

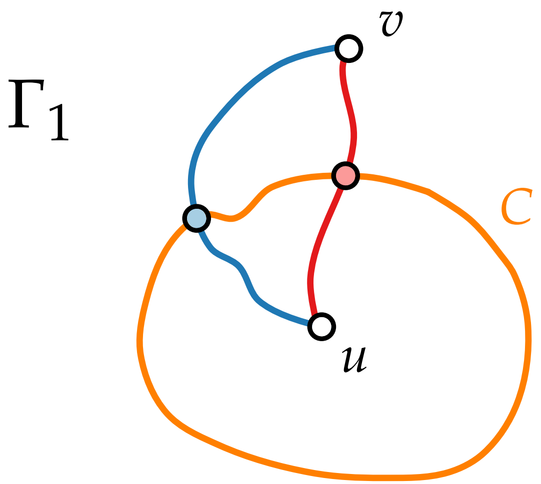

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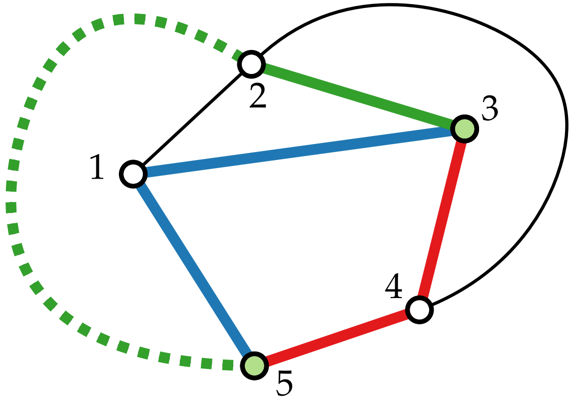

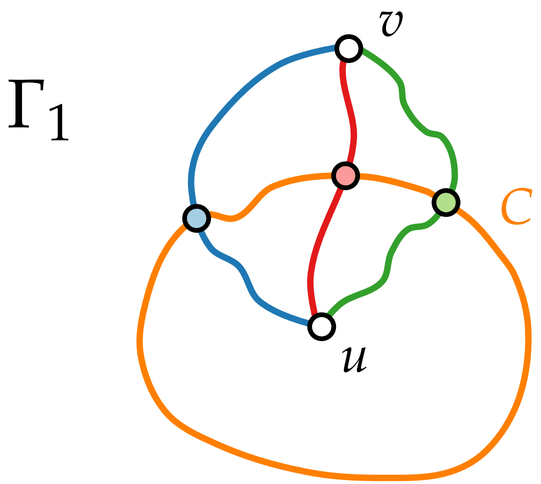

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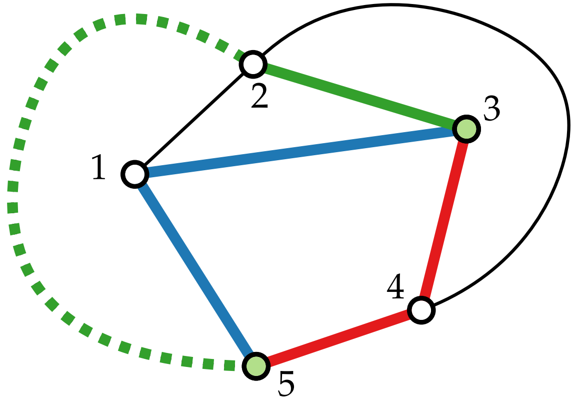

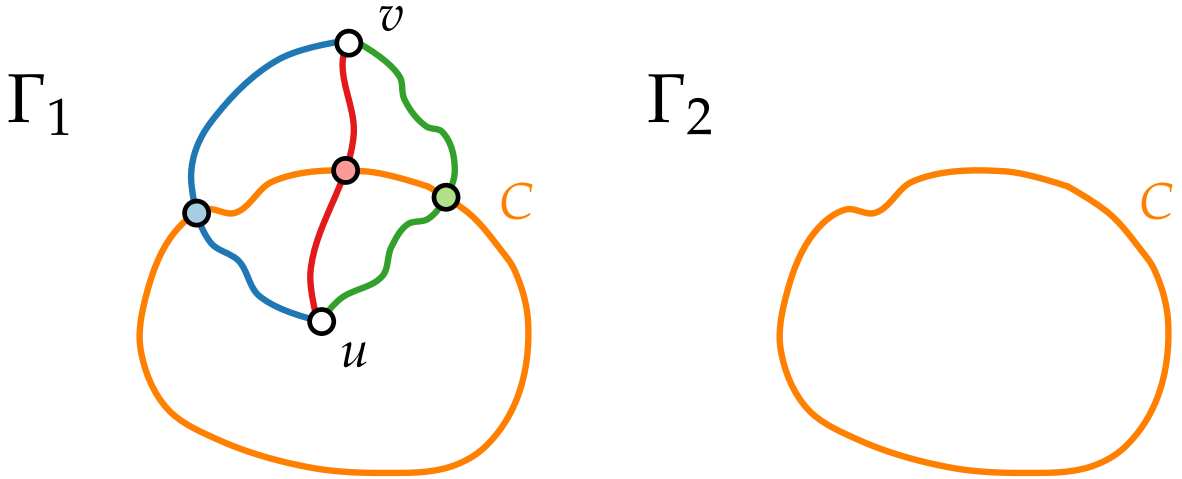

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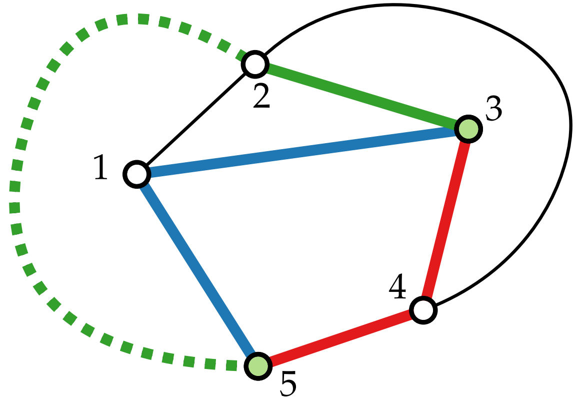

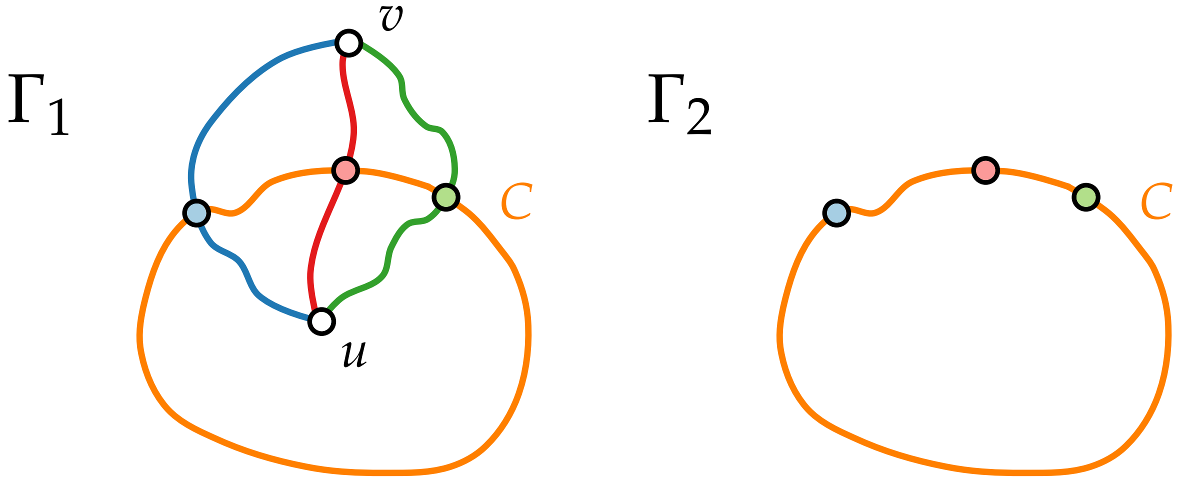

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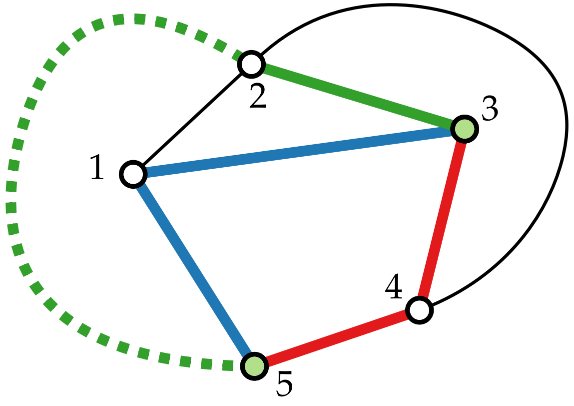

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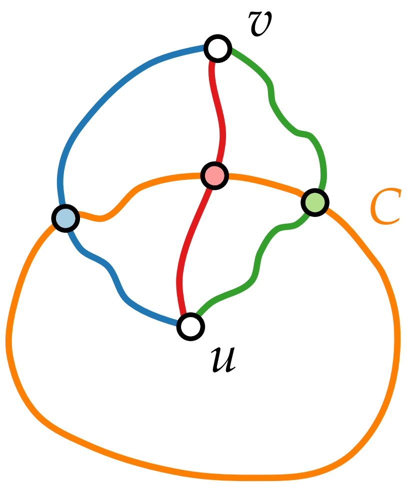

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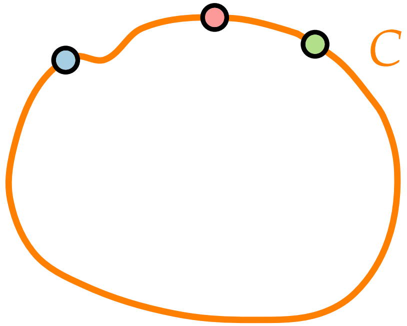

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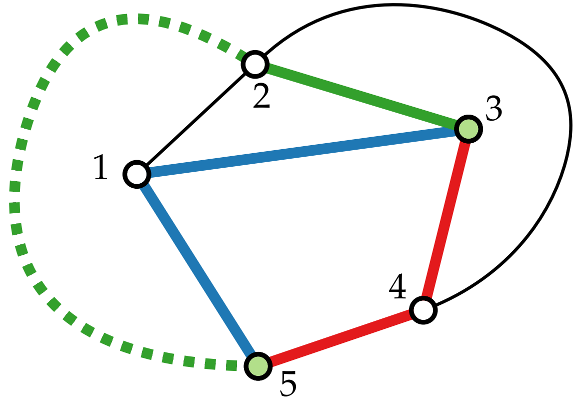

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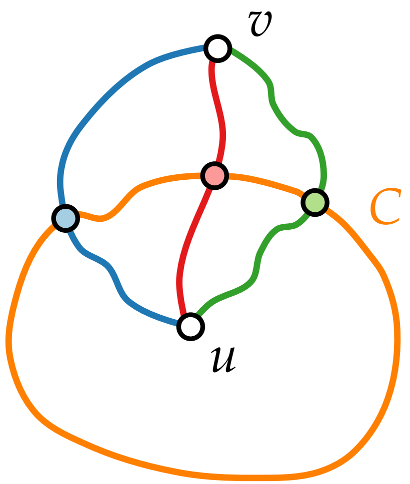

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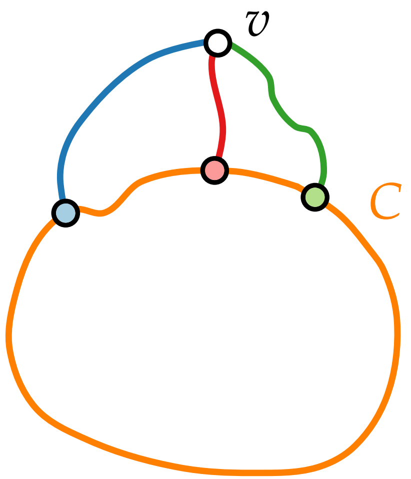

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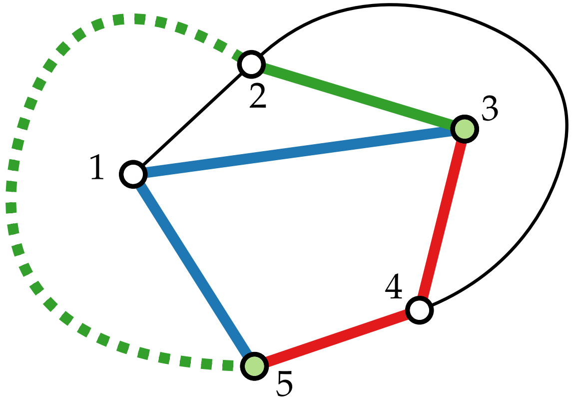

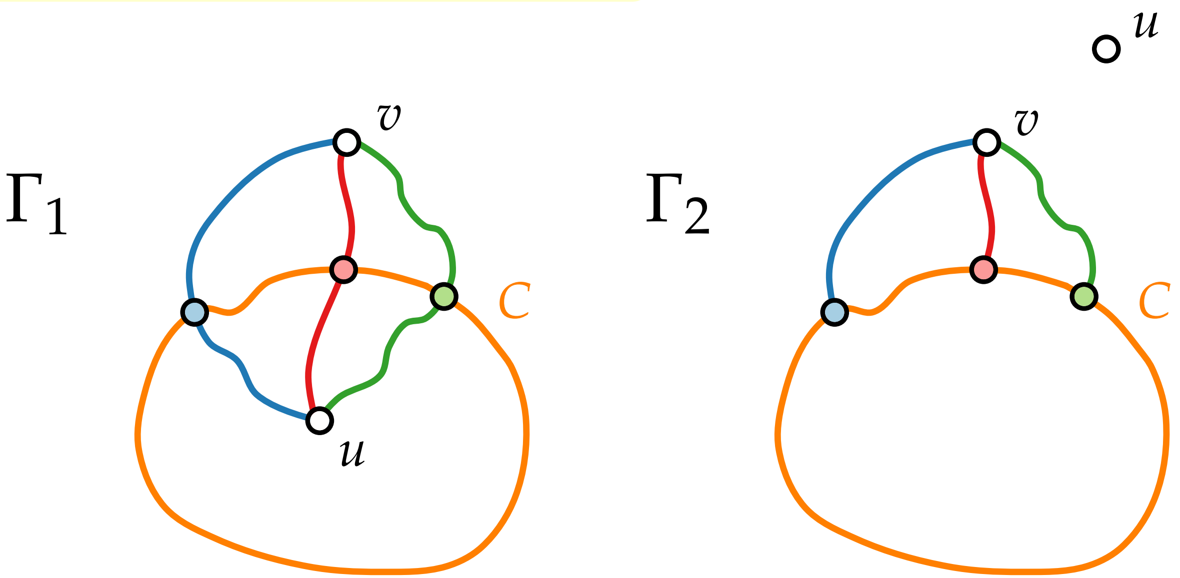

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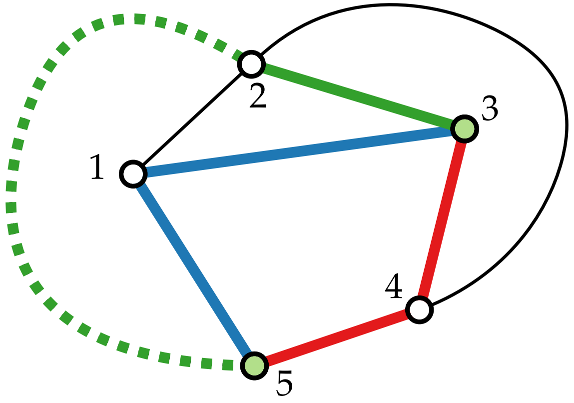

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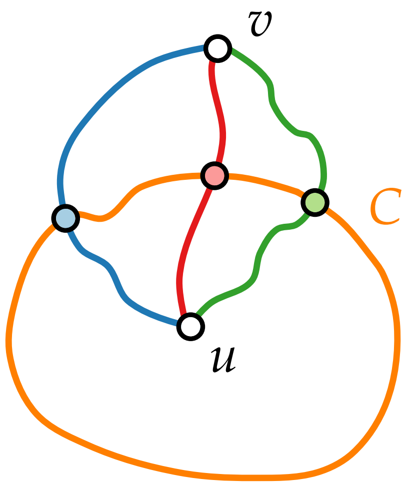

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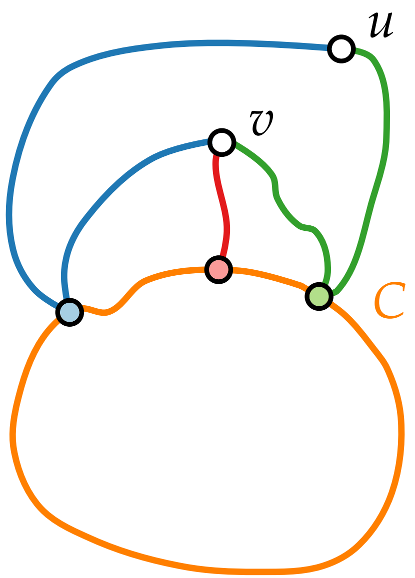

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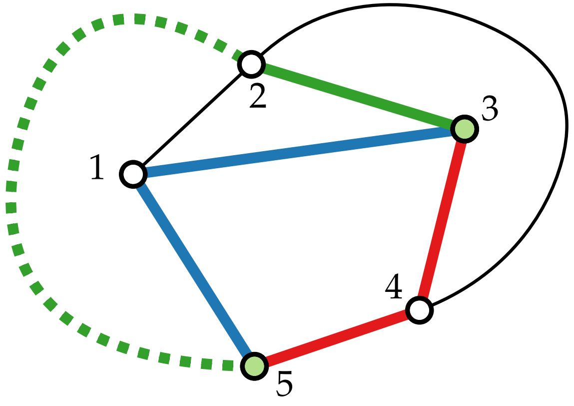

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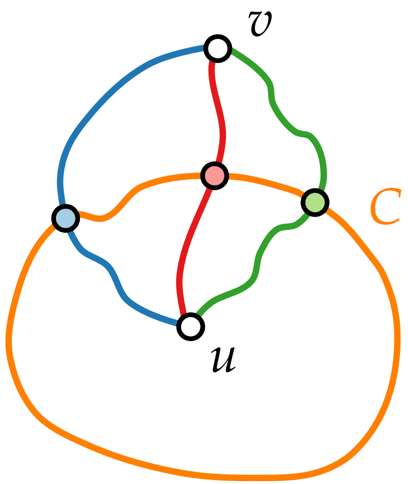

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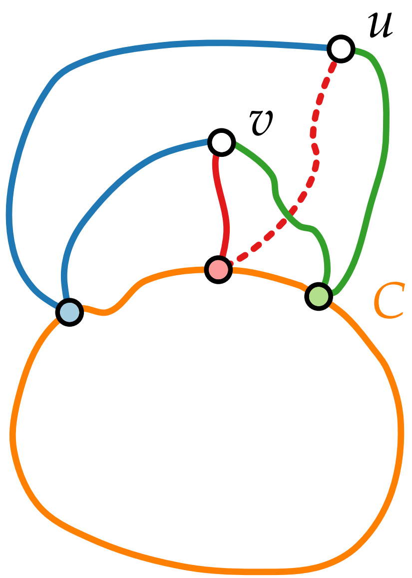

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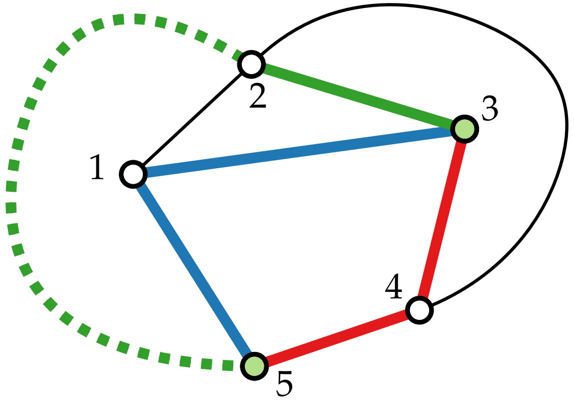

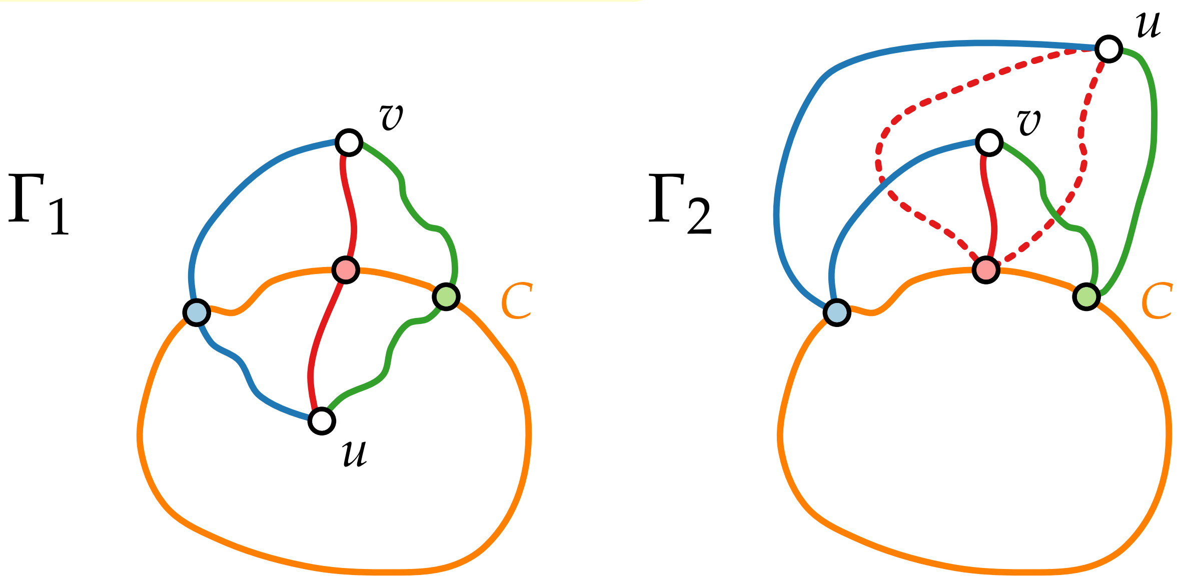

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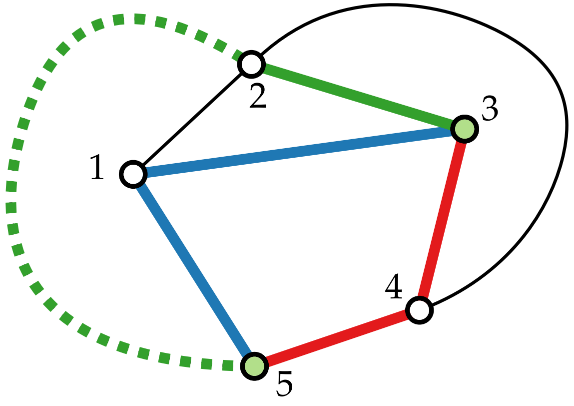

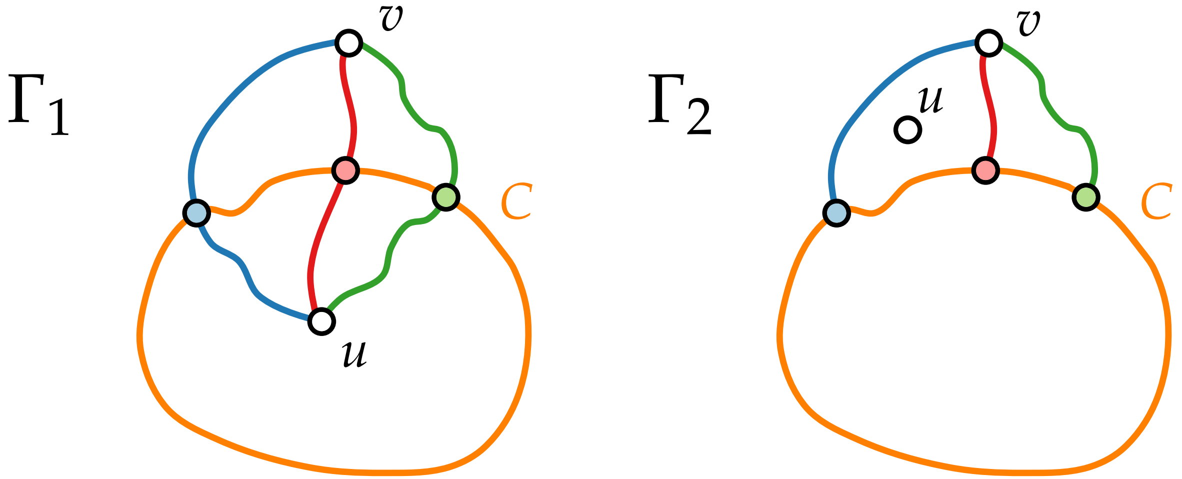

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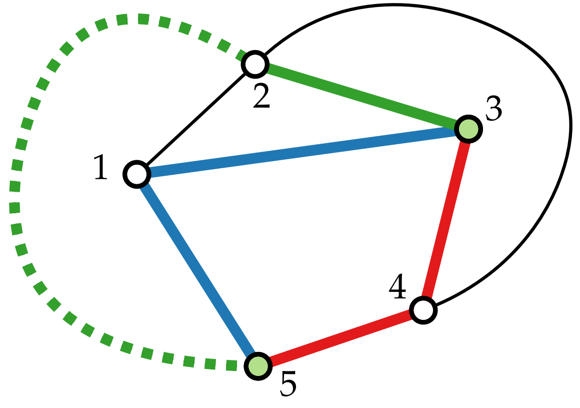

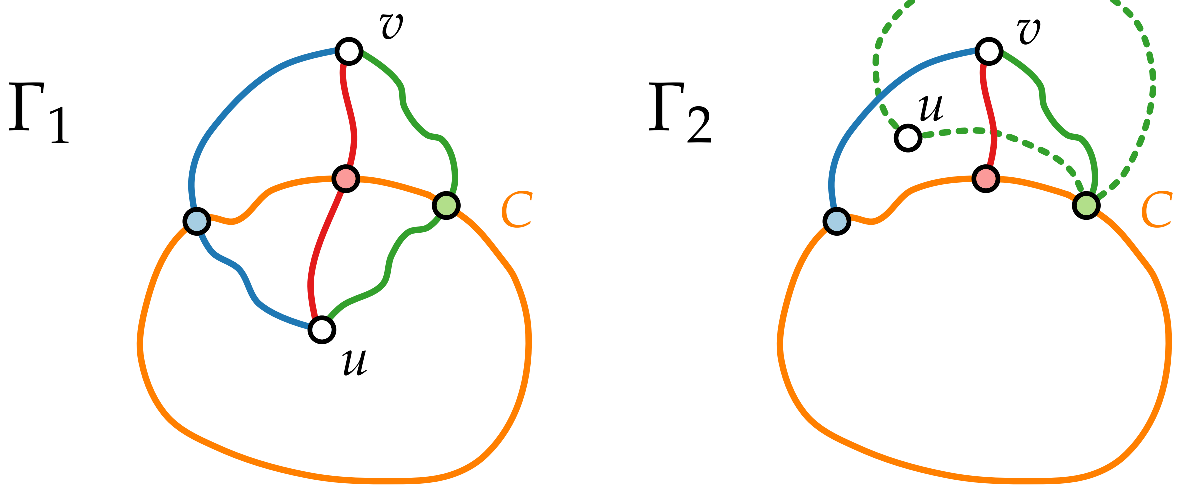

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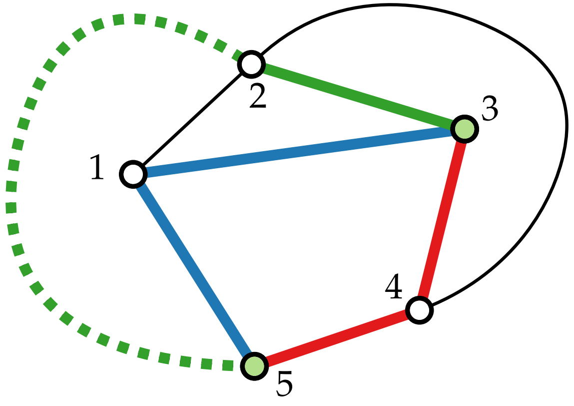
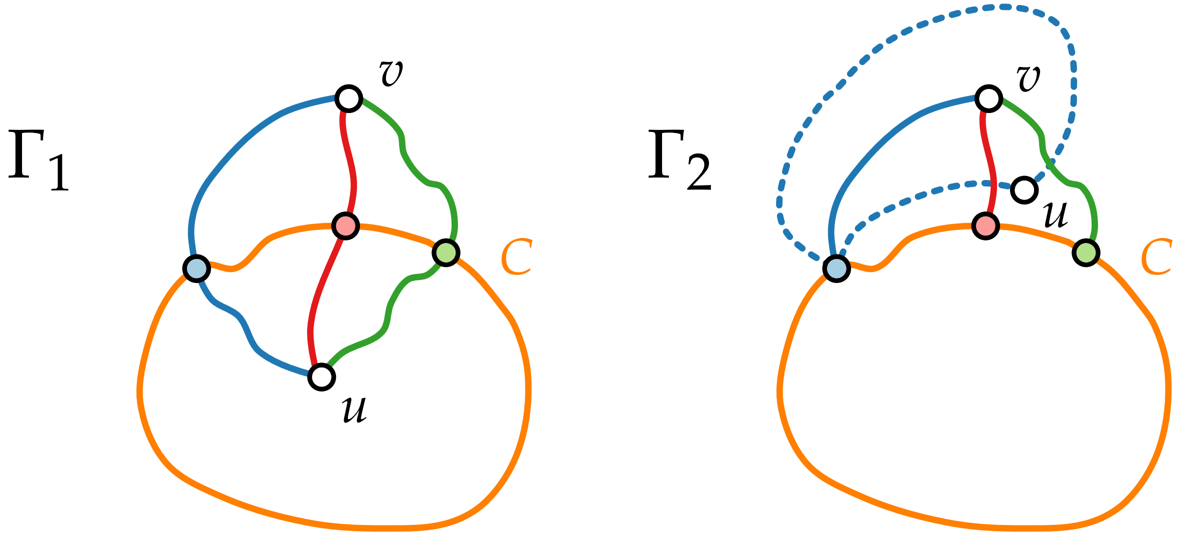
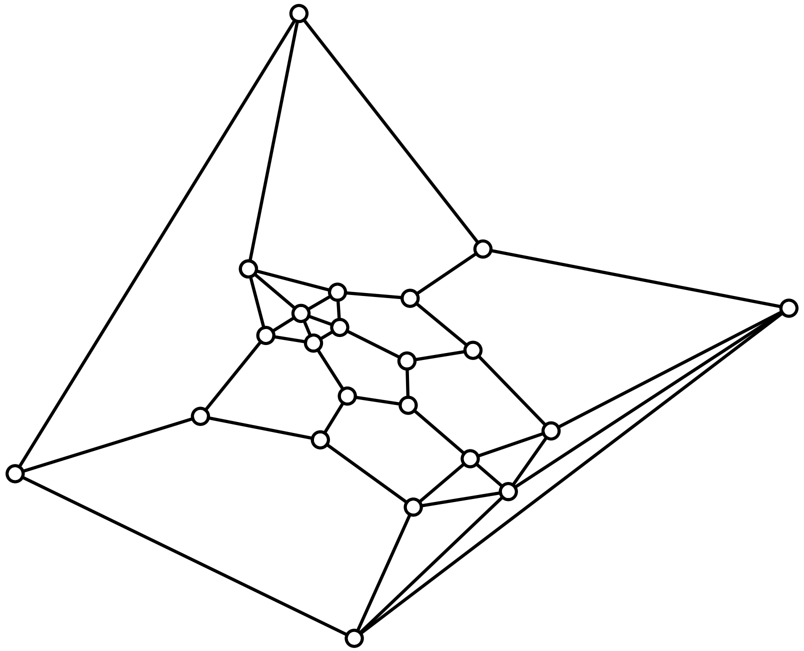




Tutte’s Theorem

Theorem. [Tutte 1963]

Let G be a 3-connected planar graph



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Theorem. [Tutte 1963]

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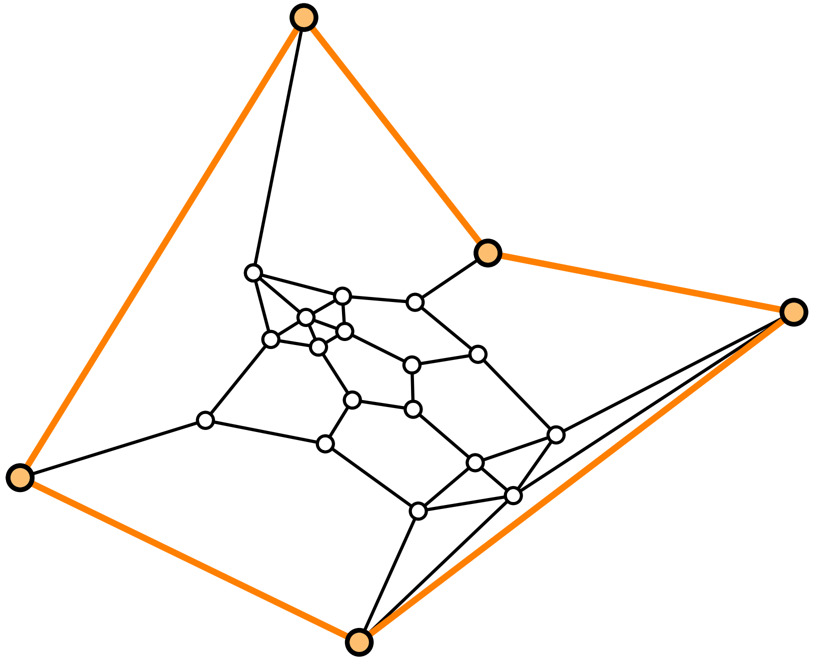

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Theorem. [Tutte 1963]

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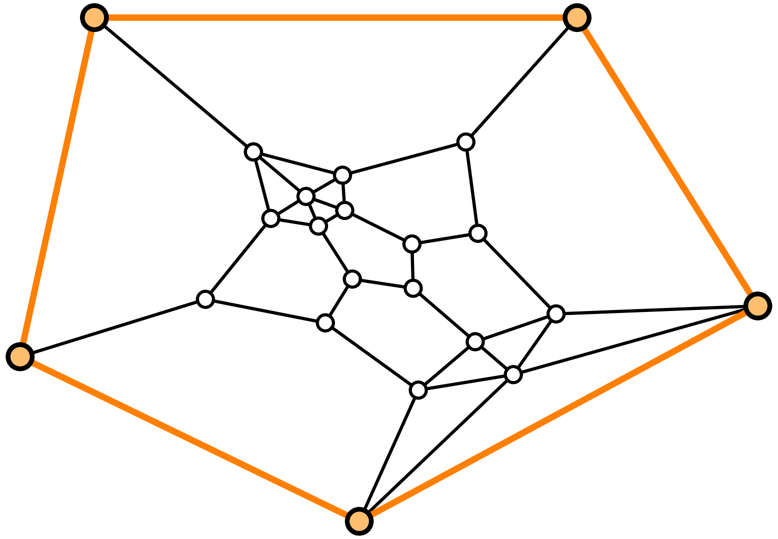

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Theorem. [Tutte 1963]

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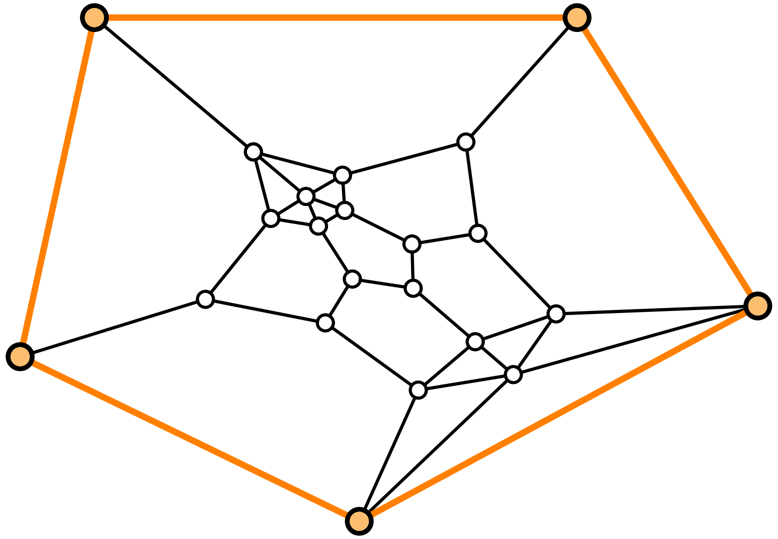
If we fix C on a strictly convex polygon, then the Tutte drawing of G is planar



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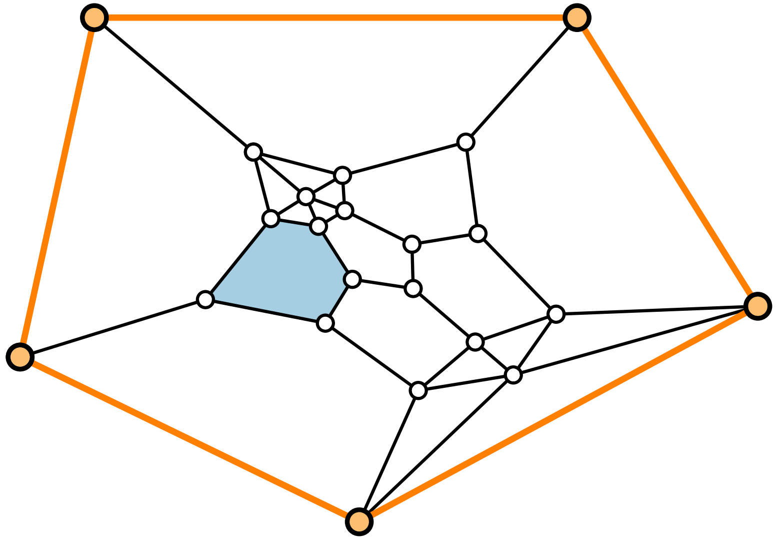
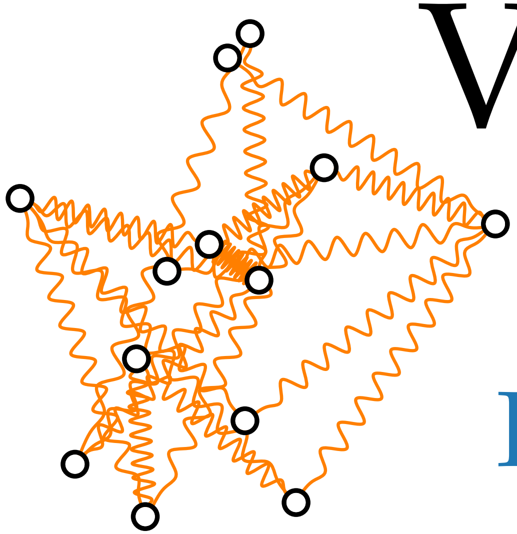
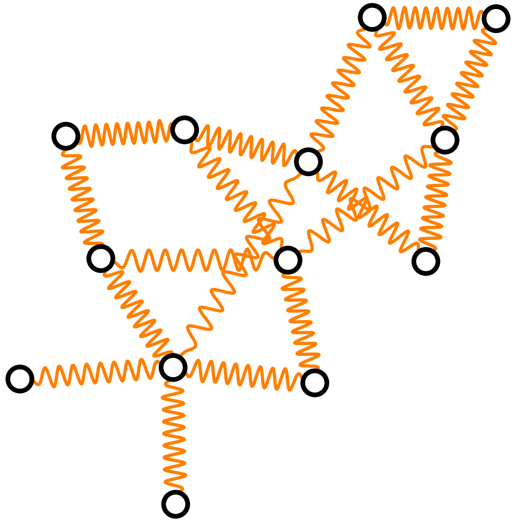


Universität Trier

Visualization of Graphs



Lecture 3:
Force-Directed Drawing Algorithms

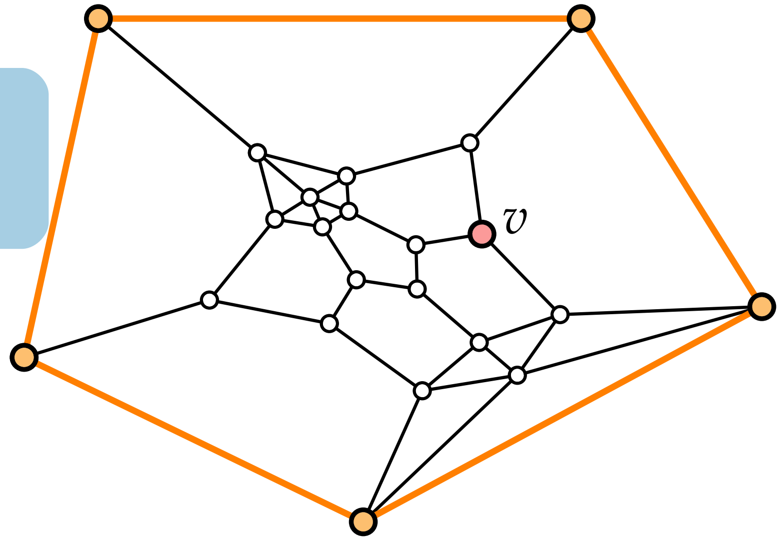


Part VI:
Proof of Tutte's Theorem

Philipp Kindermann

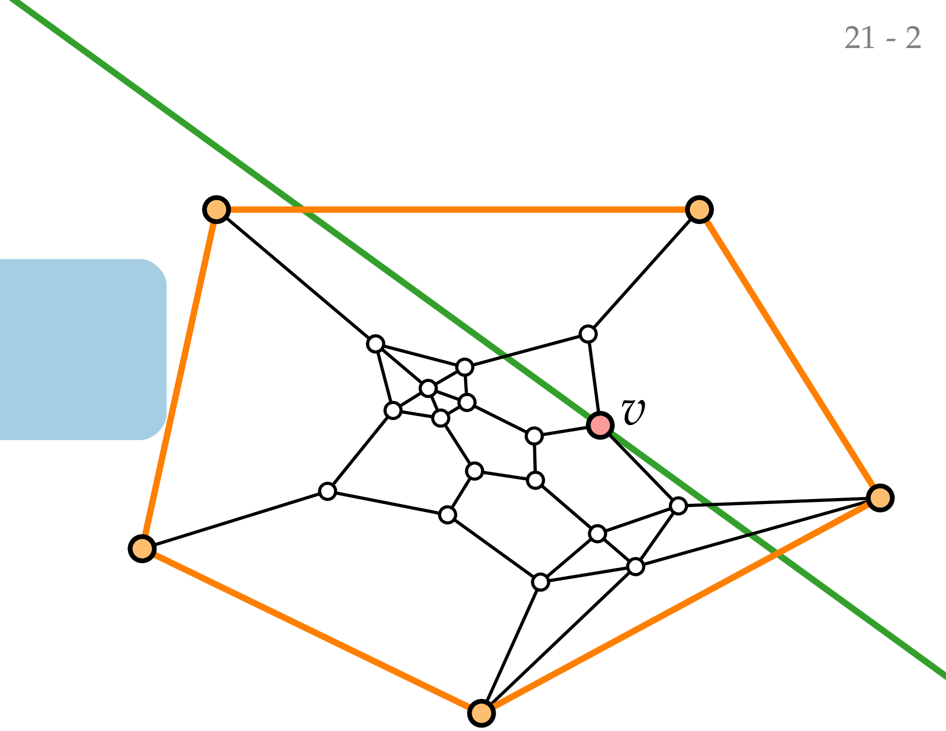
Properties of Tutte Drawings

Property 1. Let $v \in V$ free



Properties of Tutte Drawings

Property 1. Let $v \in V$ free, ℓ line through v.



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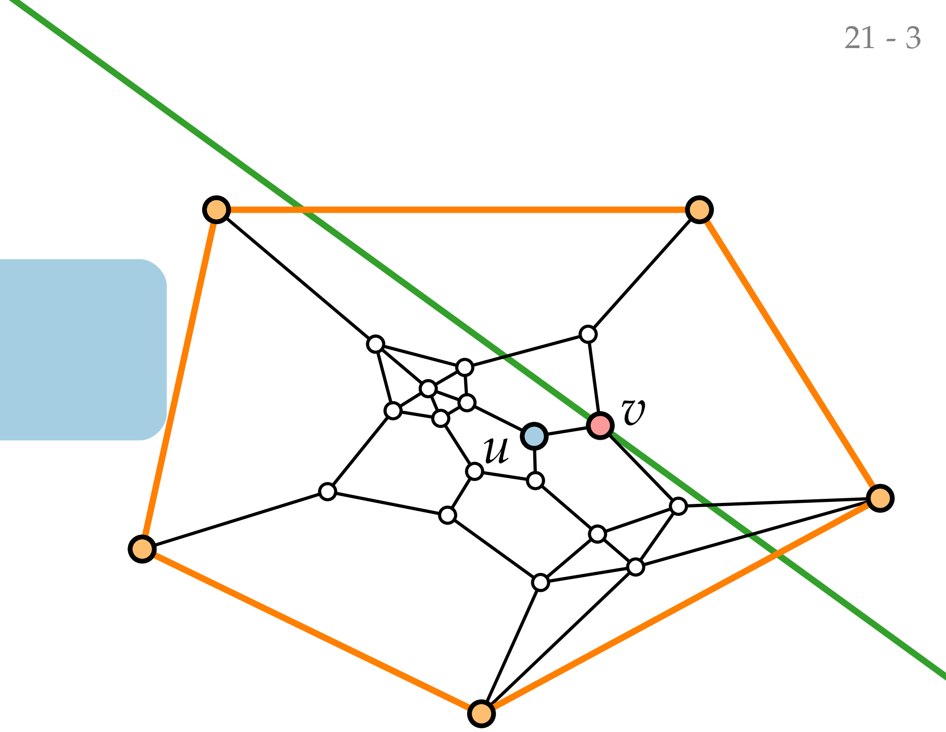

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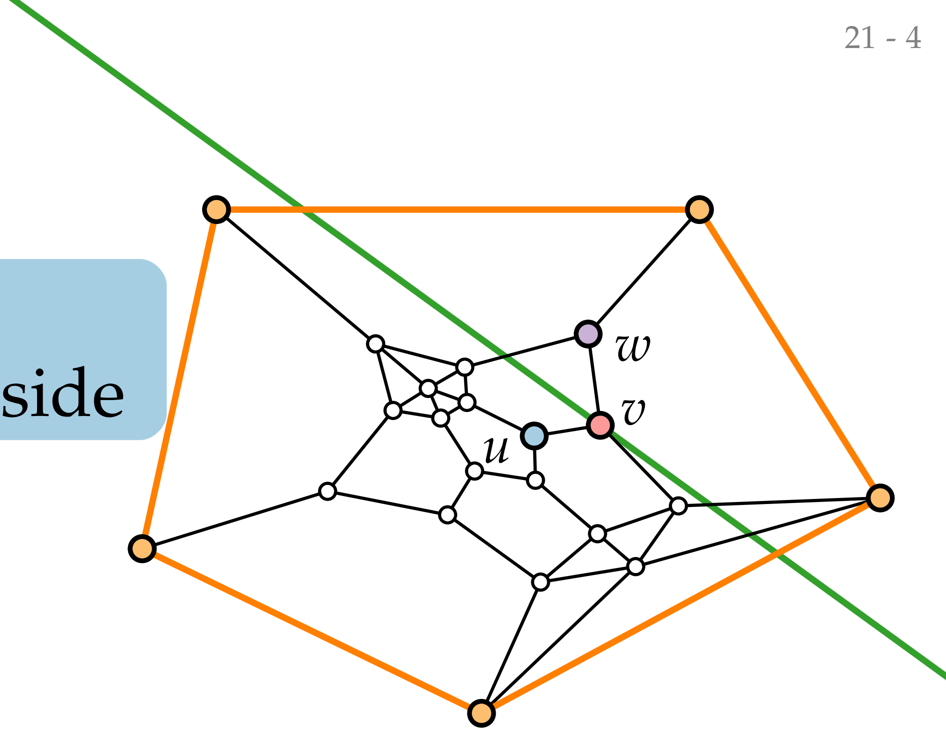
$\exists uv \in E$ on one side of $\ell \Rightarrow \exists vw \in E$ on other side



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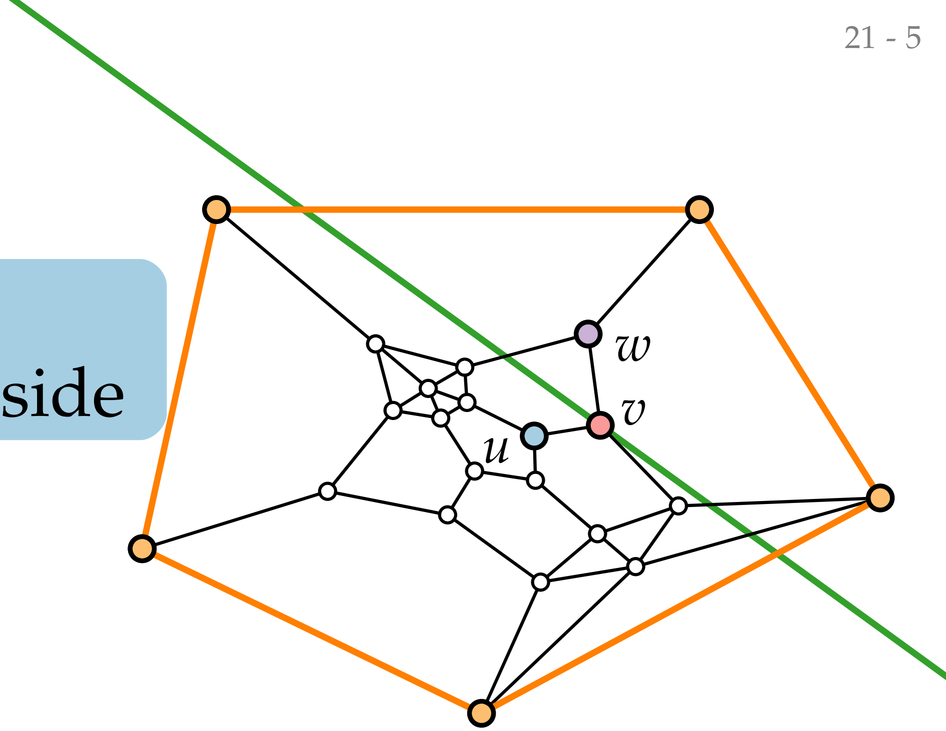

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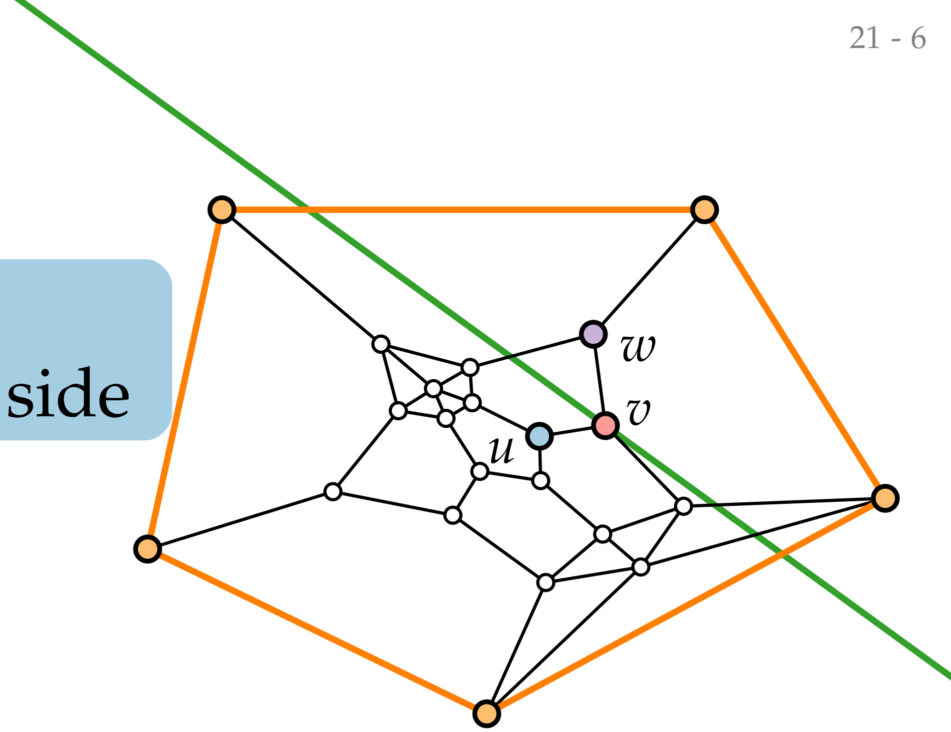
Property 2. All free vertices lie inside C.



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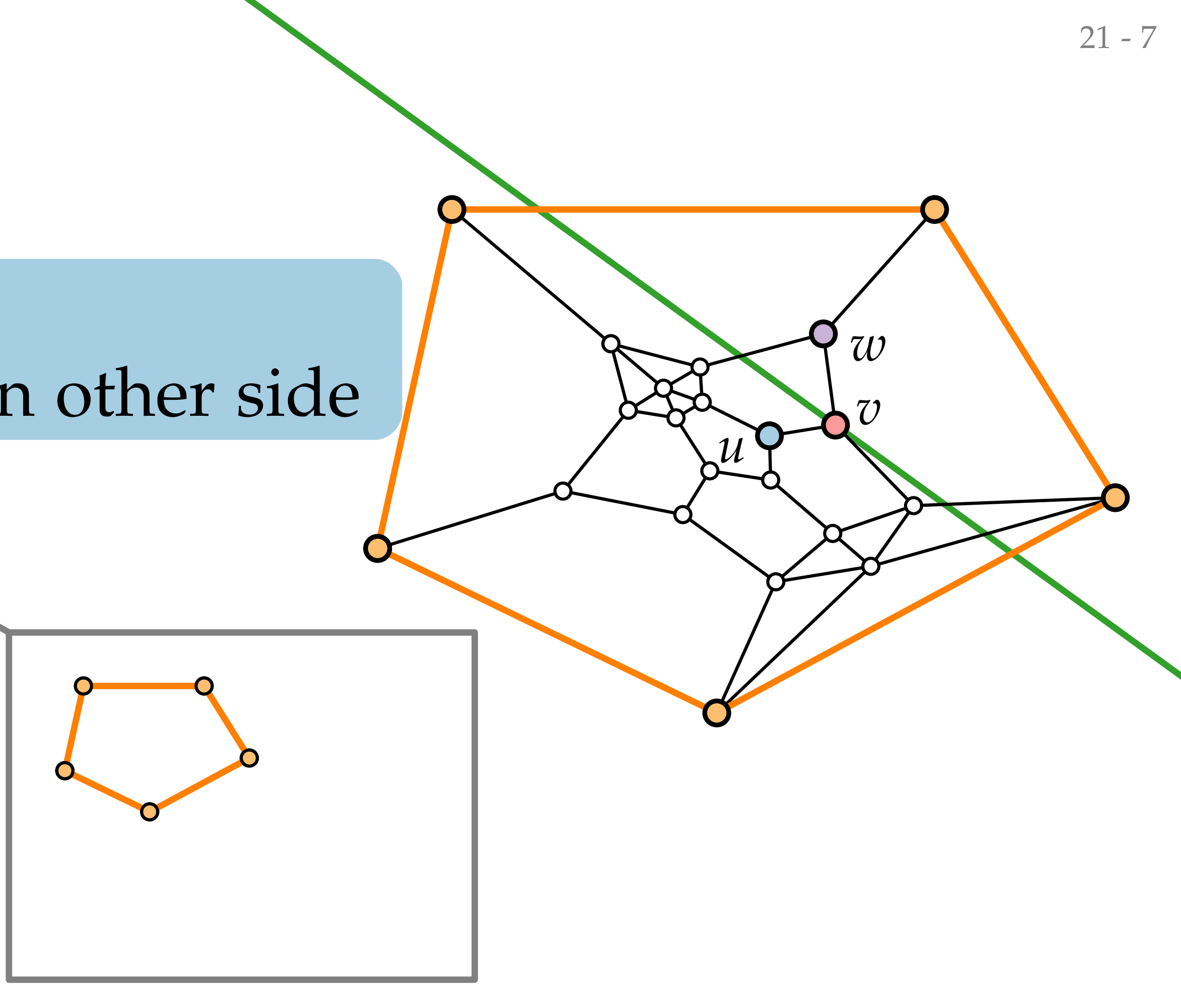

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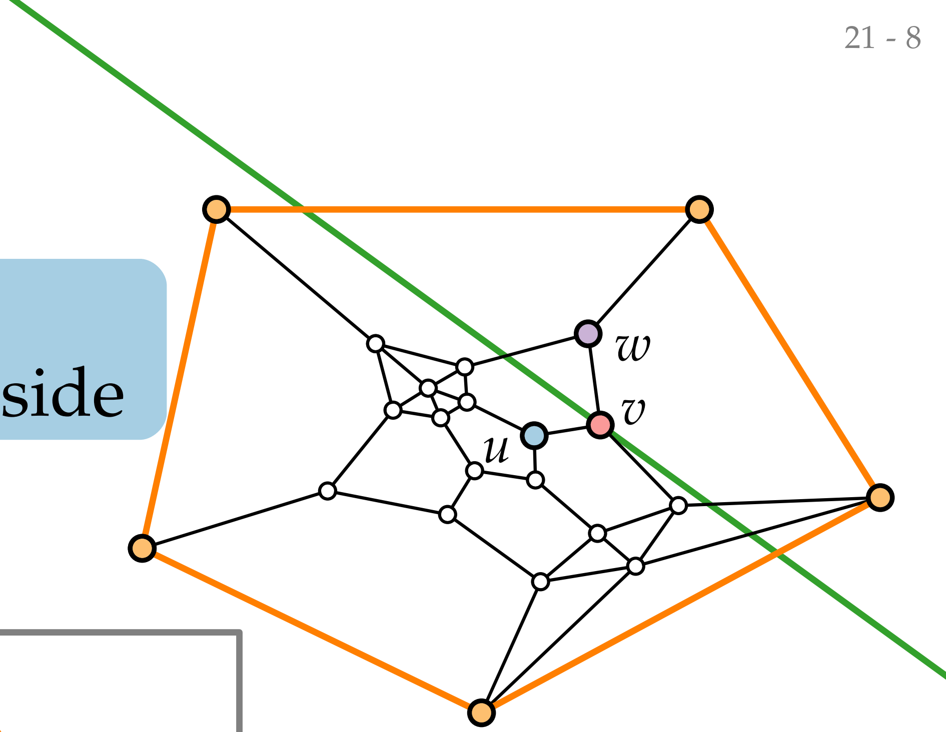

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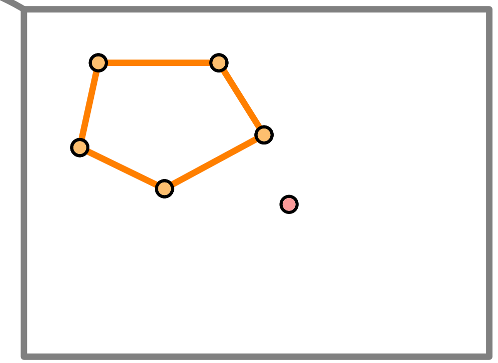

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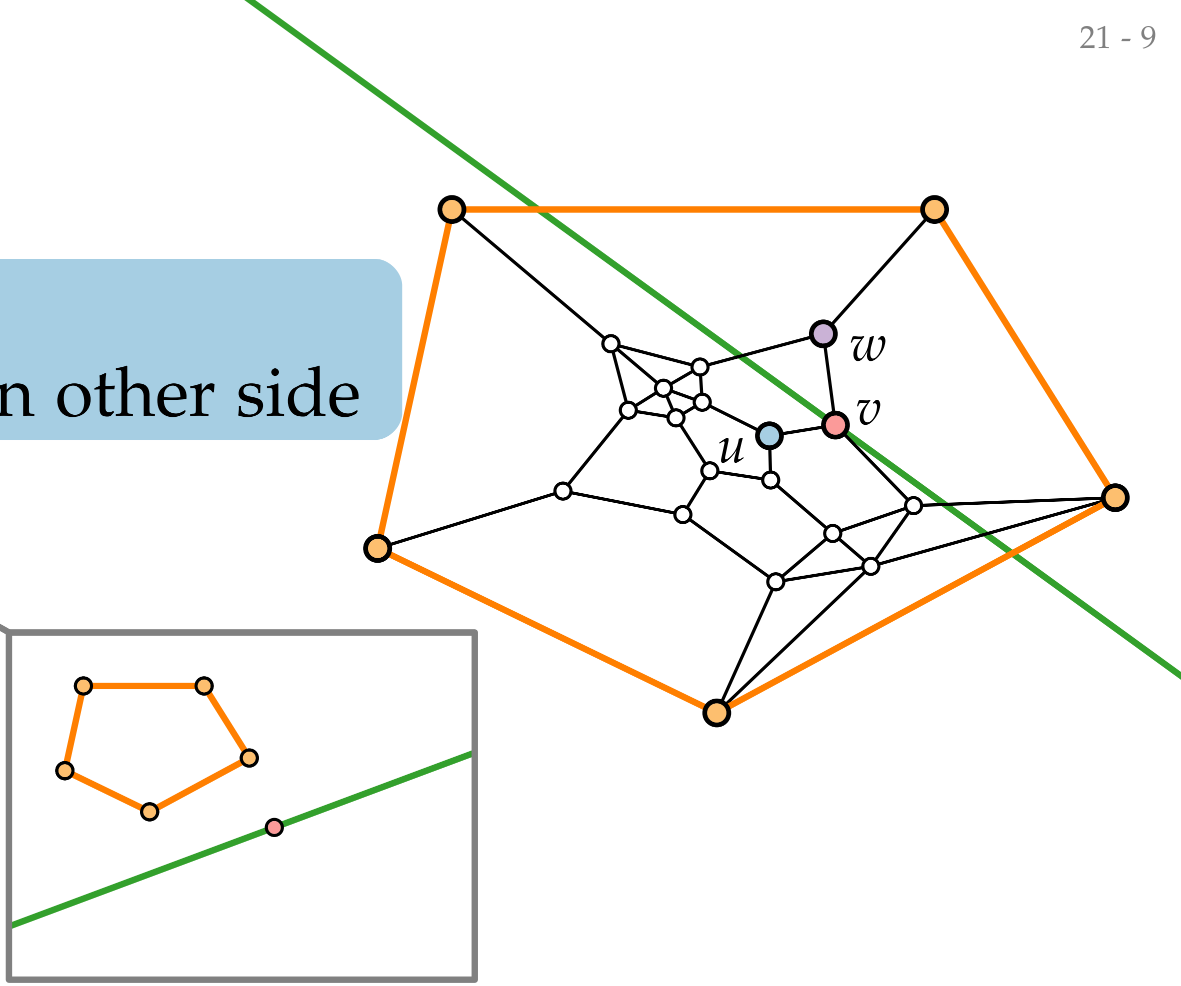

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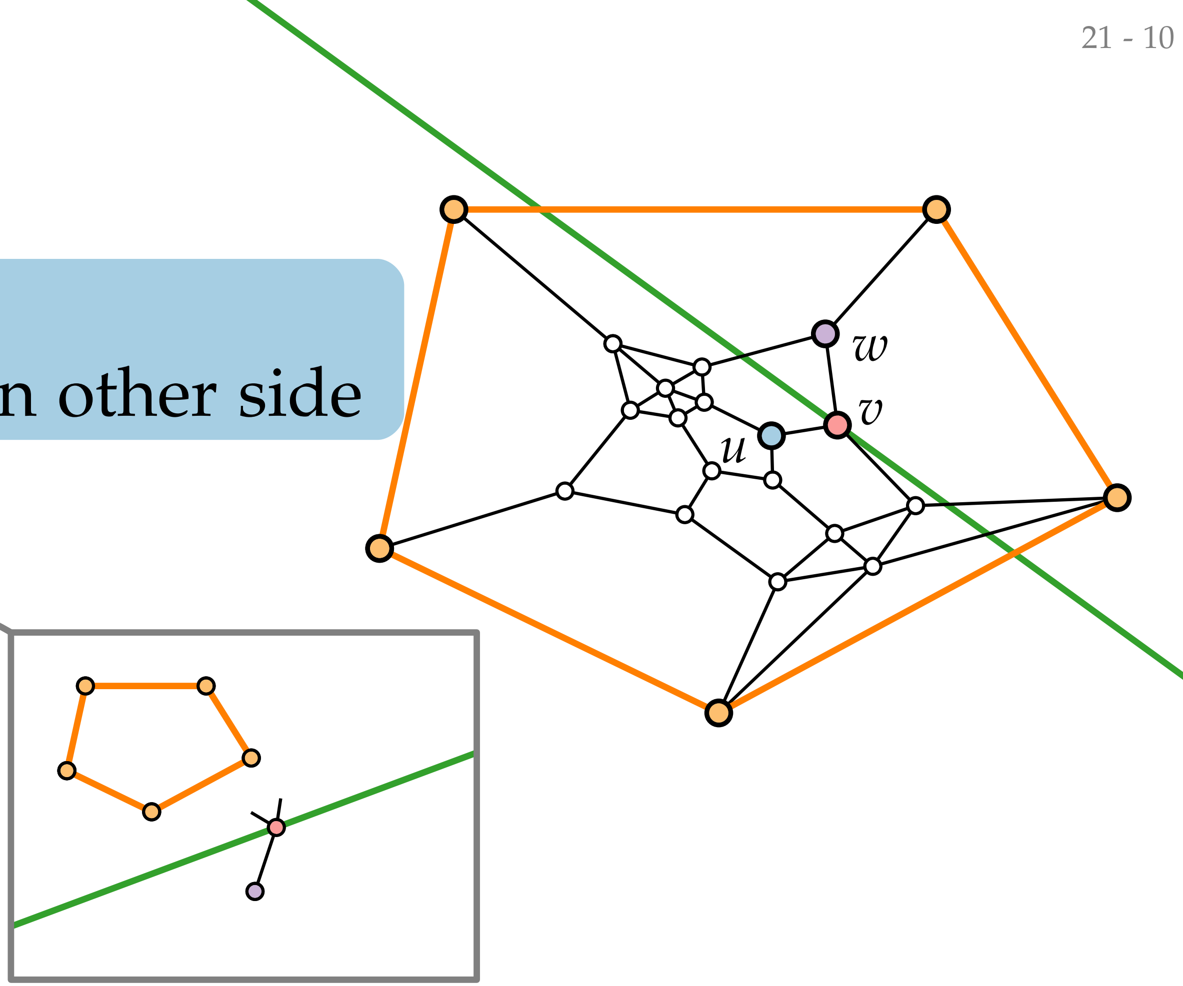

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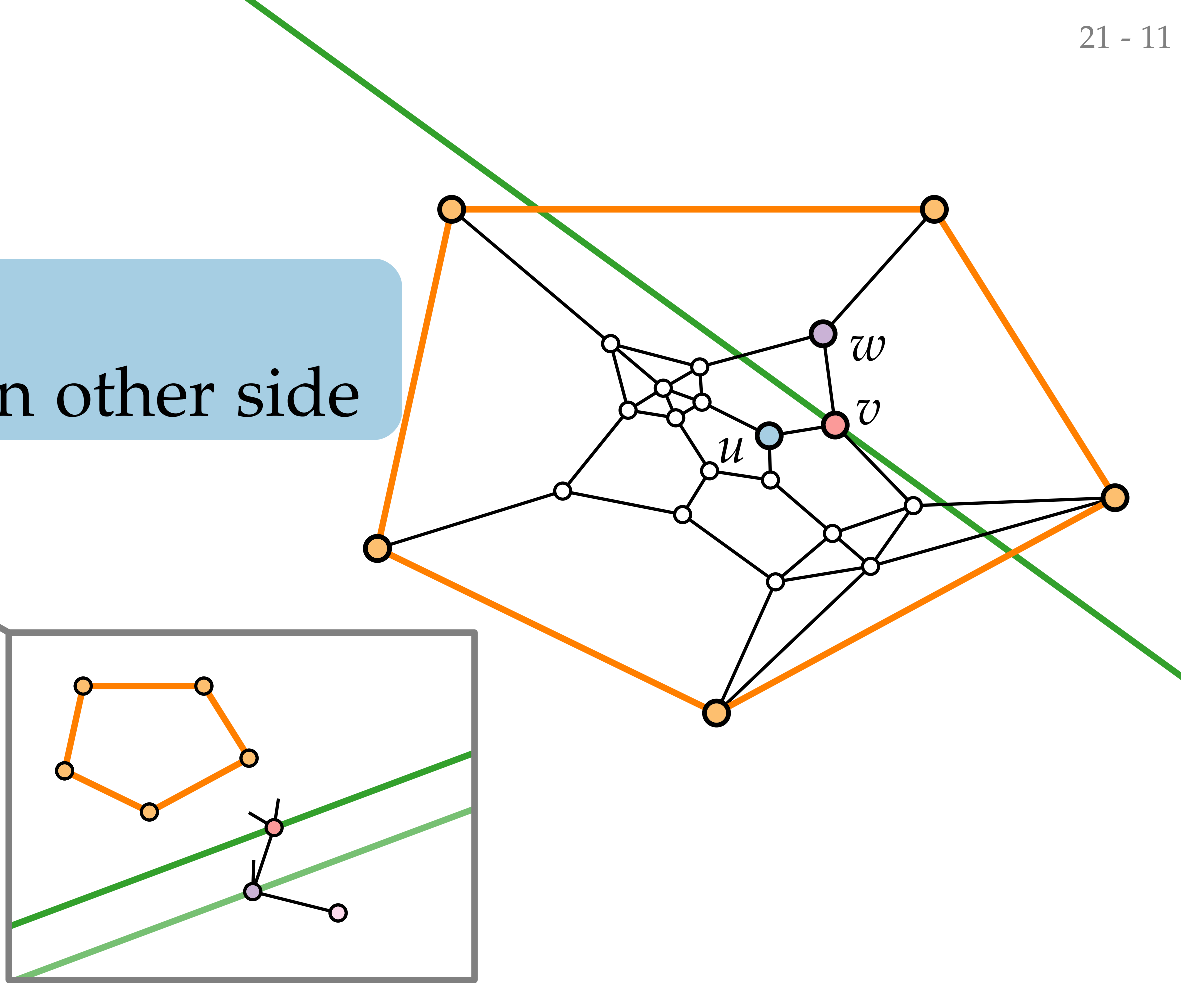

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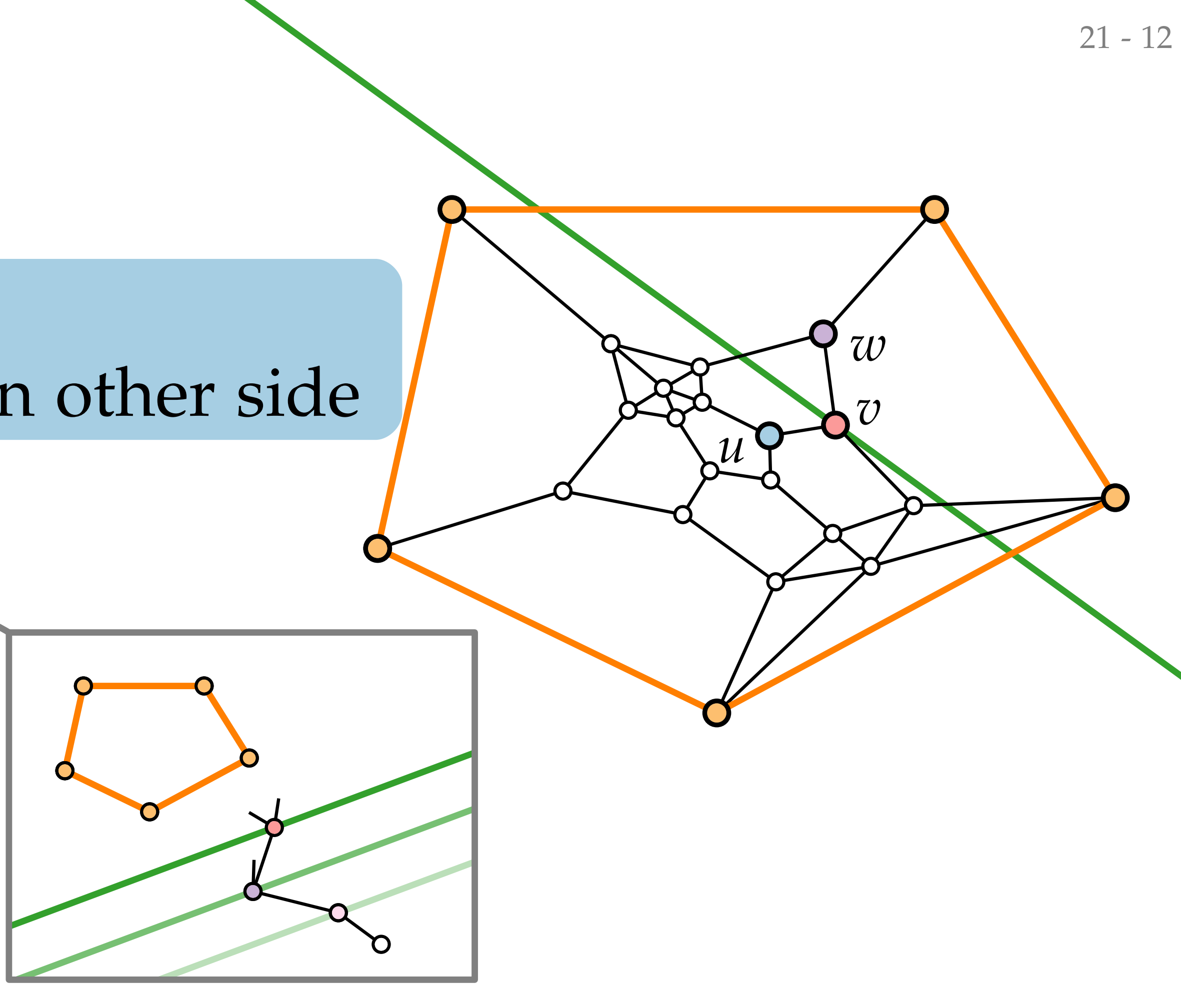

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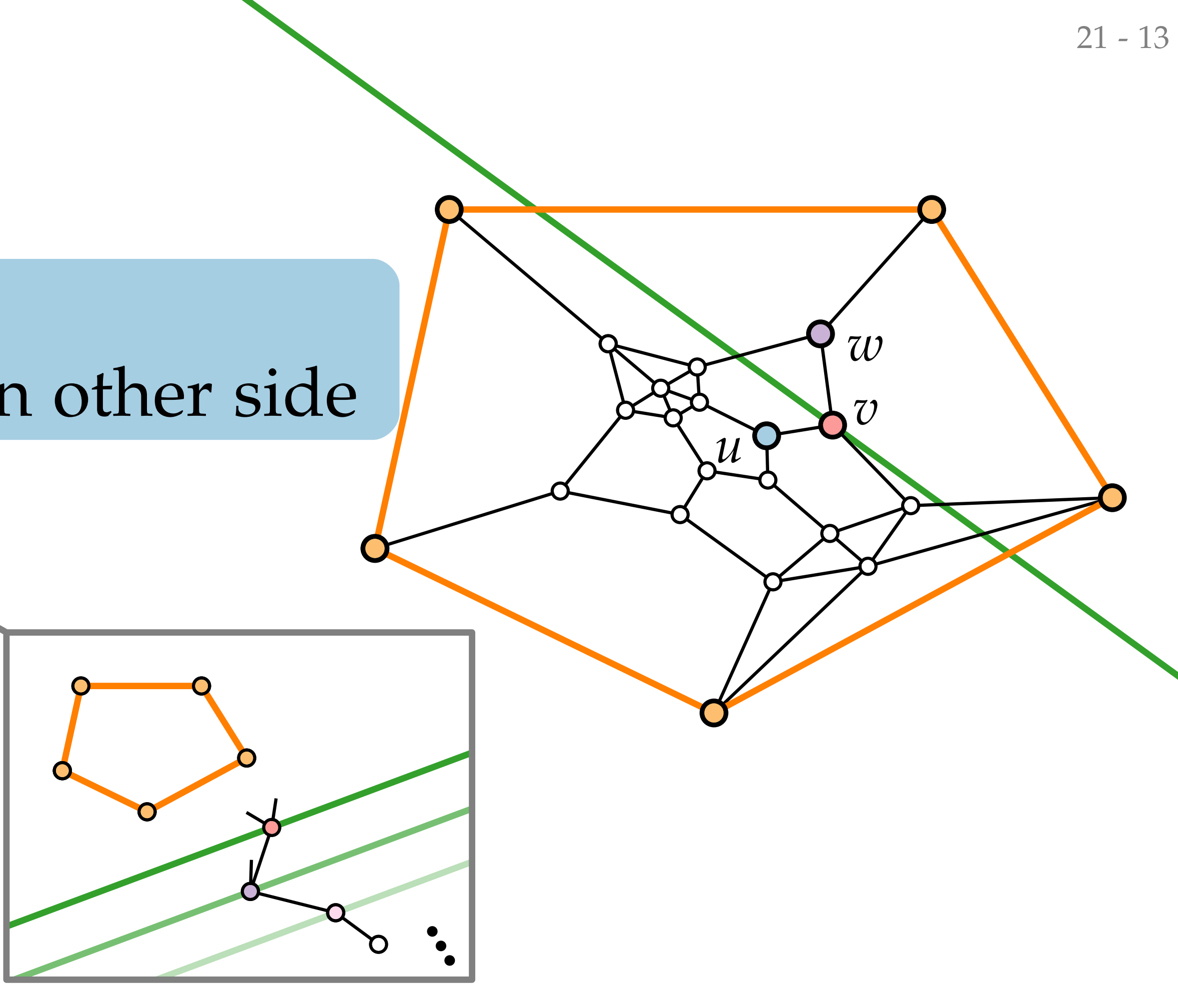

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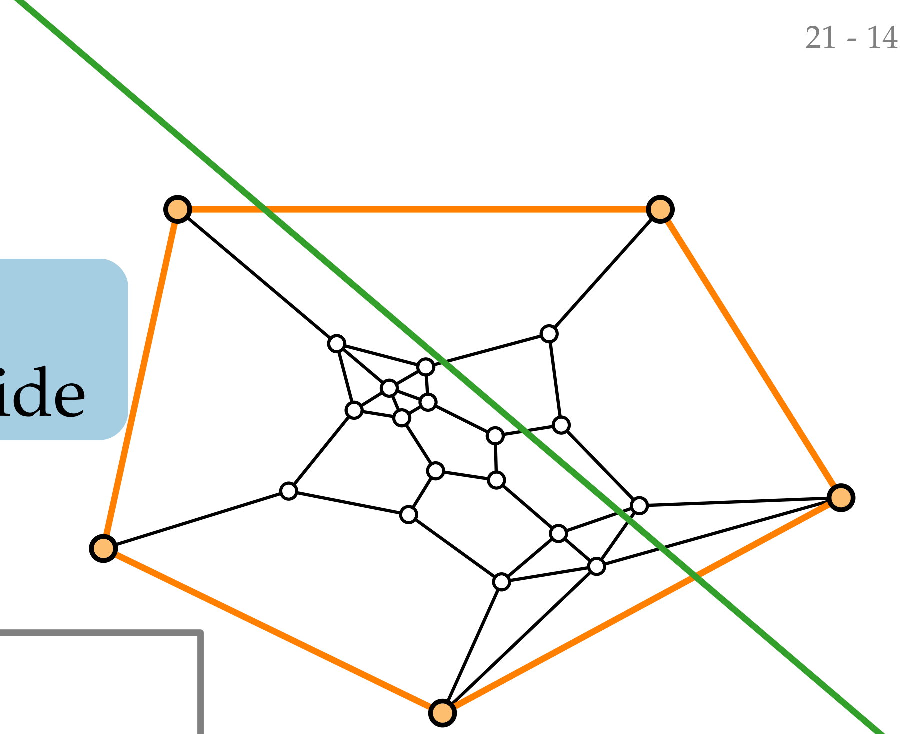

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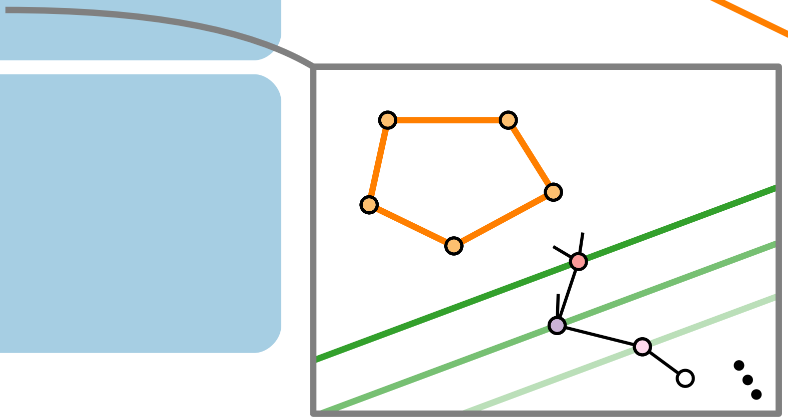

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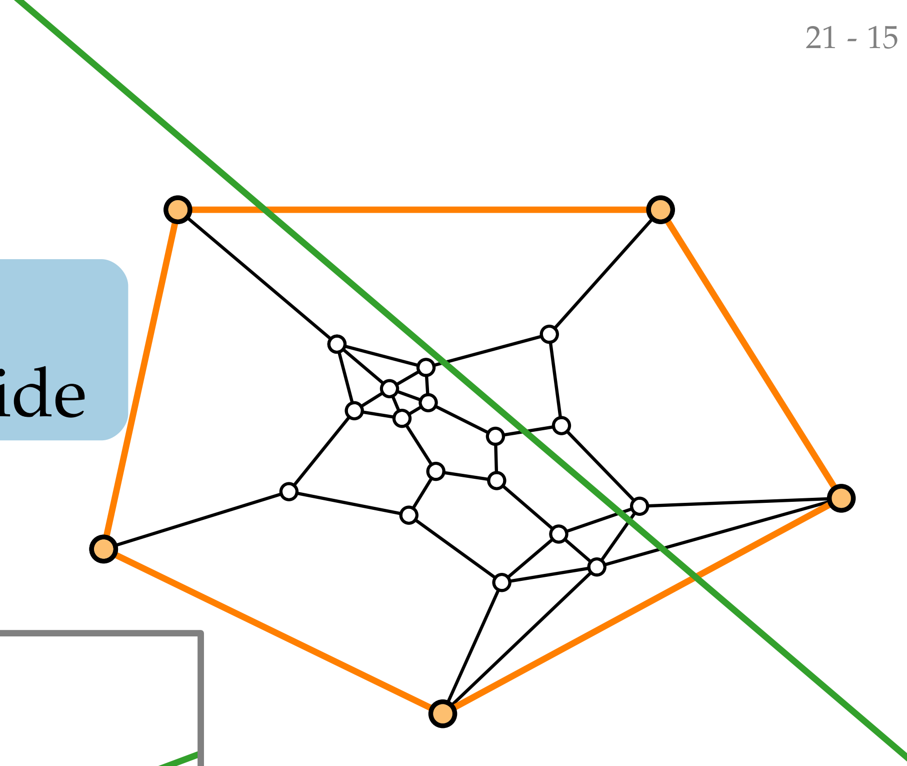

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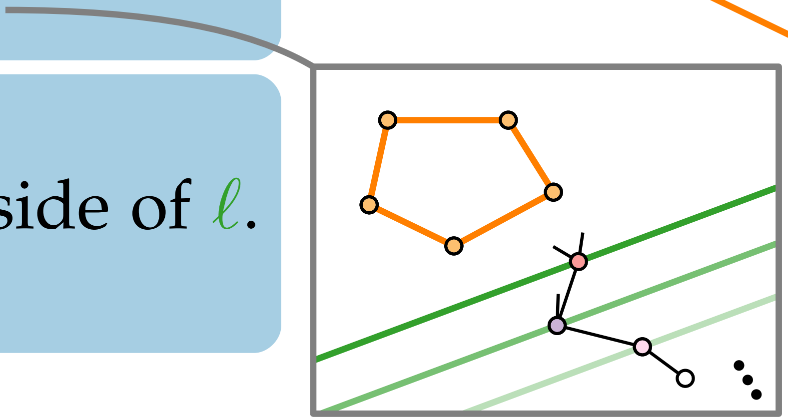

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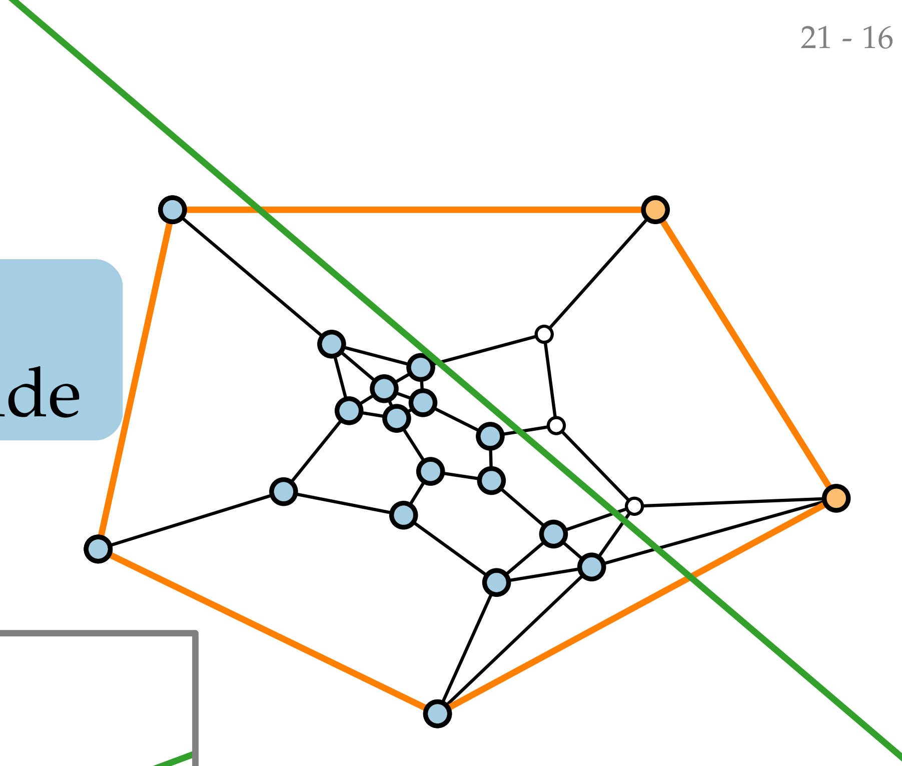

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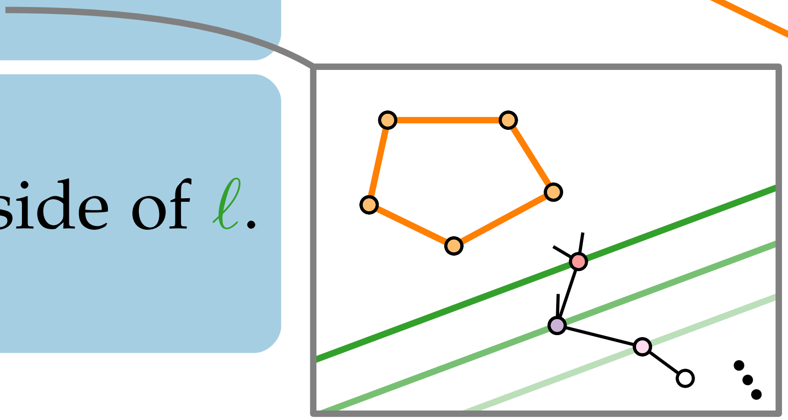

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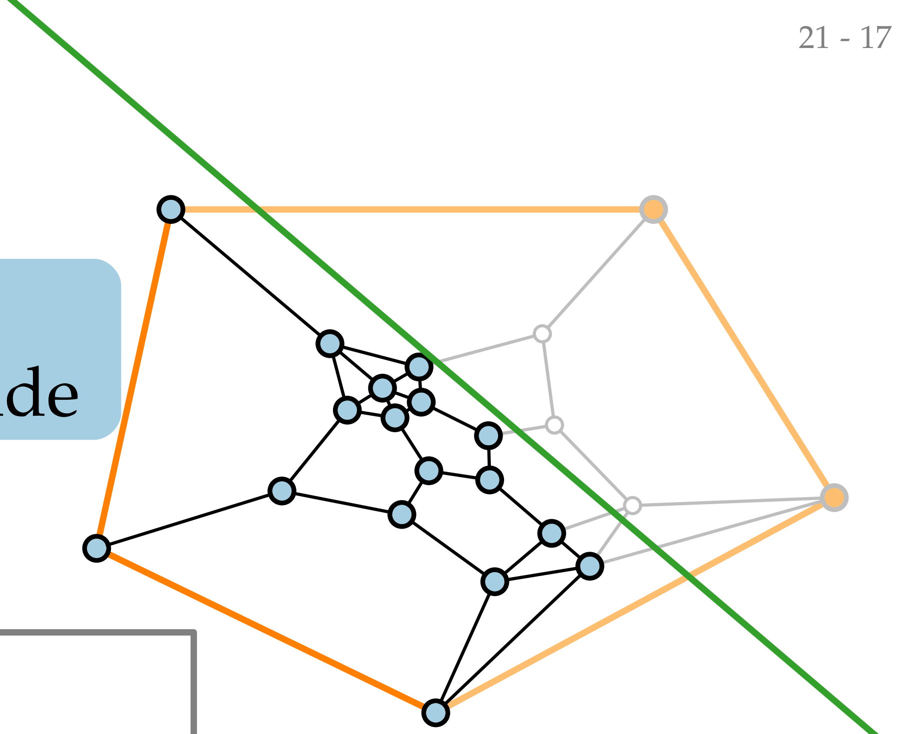

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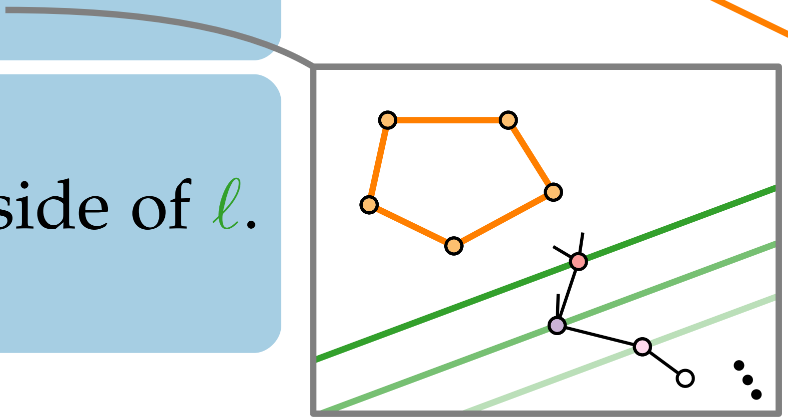

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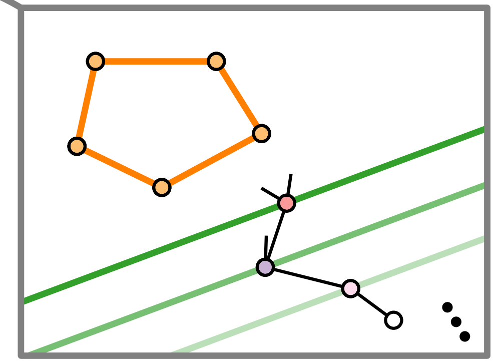

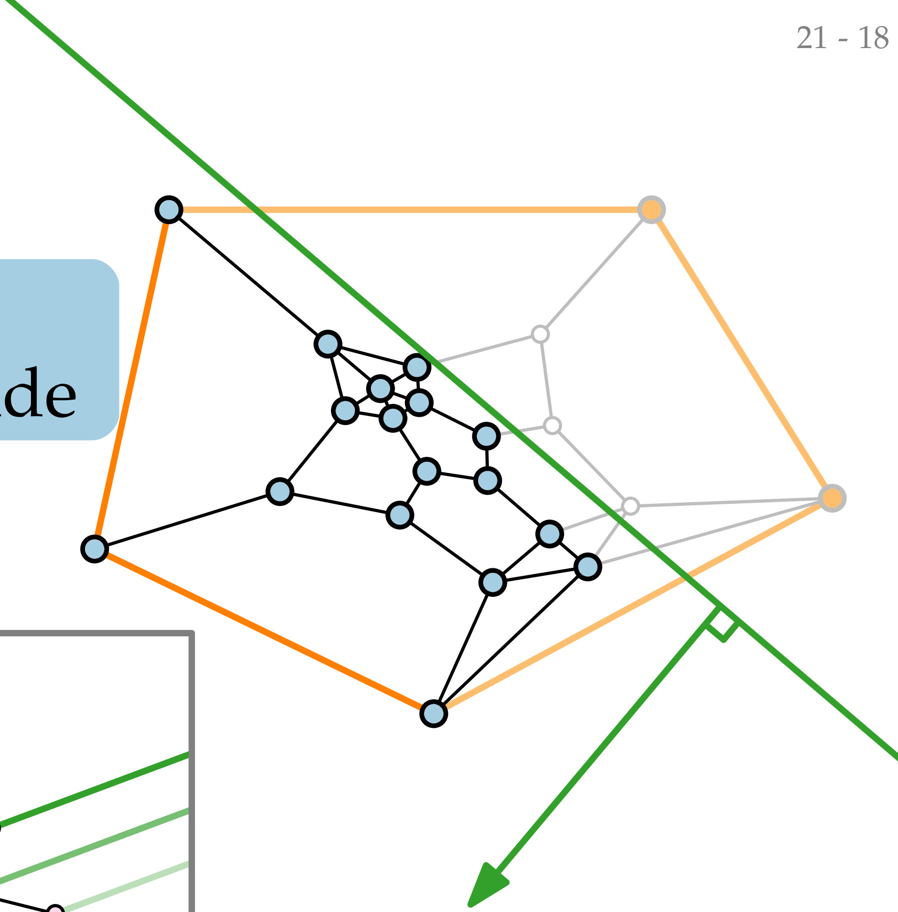

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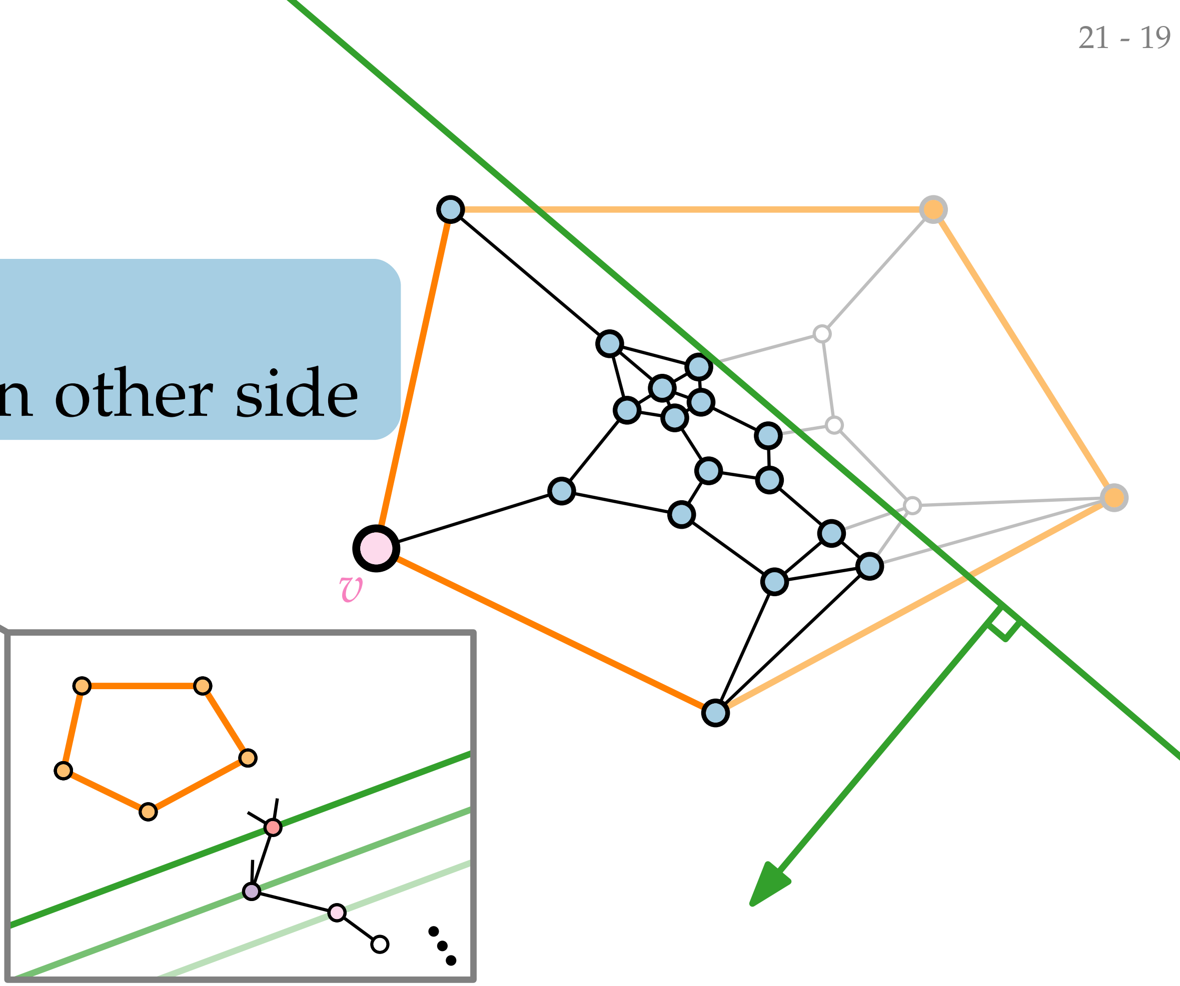
v furthest away from ℓ



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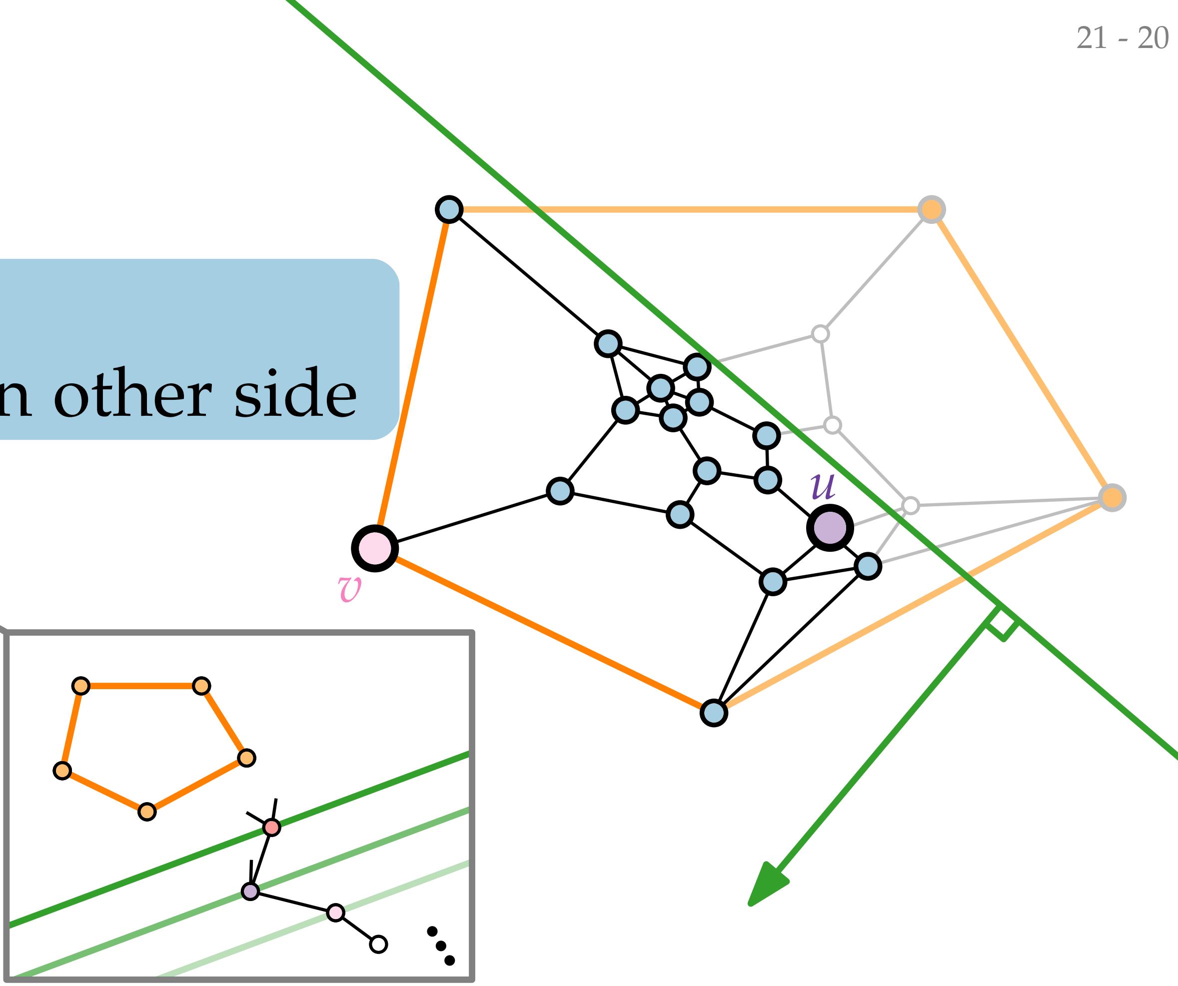
v furthest away from ℓ
Pick any vertex u



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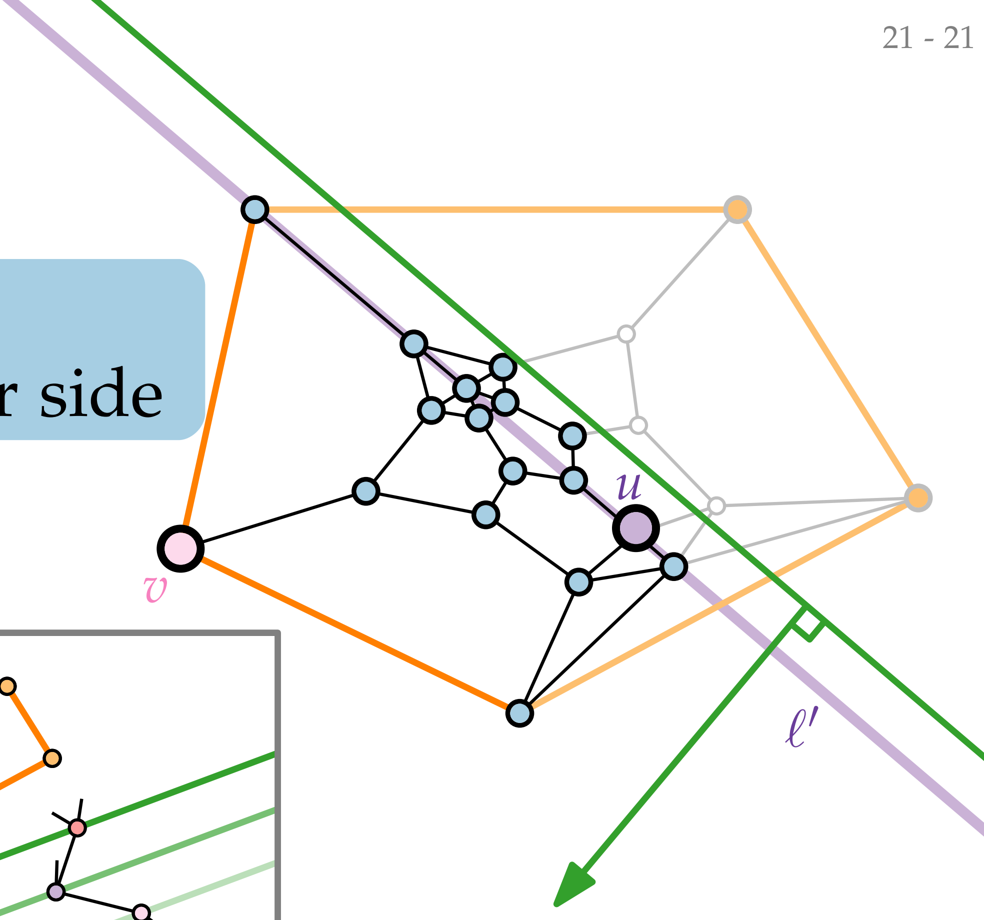

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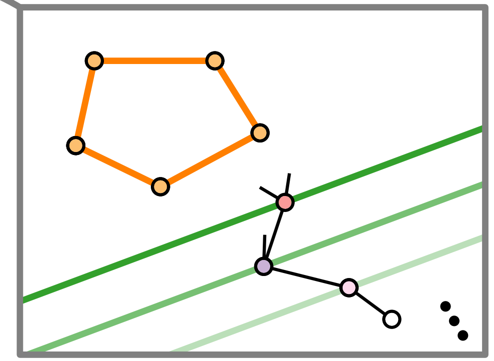

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v furthest away from ℓ
Pick any vertex u, ℓ' parallel to ℓ throught u

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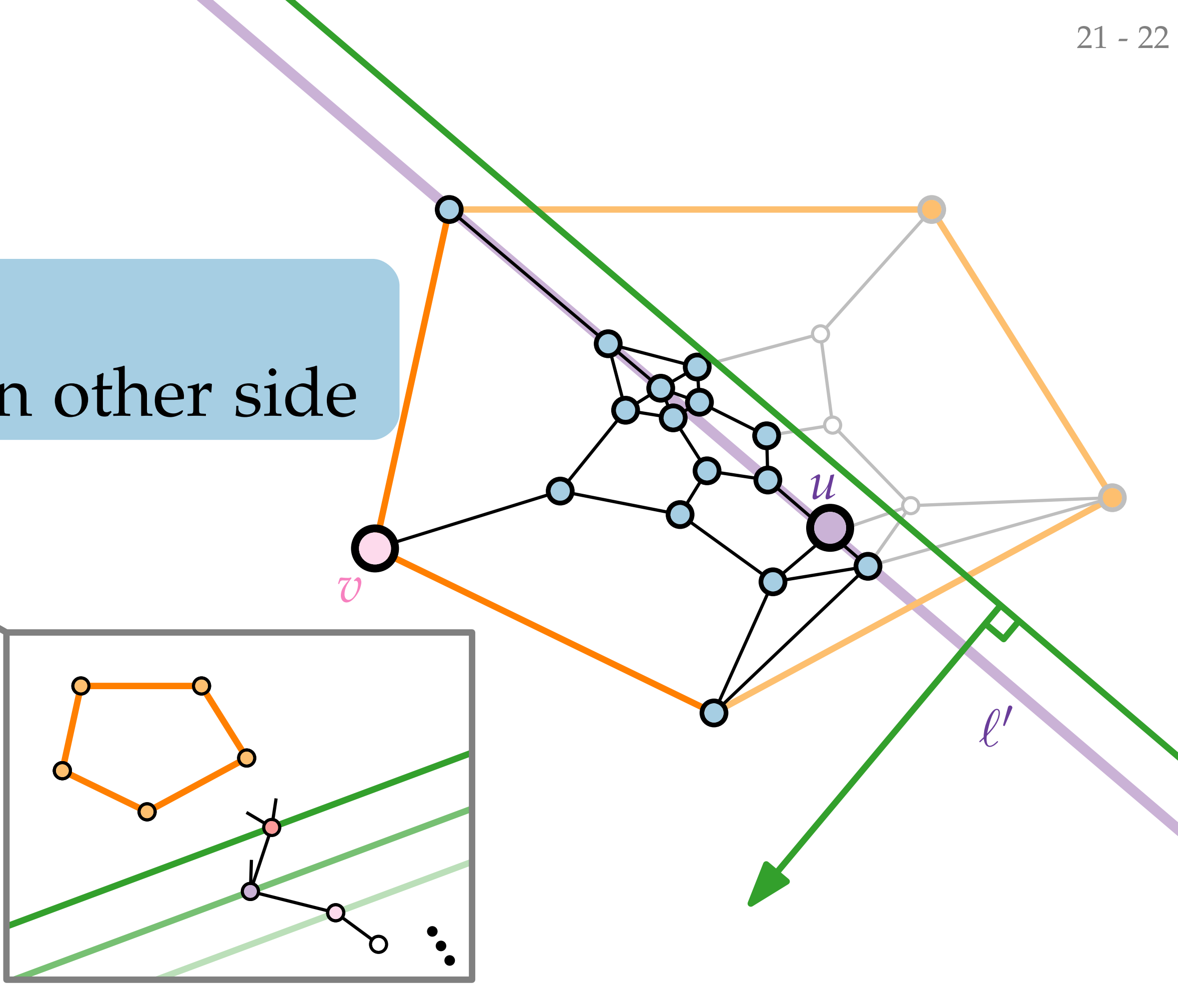
Pick any vertex u , ℓ' parallel to ℓ throught u
G connected, v not on ℓ'



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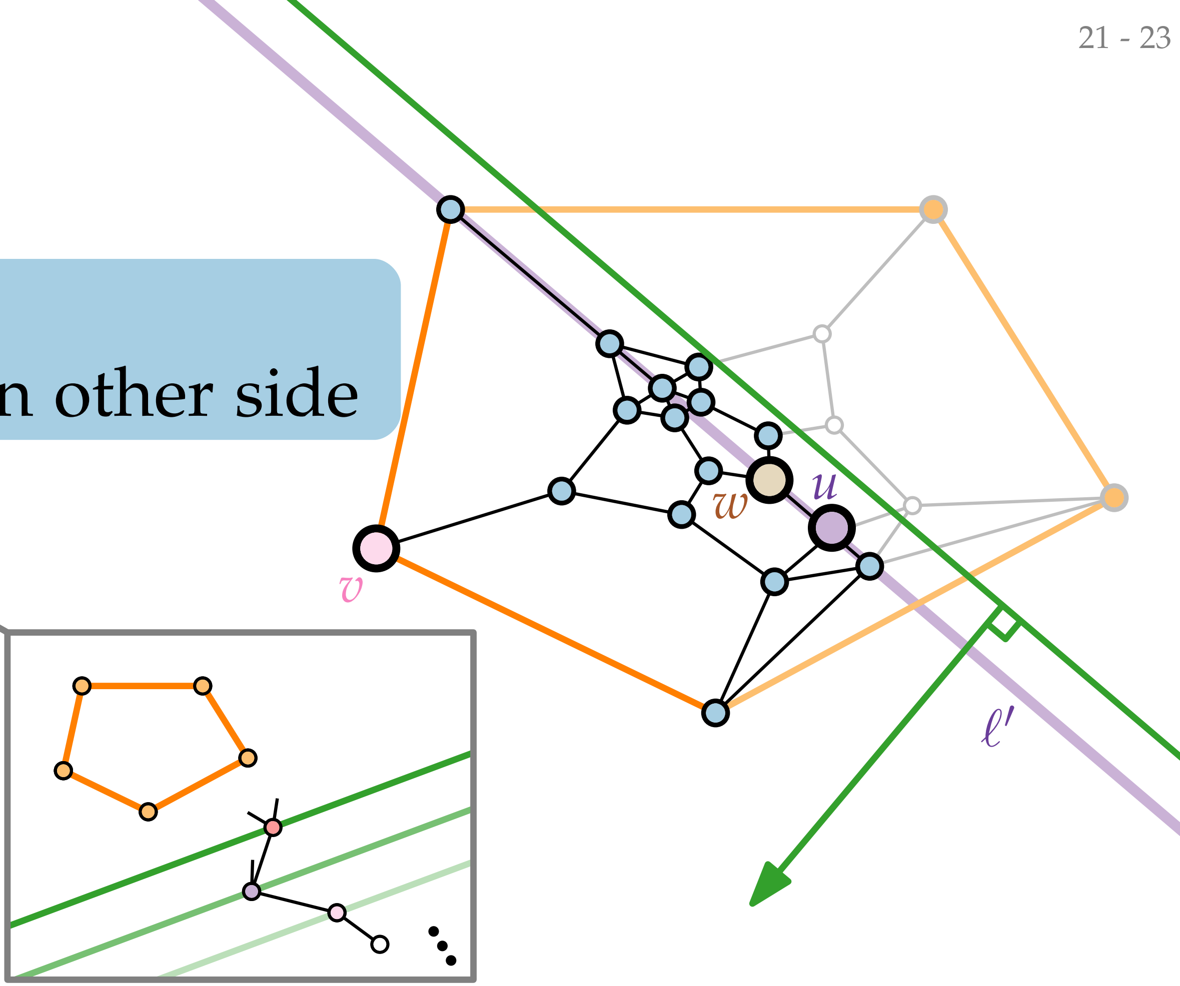

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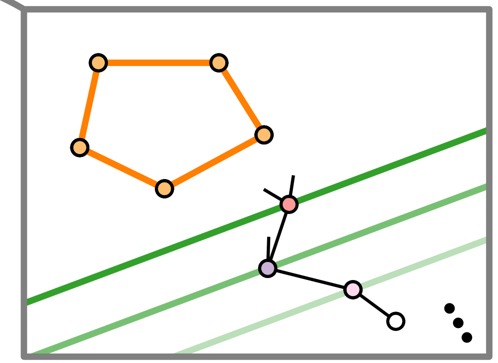

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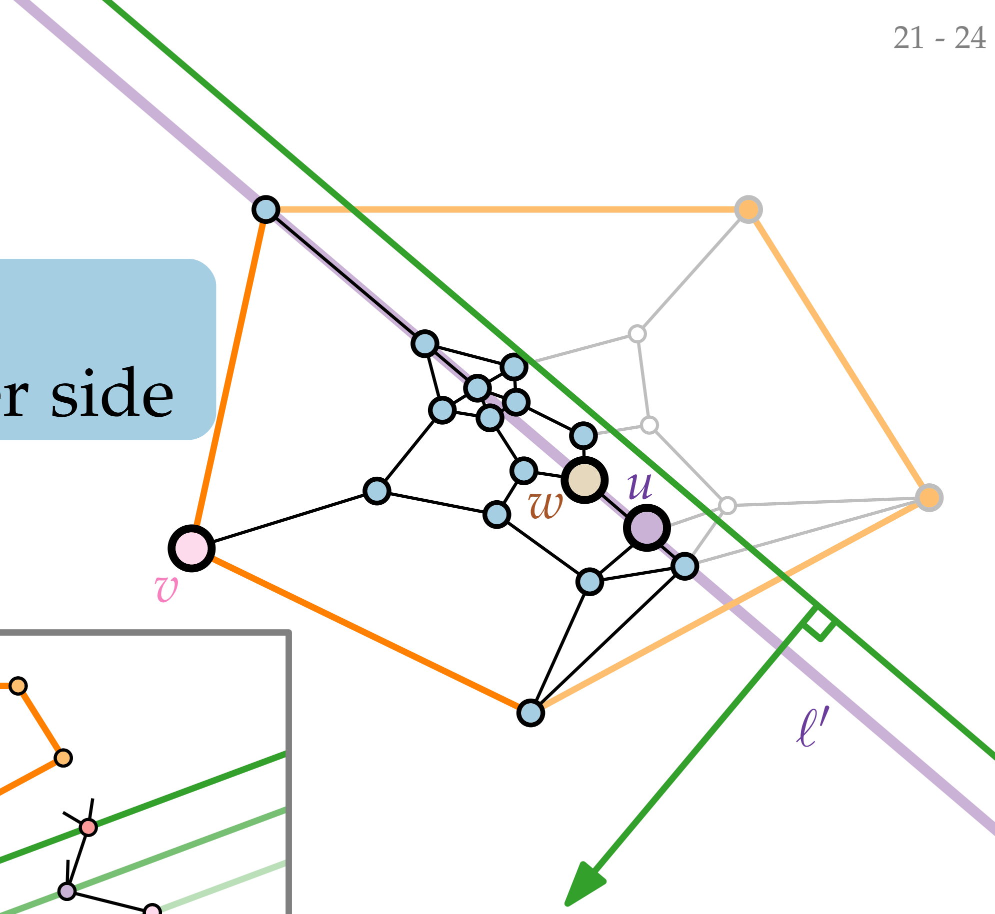

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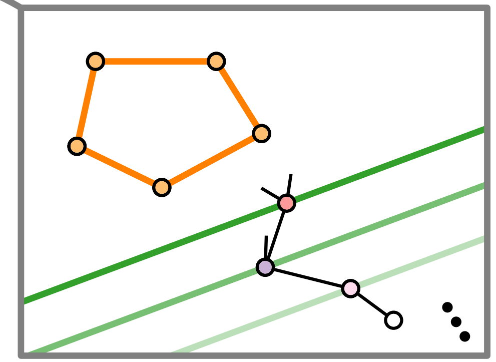

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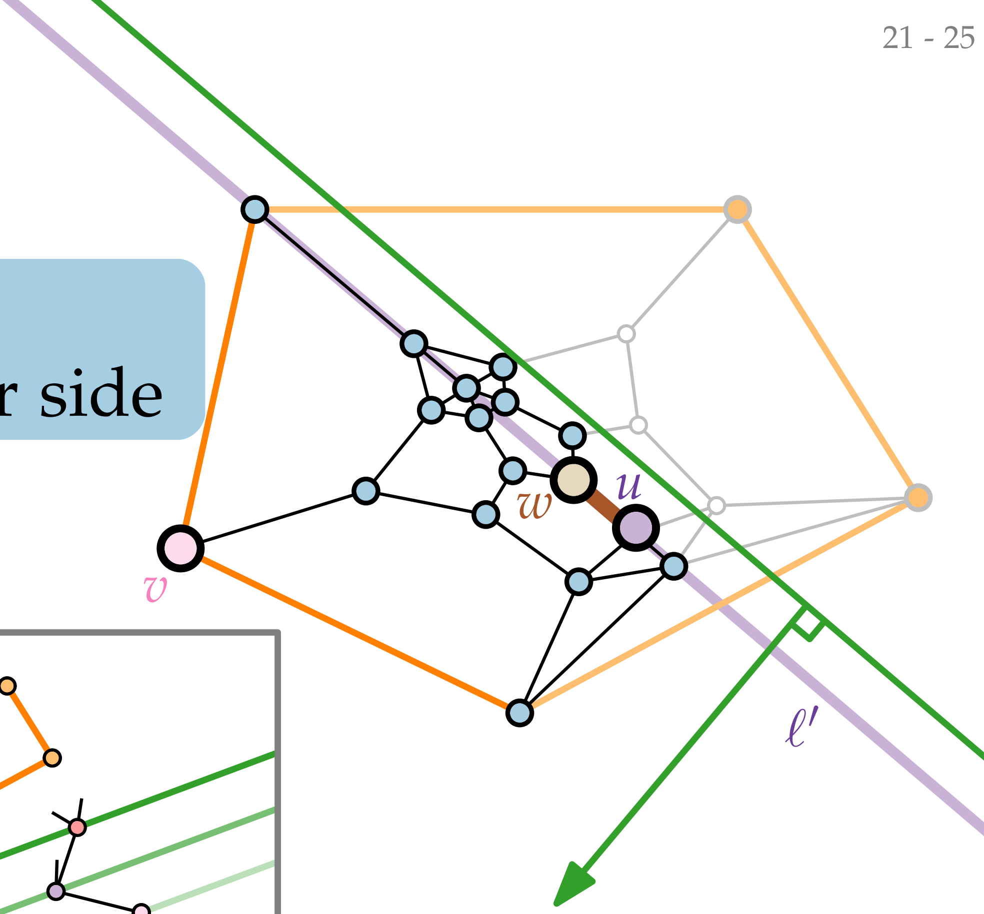

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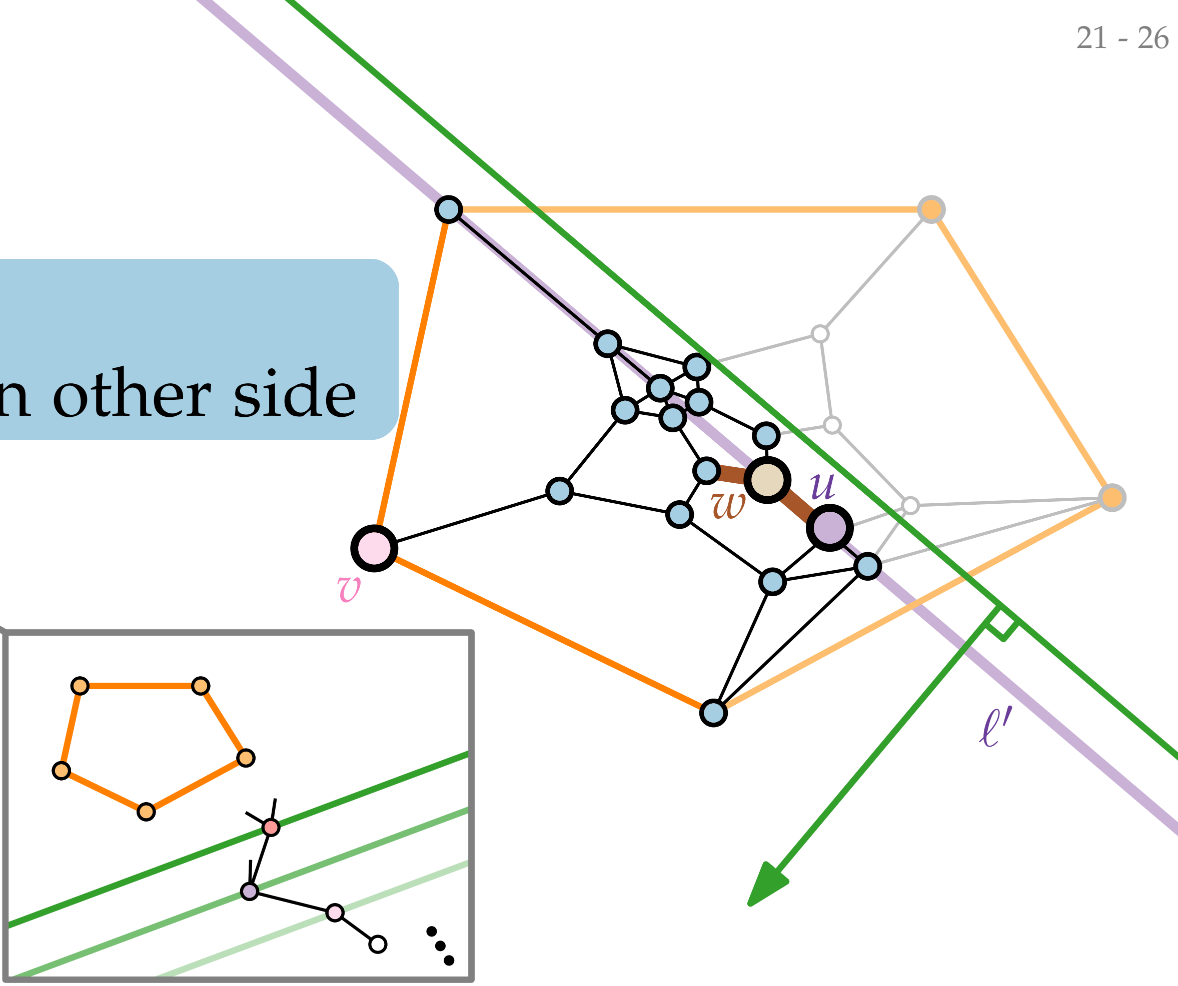

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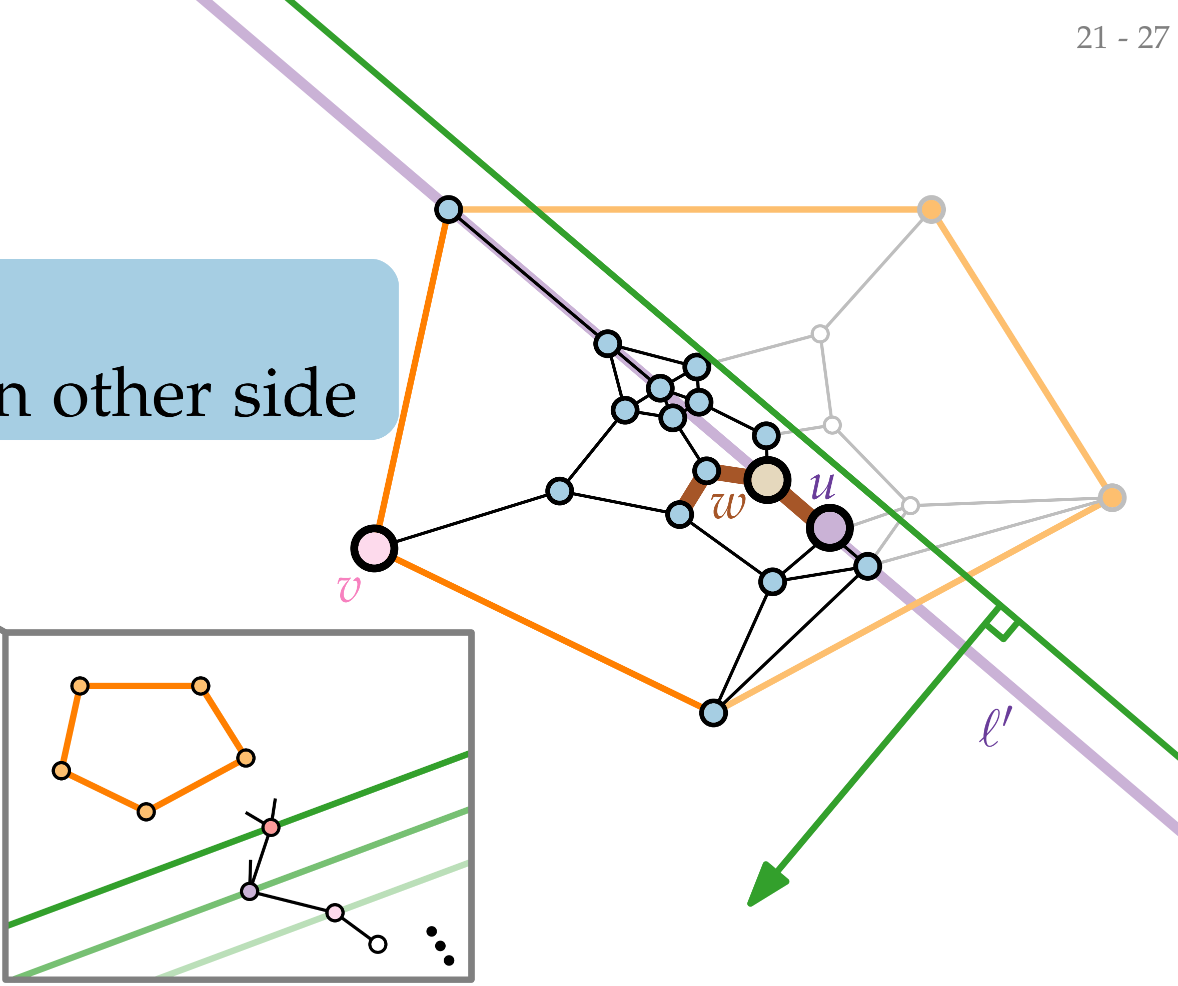

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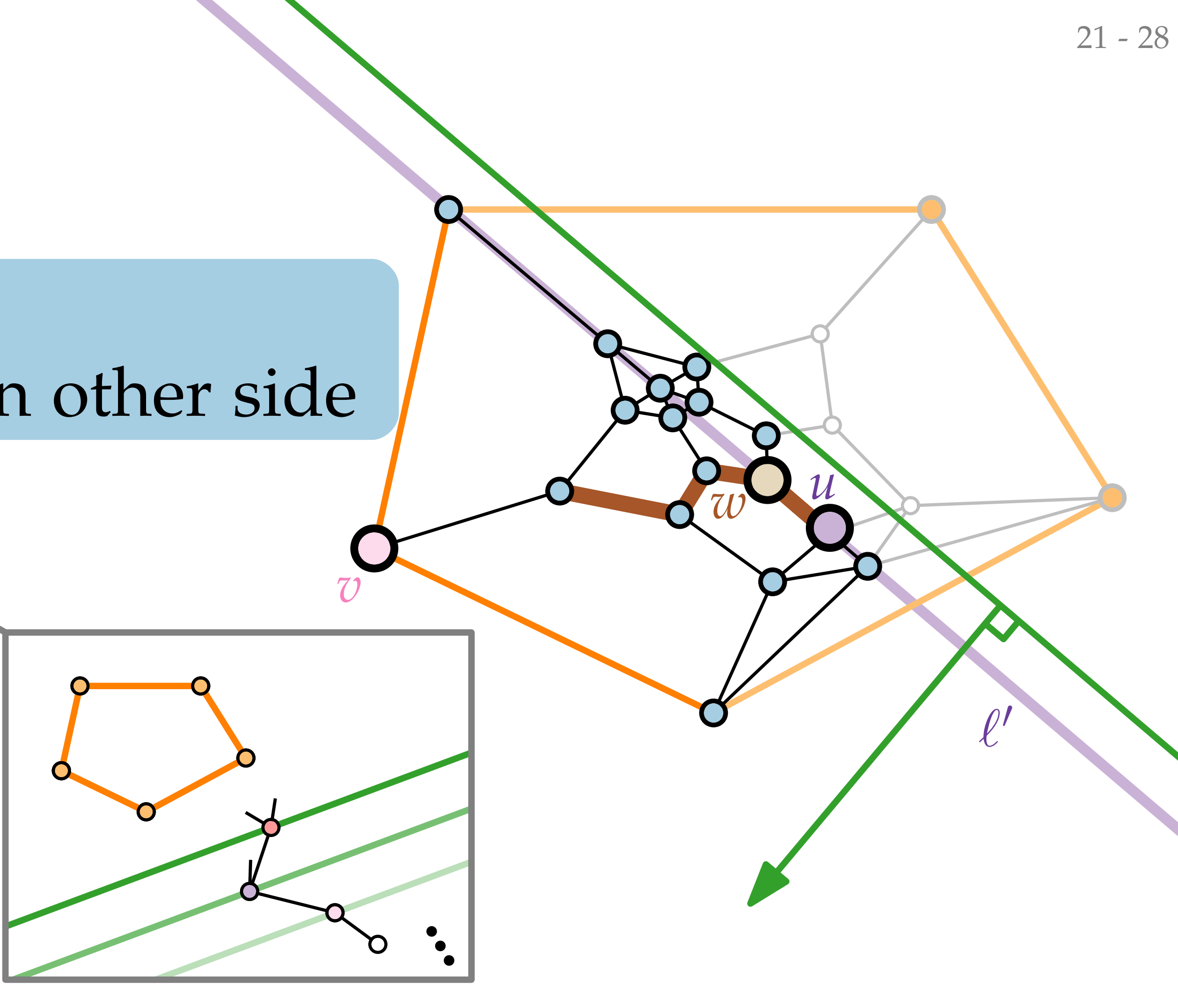

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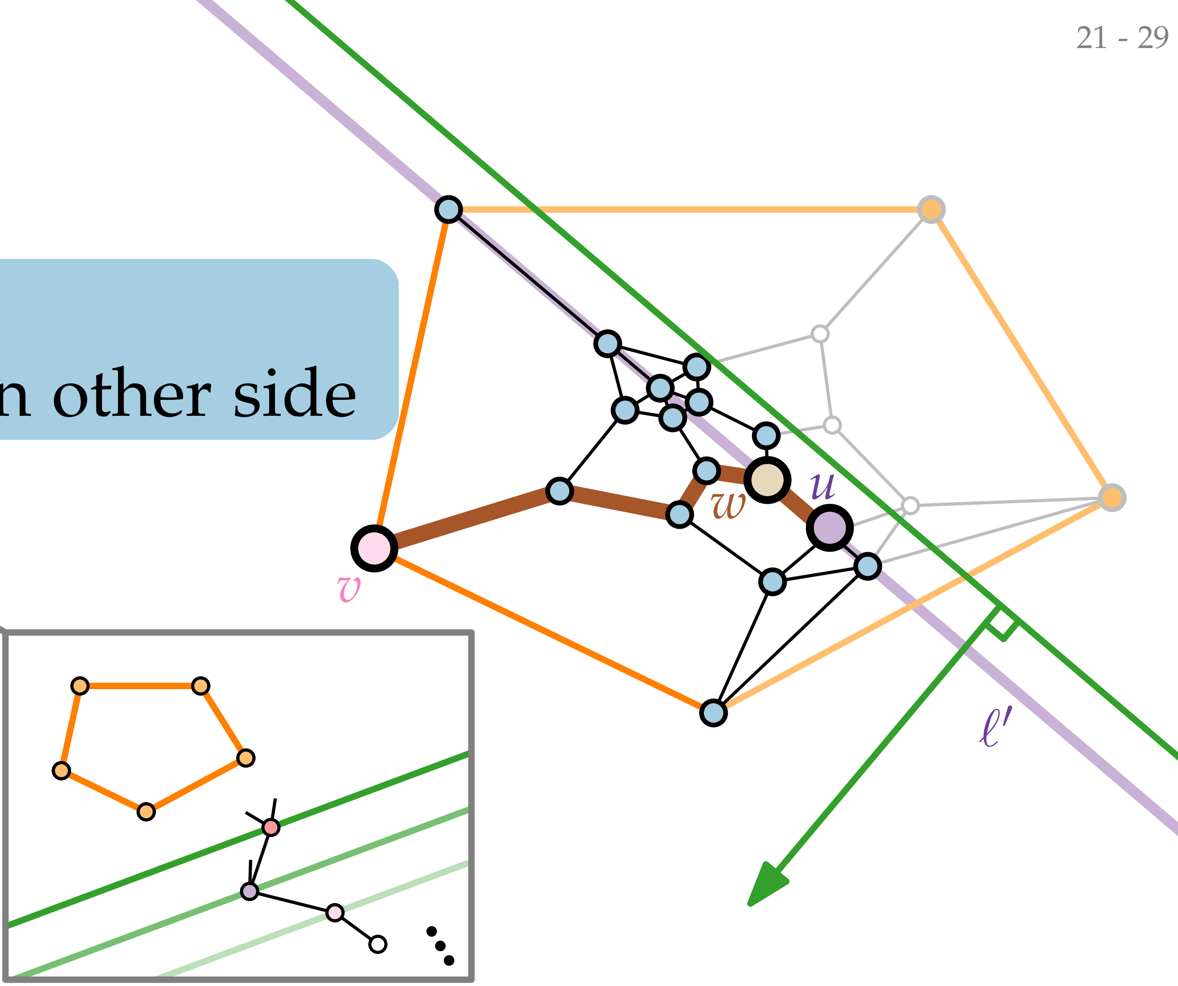

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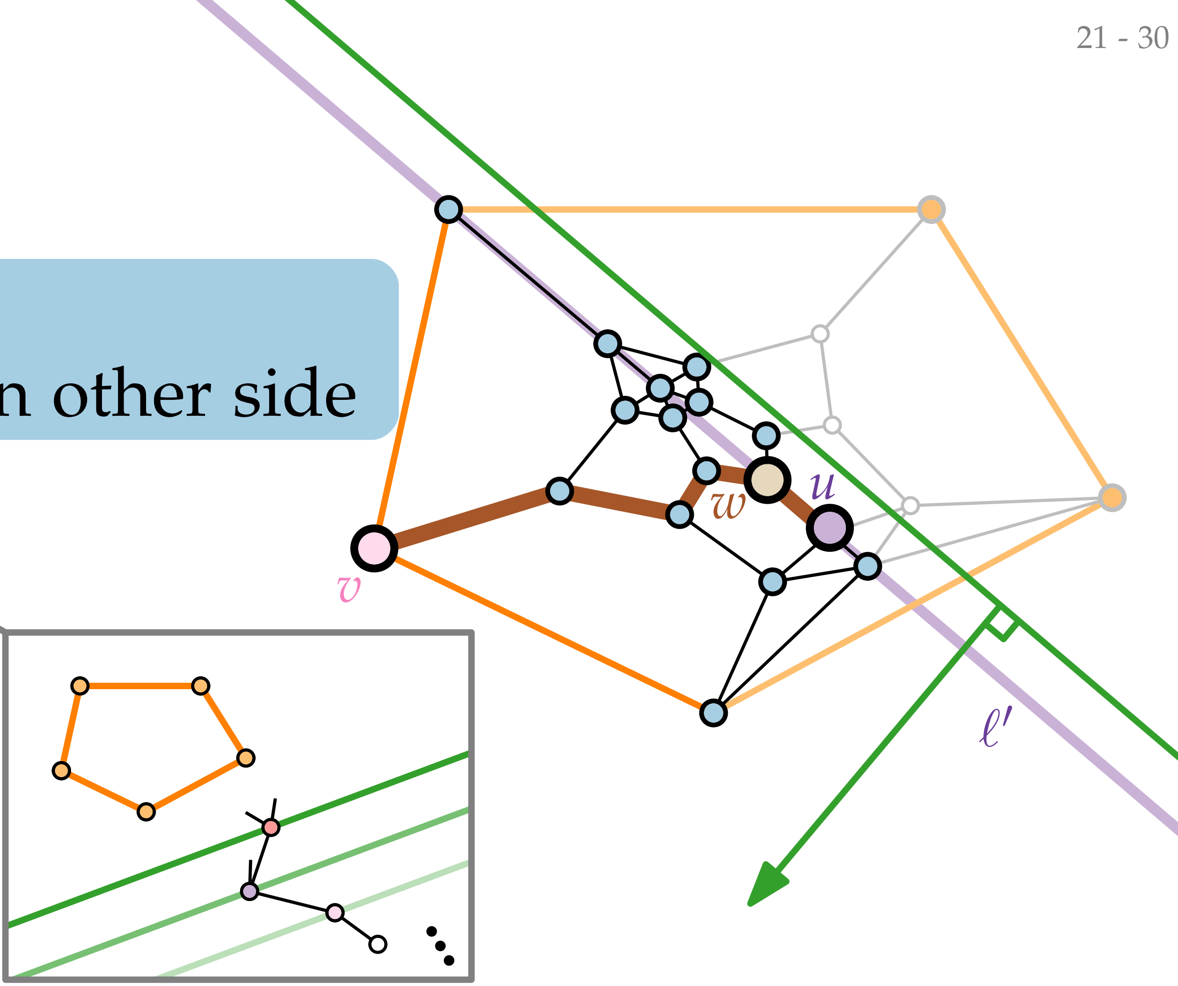

Property 4. No vertex is collinear with all of its neighbors.

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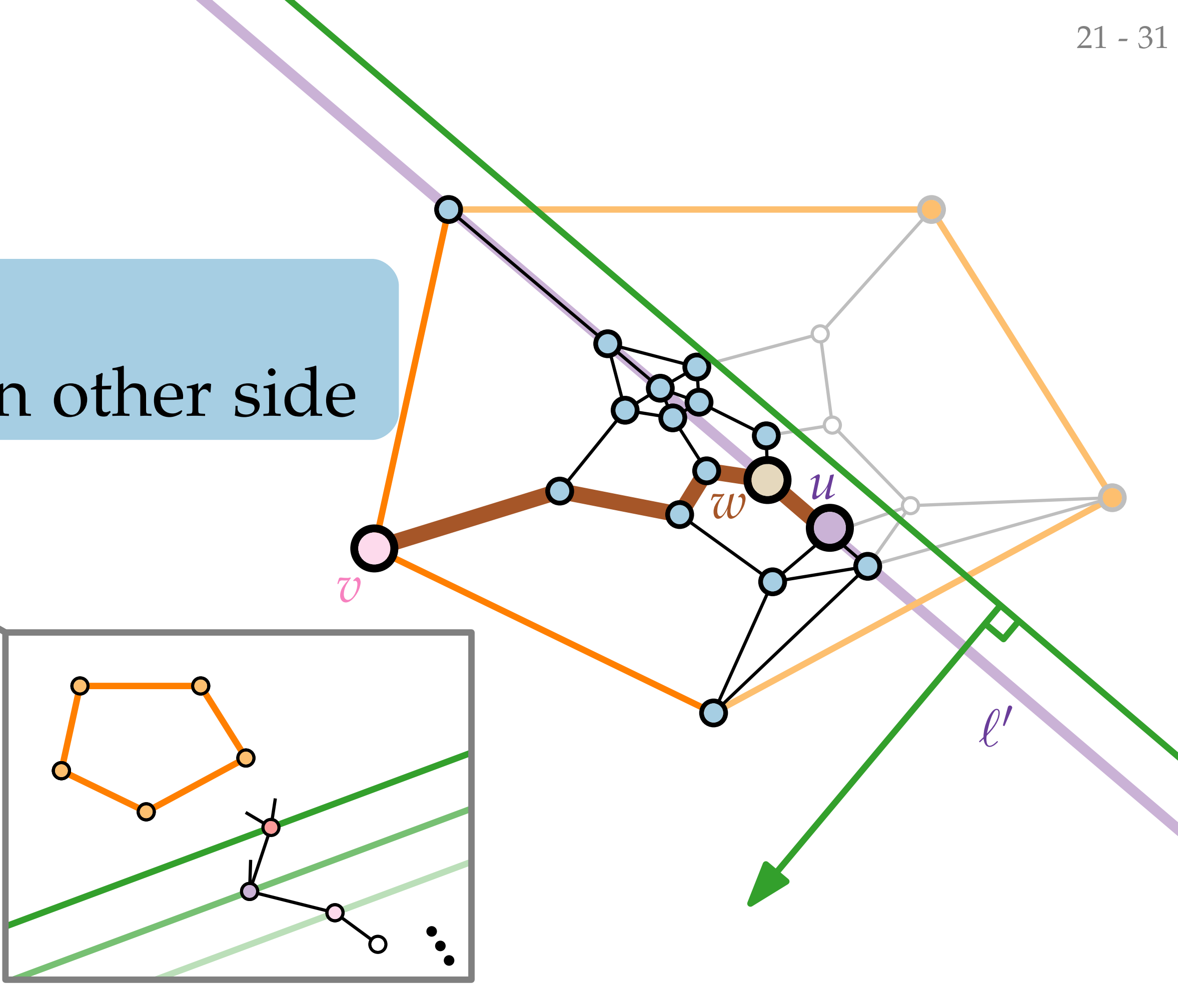

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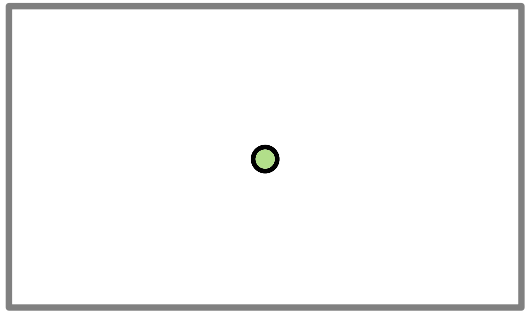

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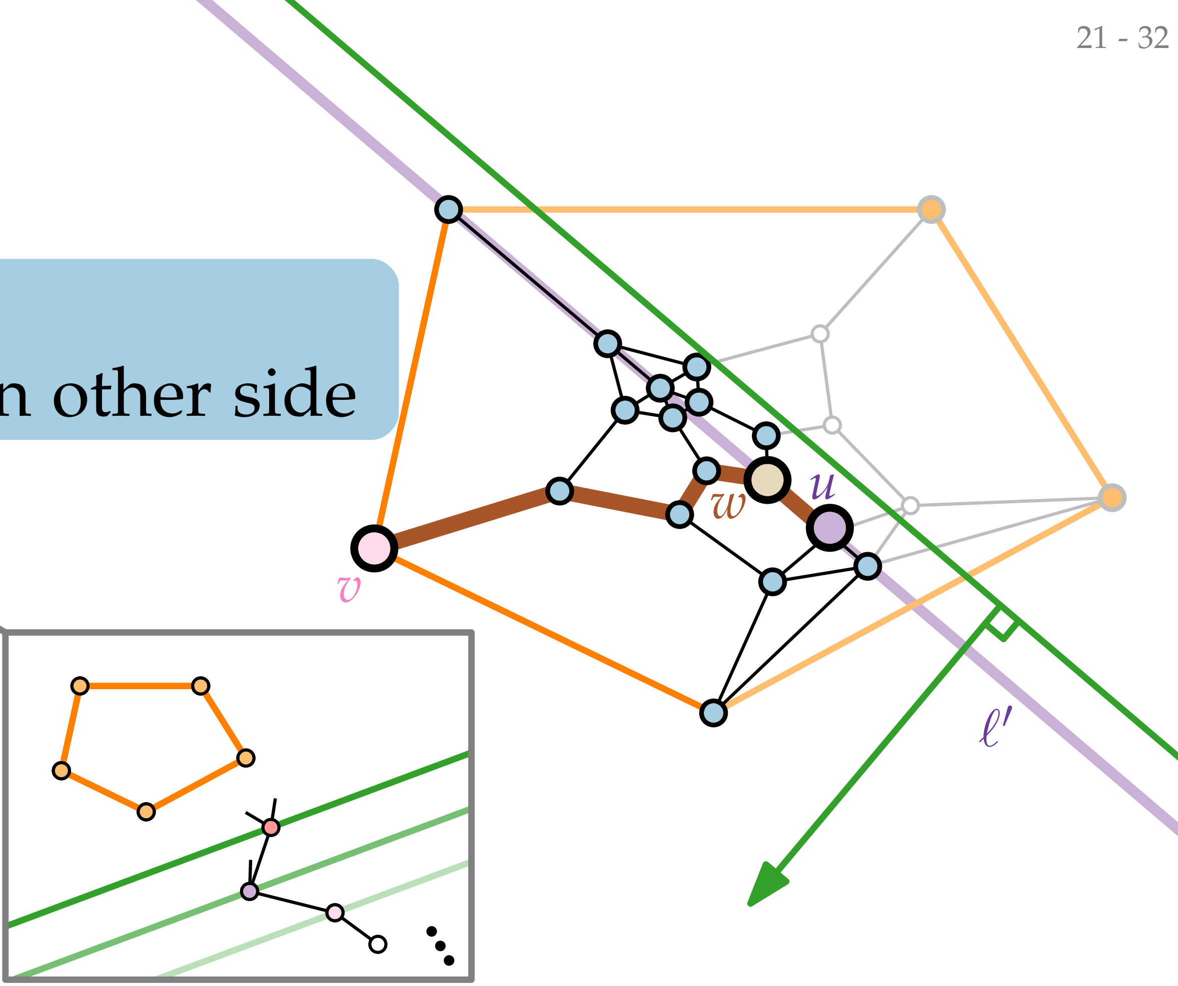

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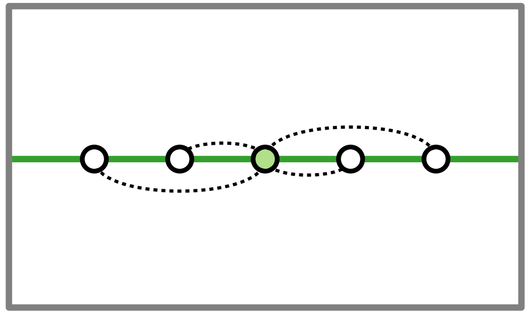

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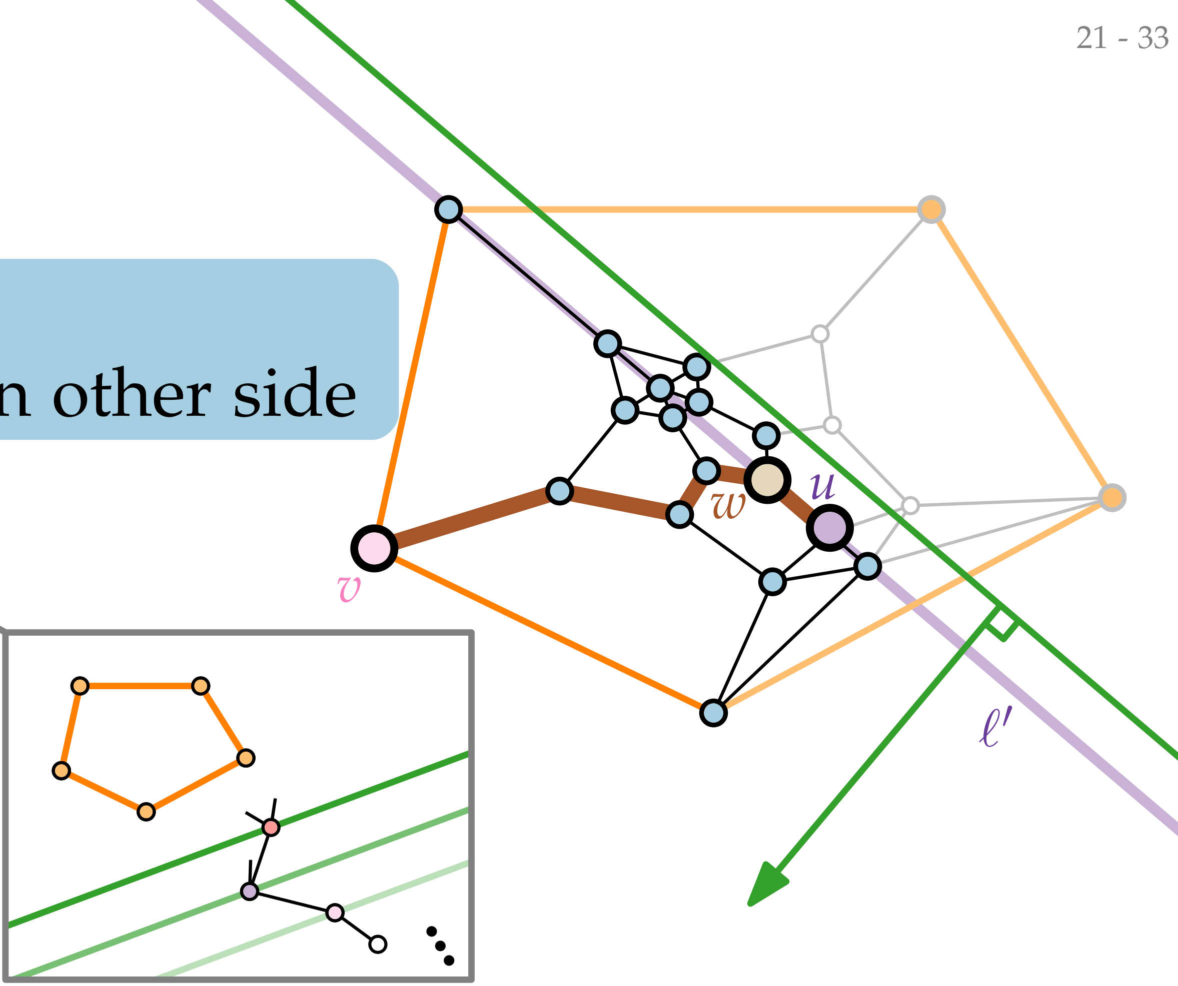
$\exists uv \in E$ on one side of $\ell \Rightarrow \exists vw \in E$ on other side

Otherwise, all forces to same side …

Property 2. All free vertices lie inside C.



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v furthest away from ℓ
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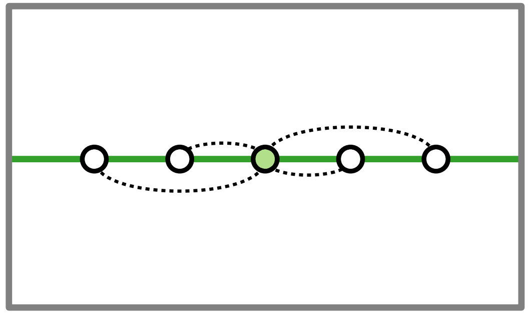
Property 4. No vertex is collinear with all of its neighbors.

Not all vertices collinear



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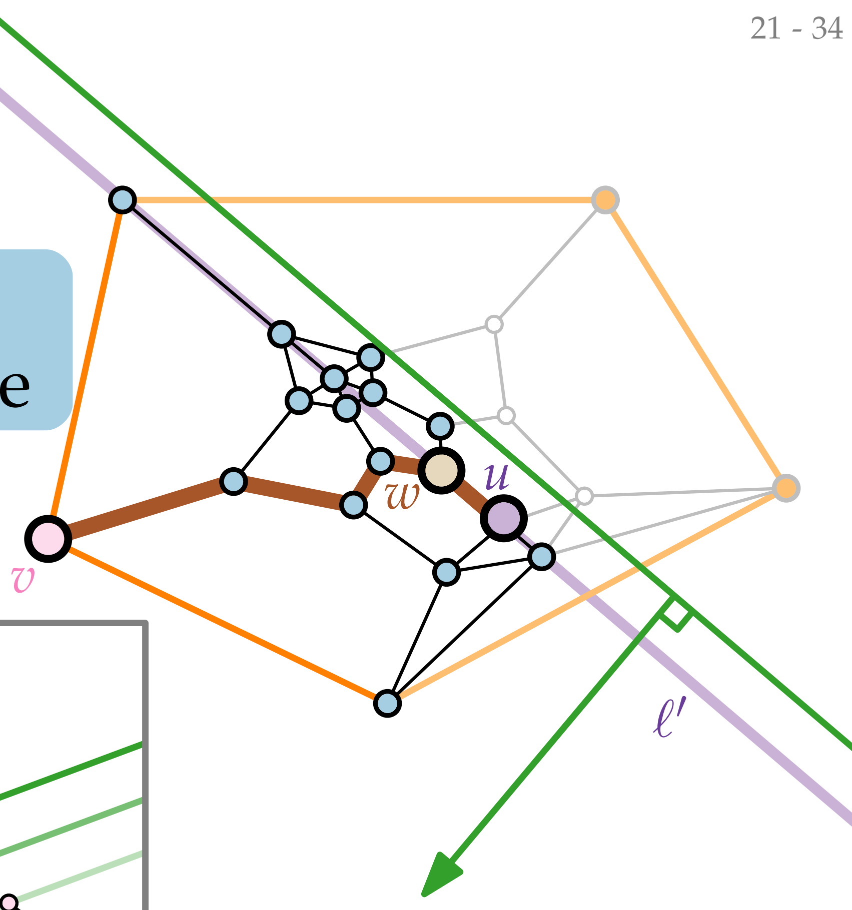

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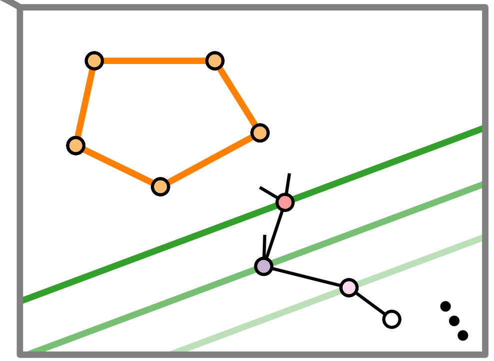

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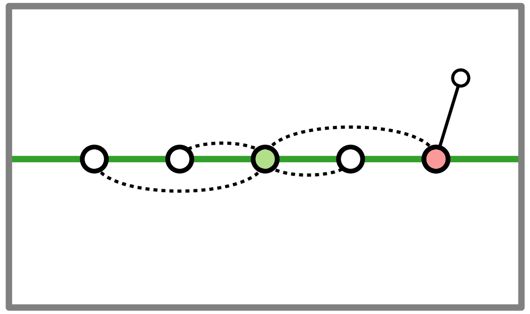

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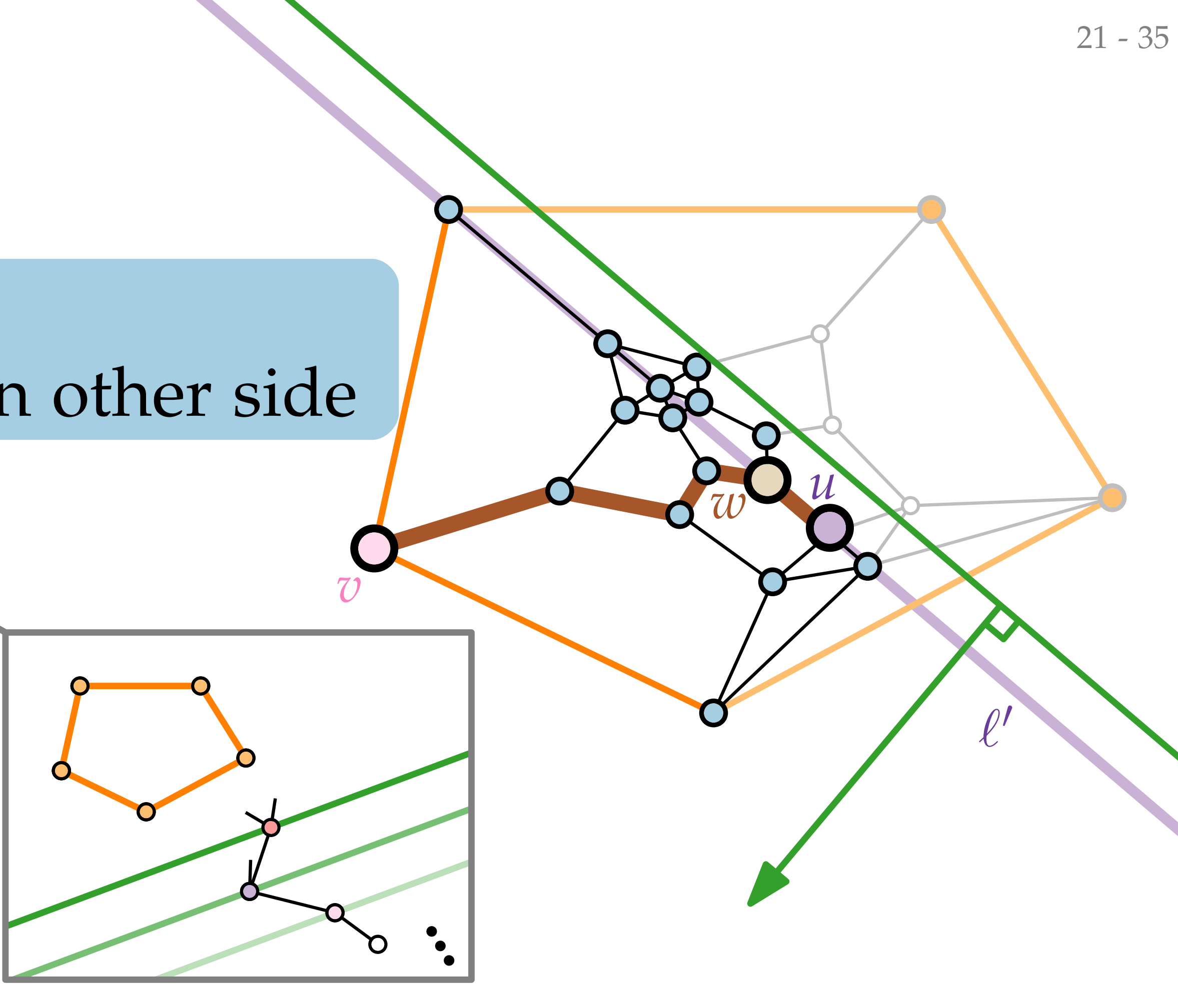

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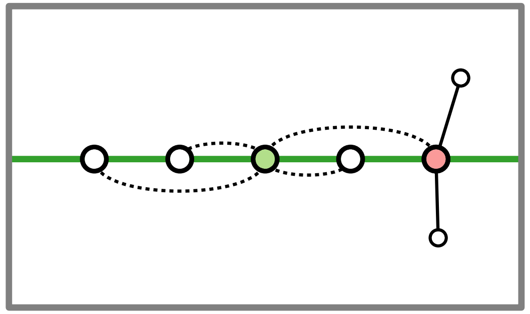

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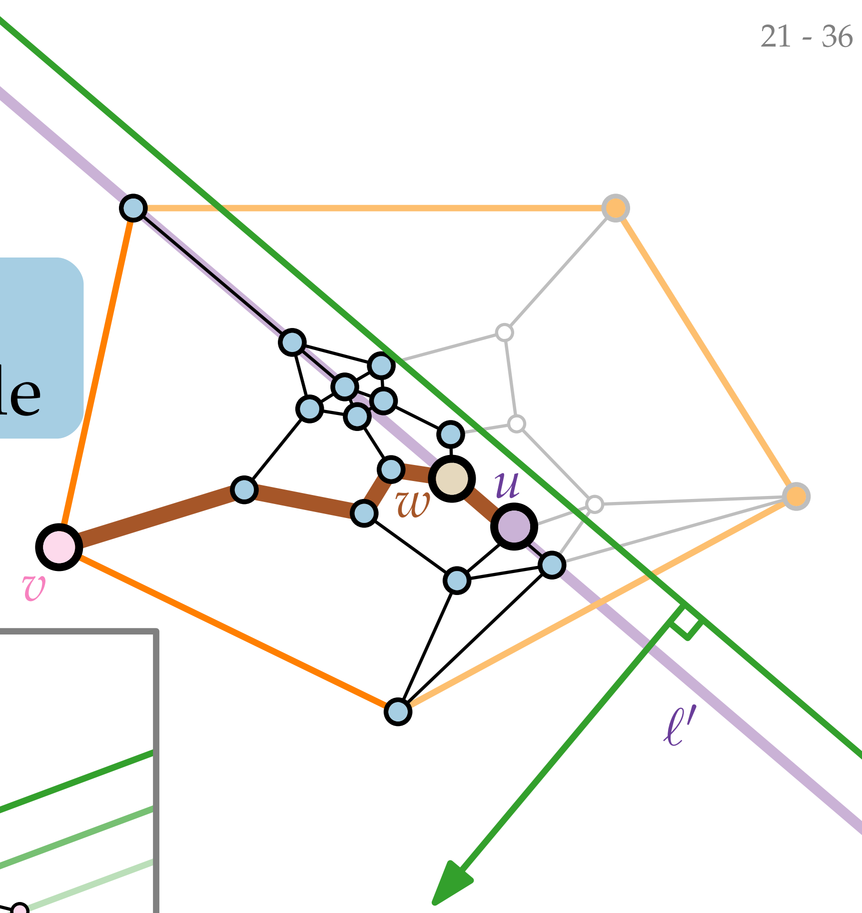

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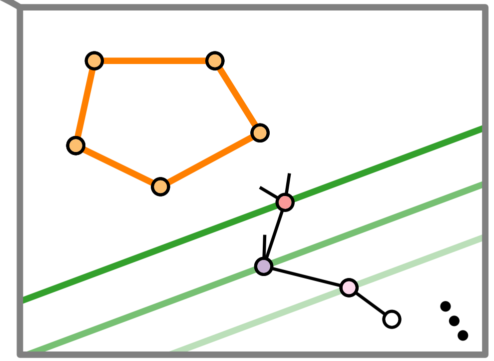

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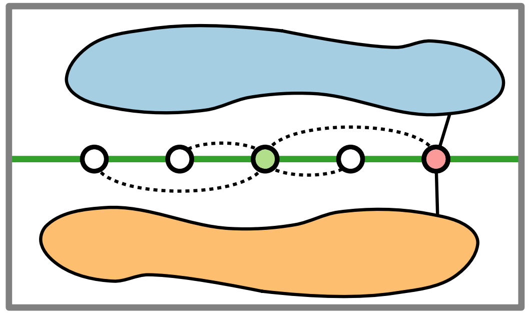

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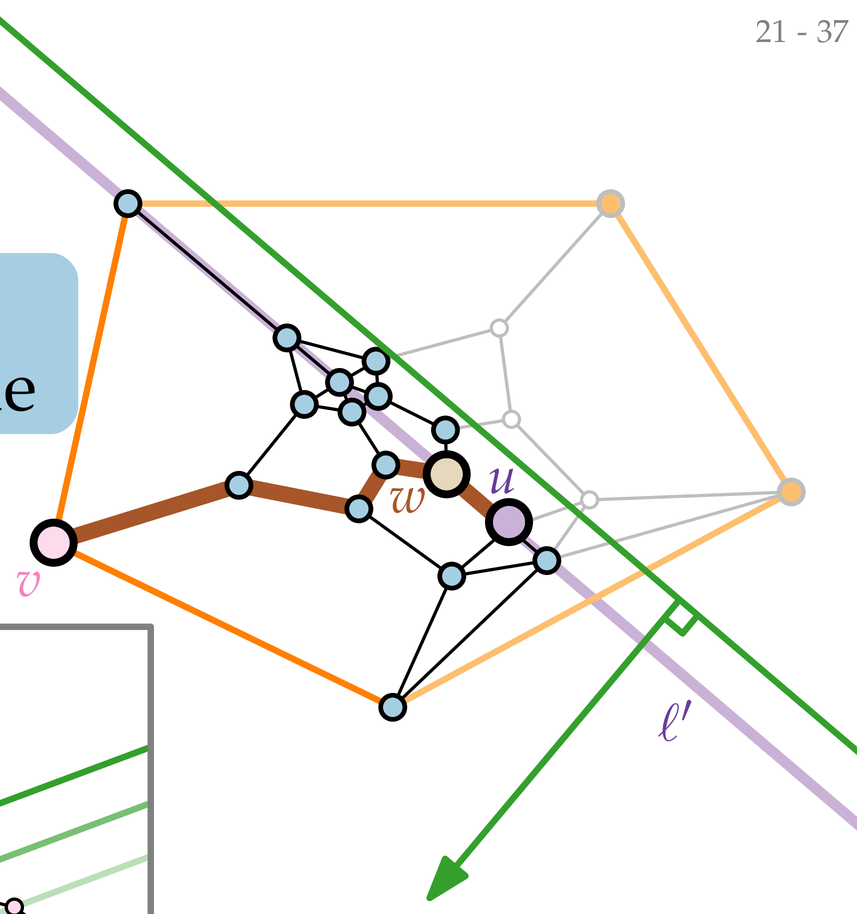

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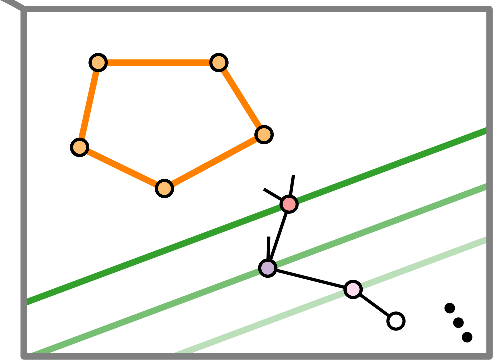

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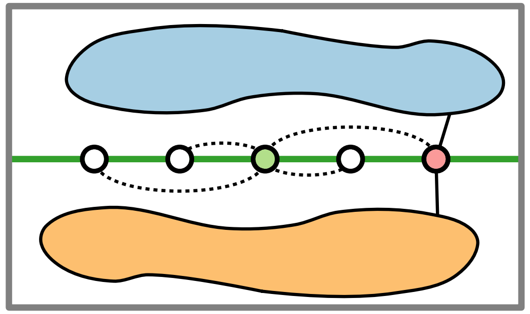

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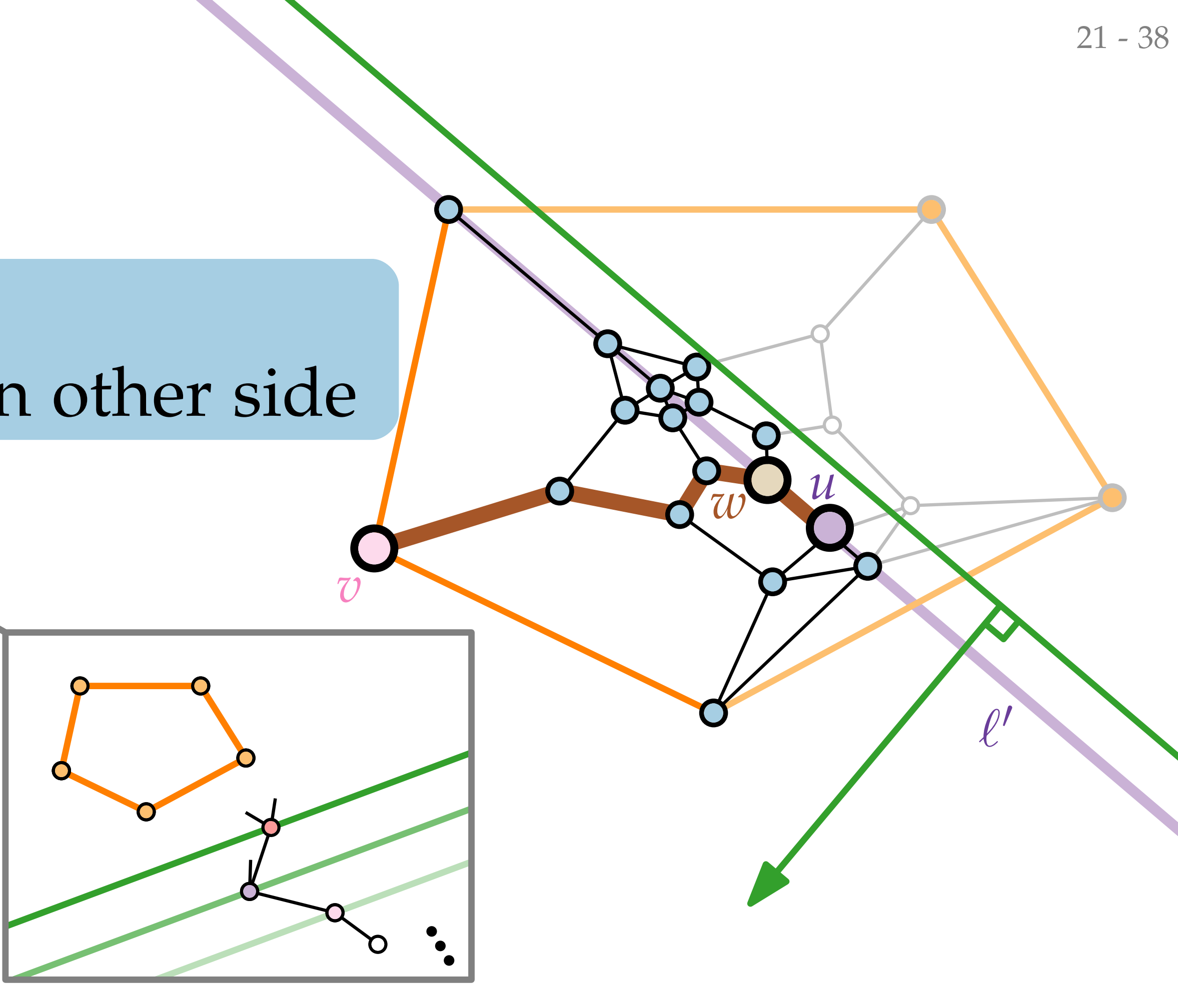

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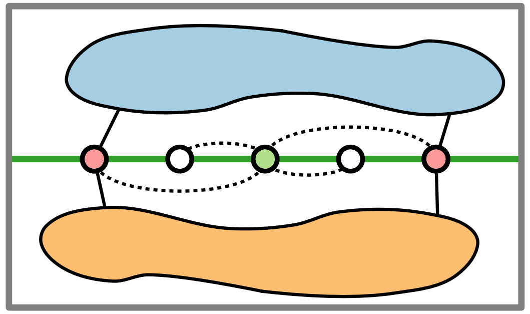

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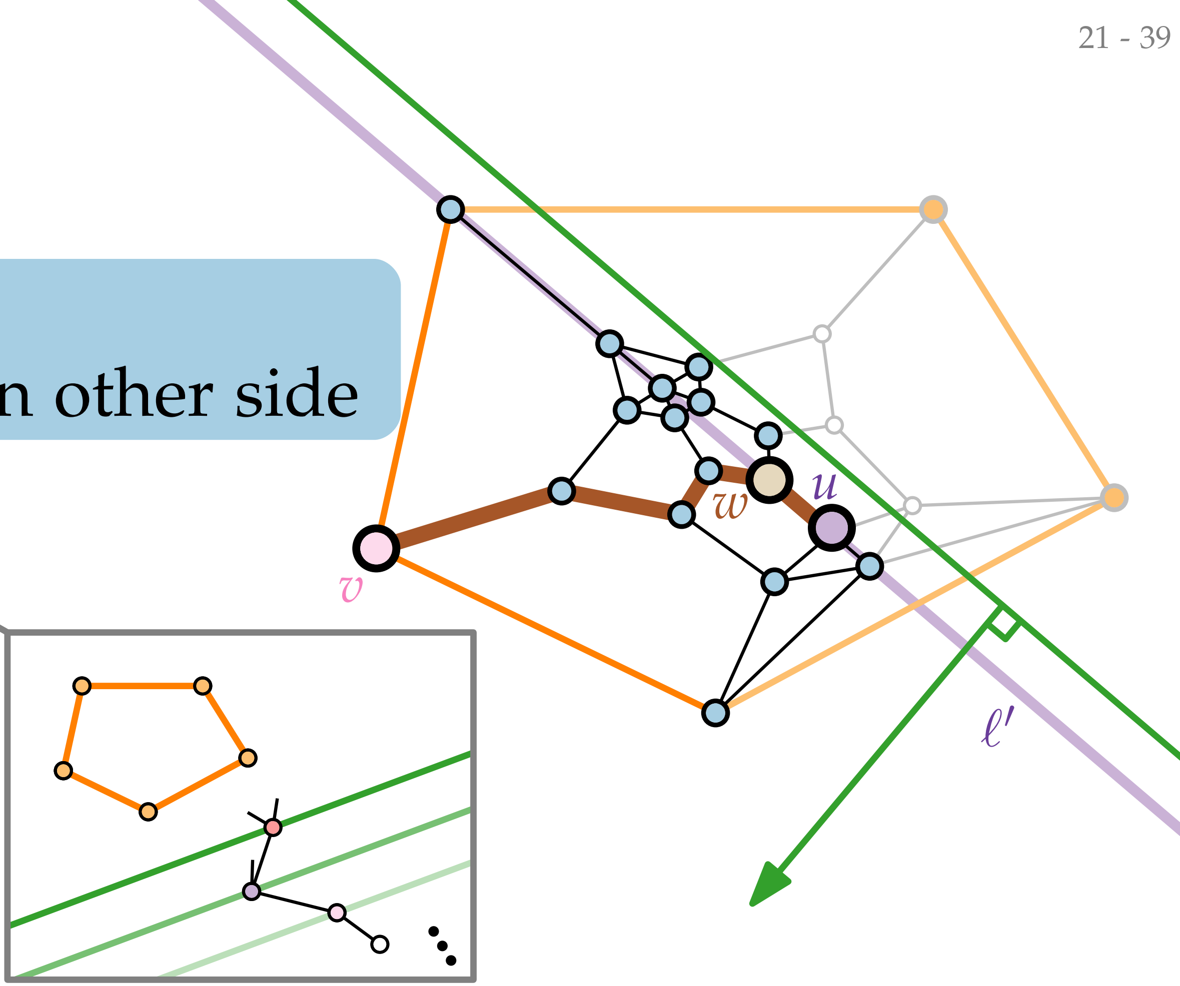

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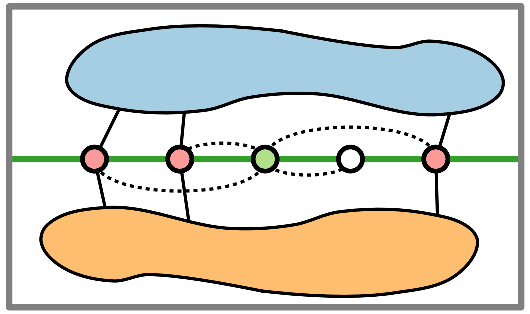

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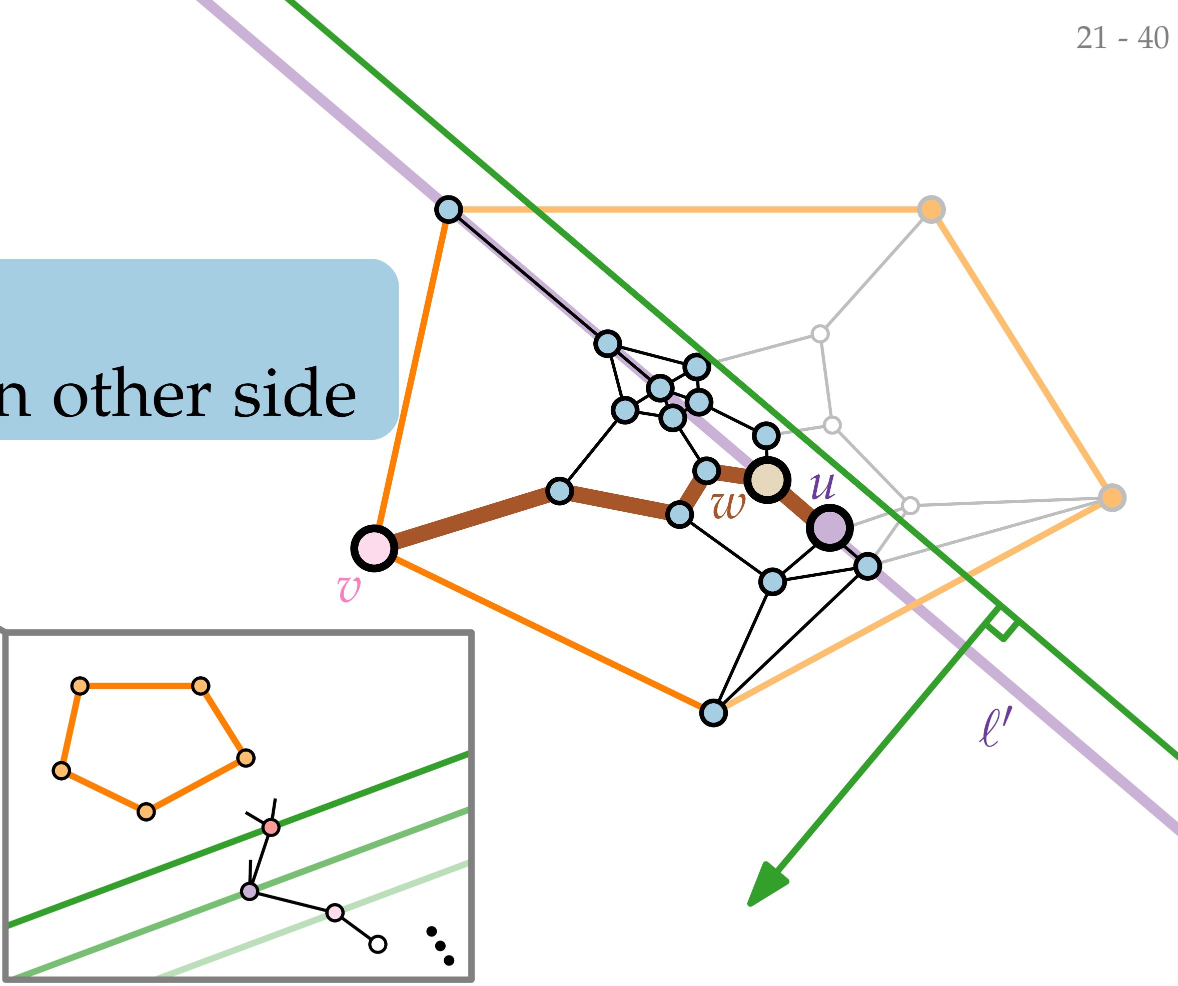

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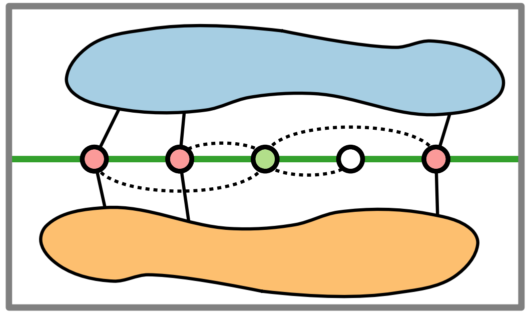

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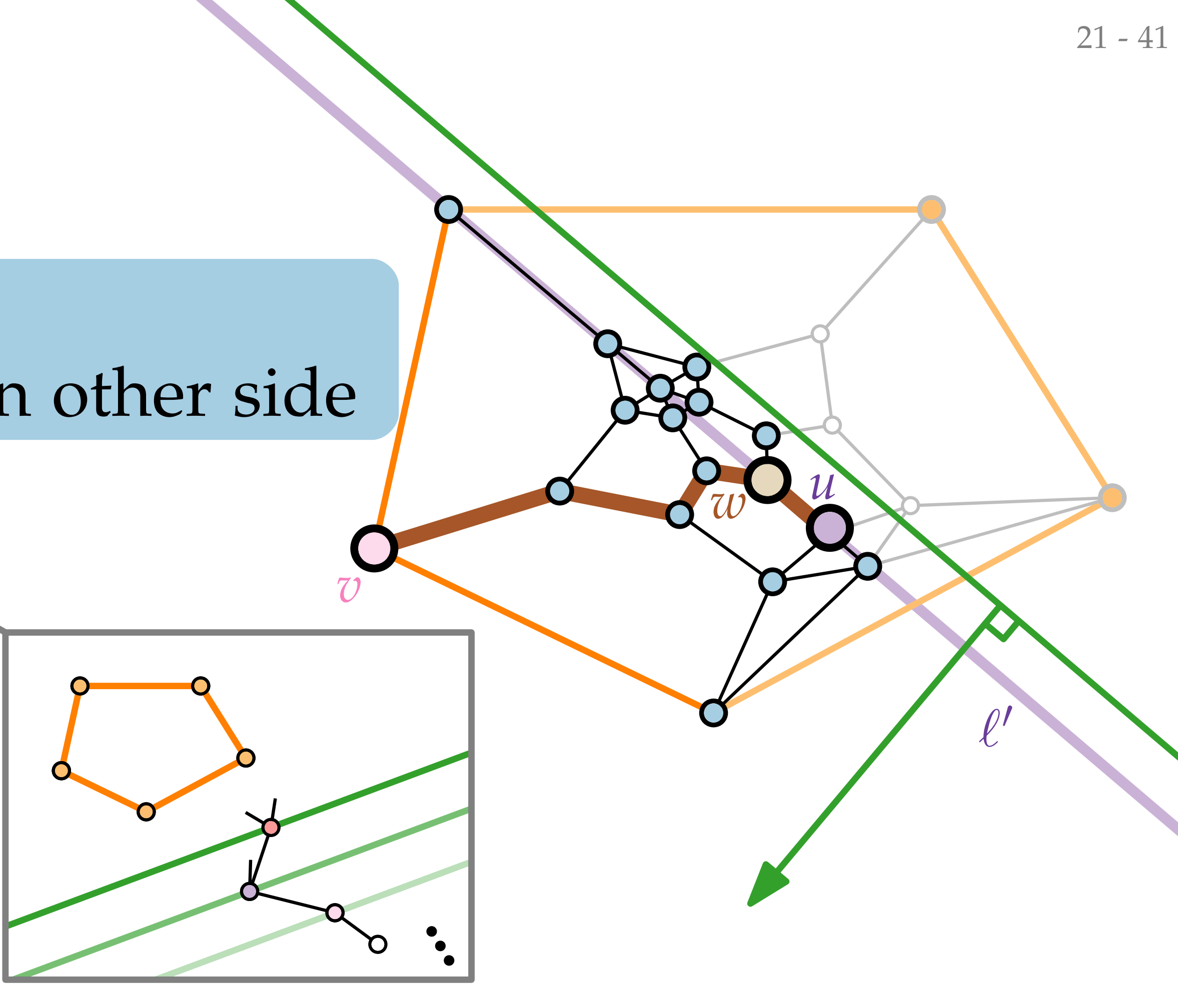

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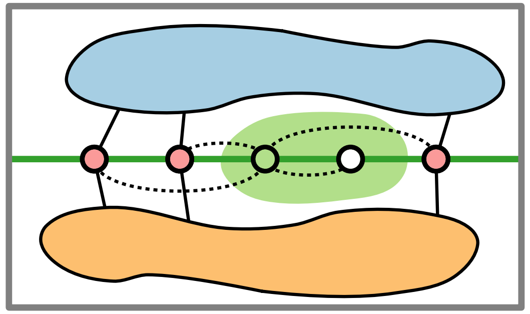

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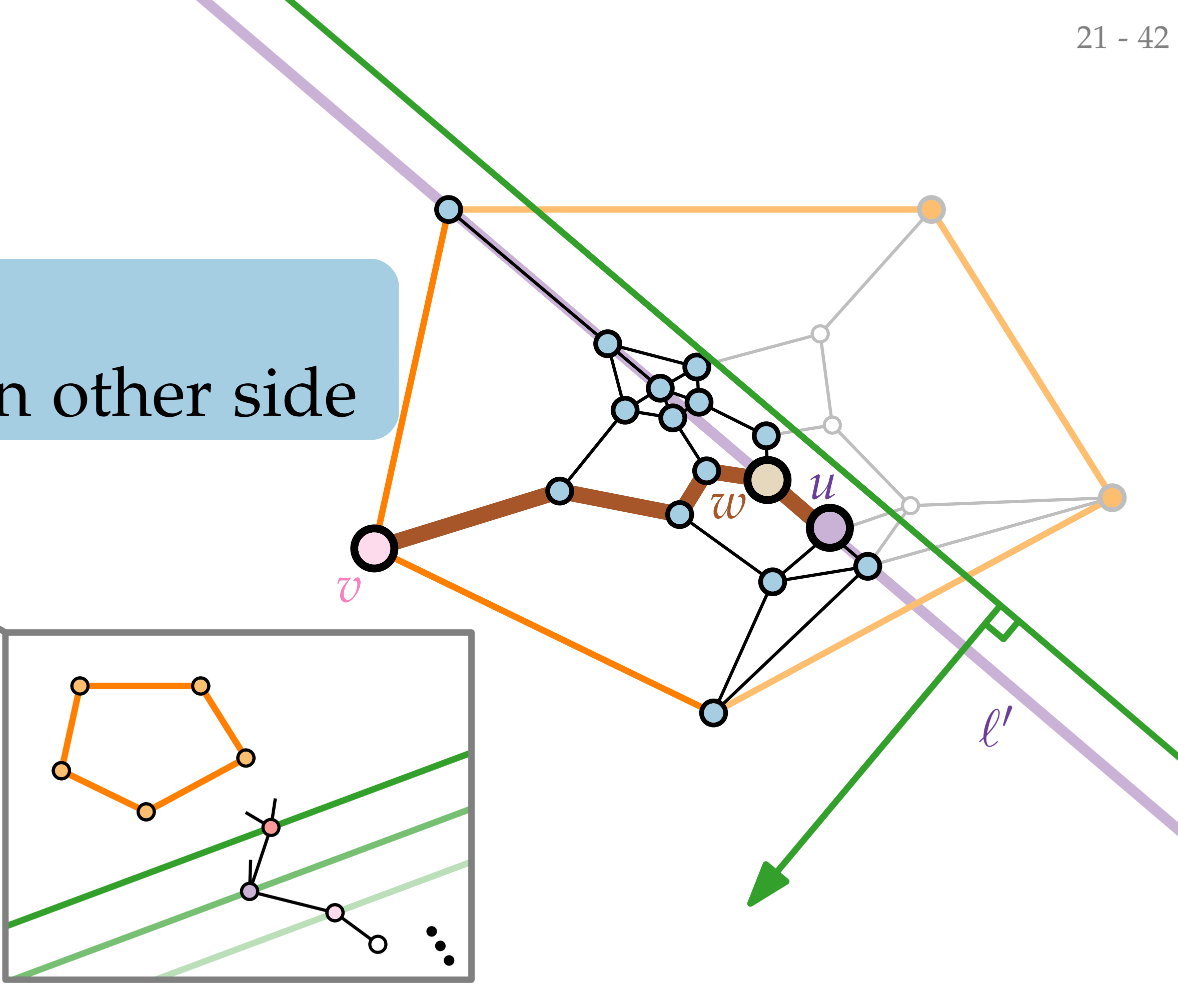

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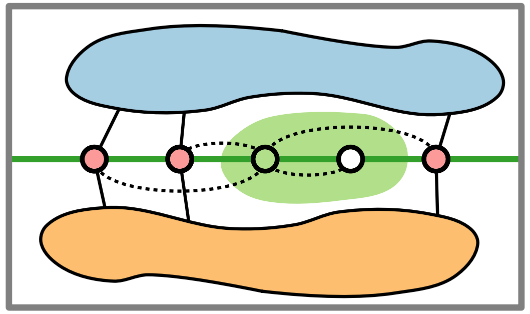

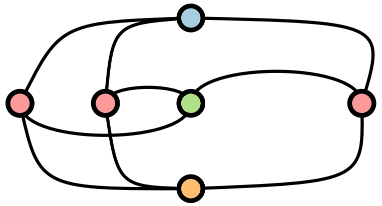

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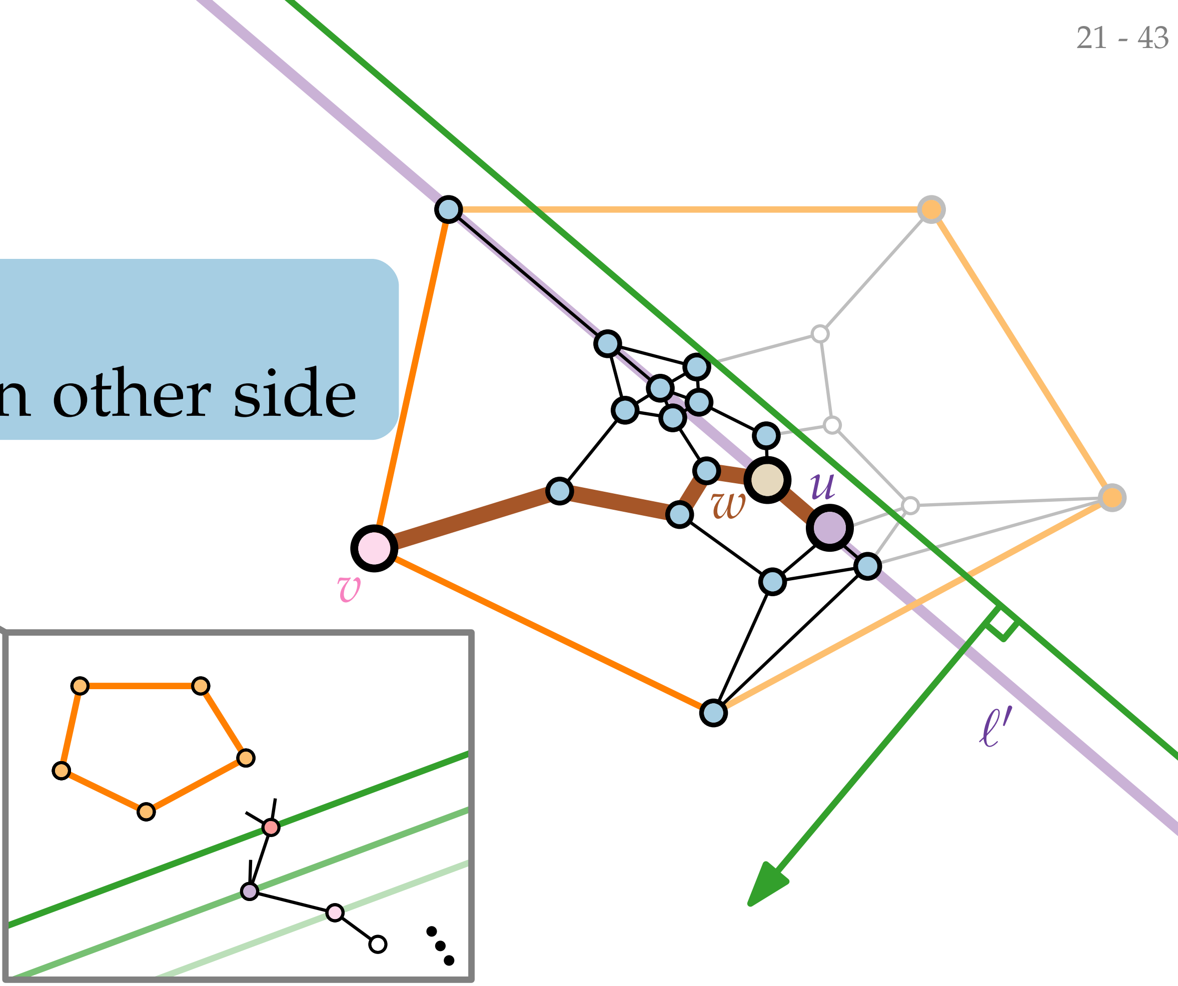

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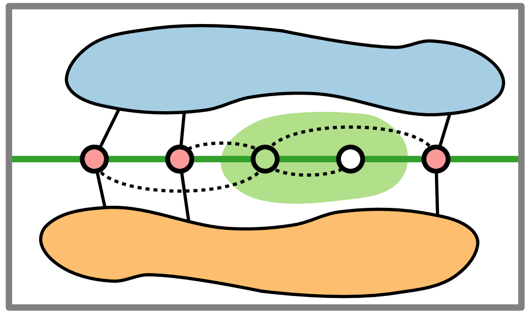

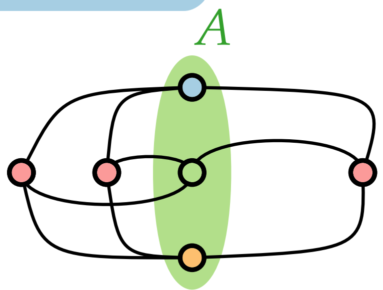

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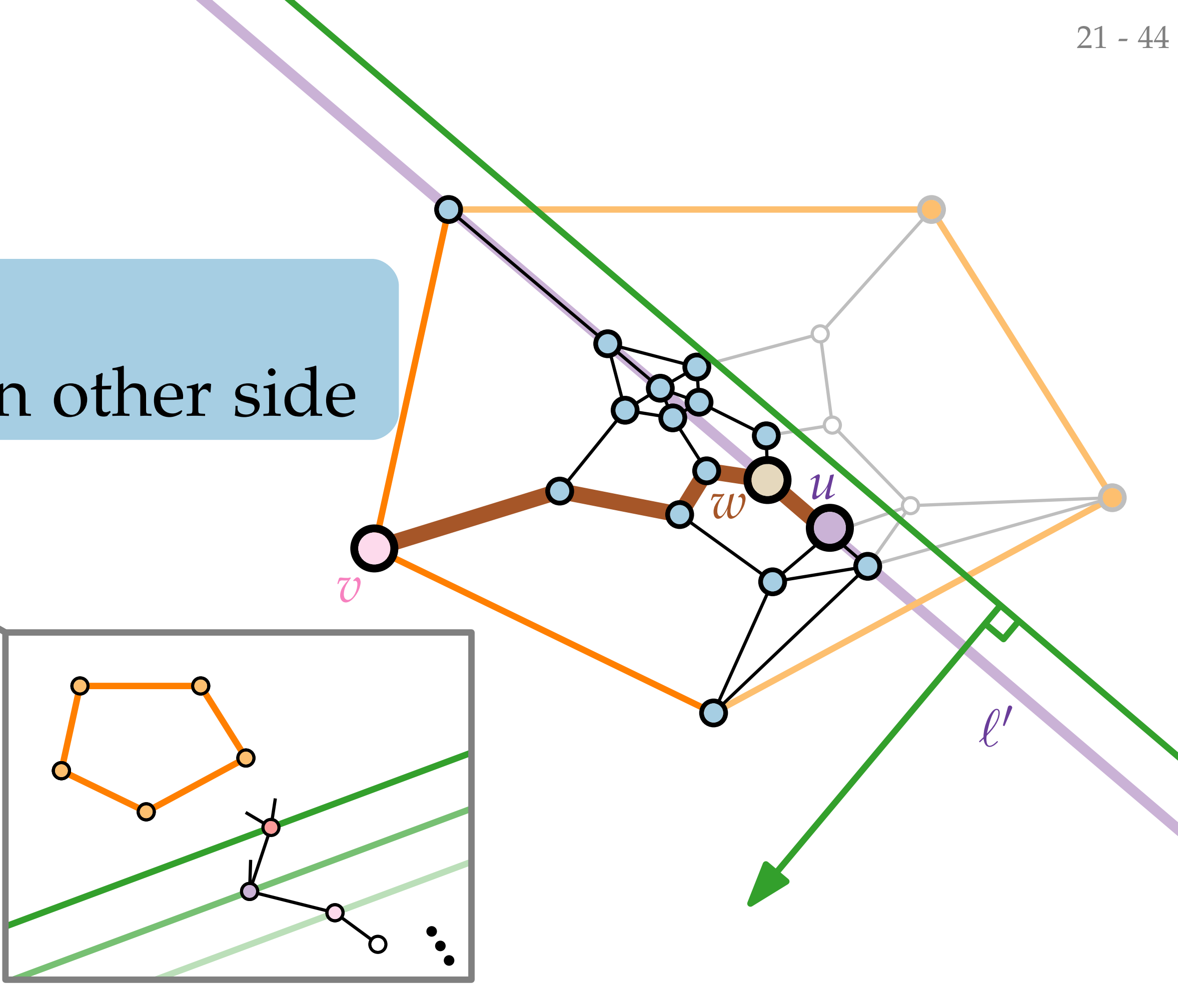

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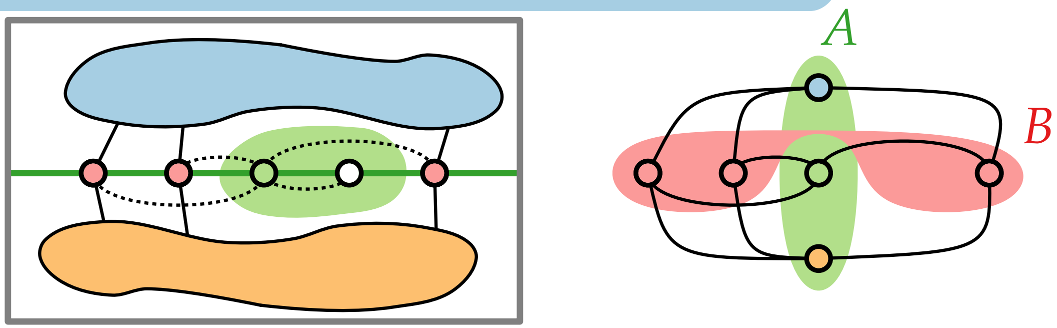

Proof of Tutte’s Theorem

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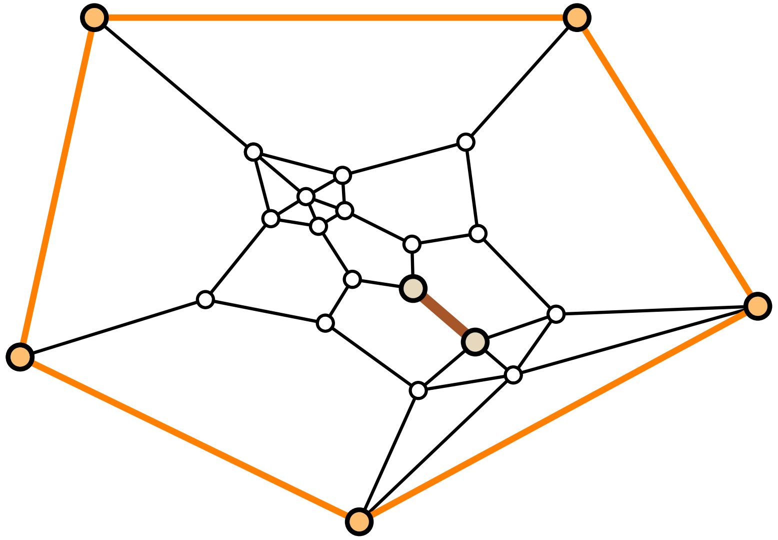

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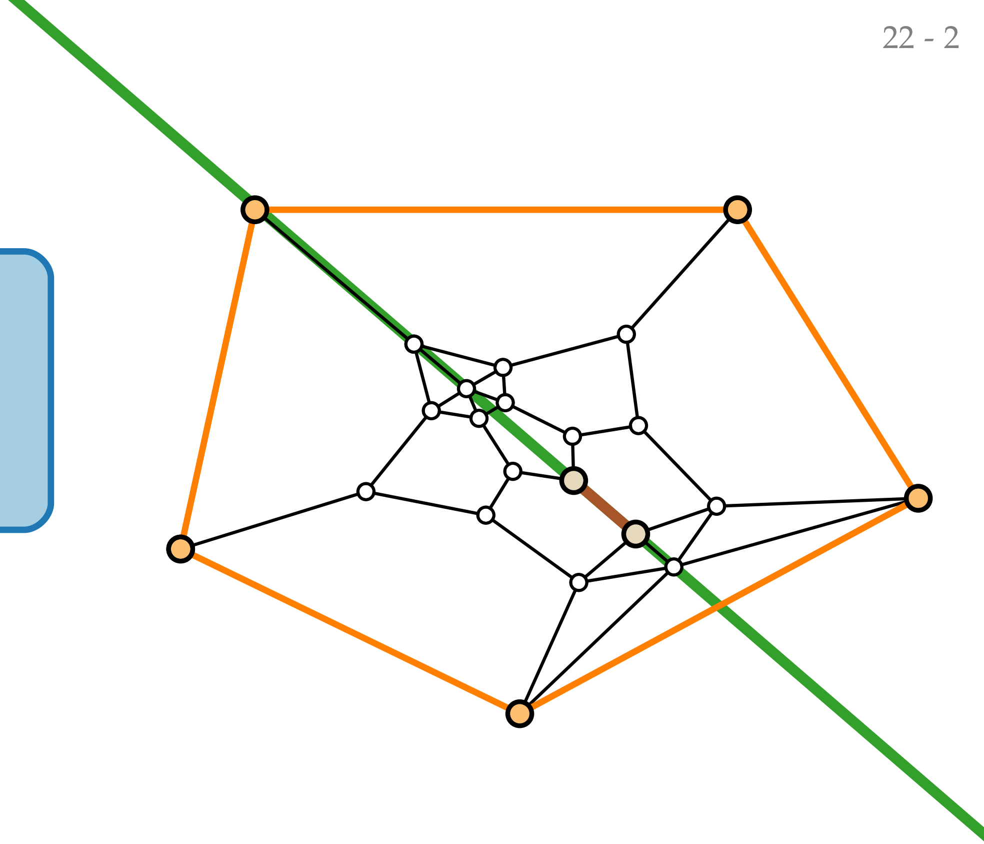

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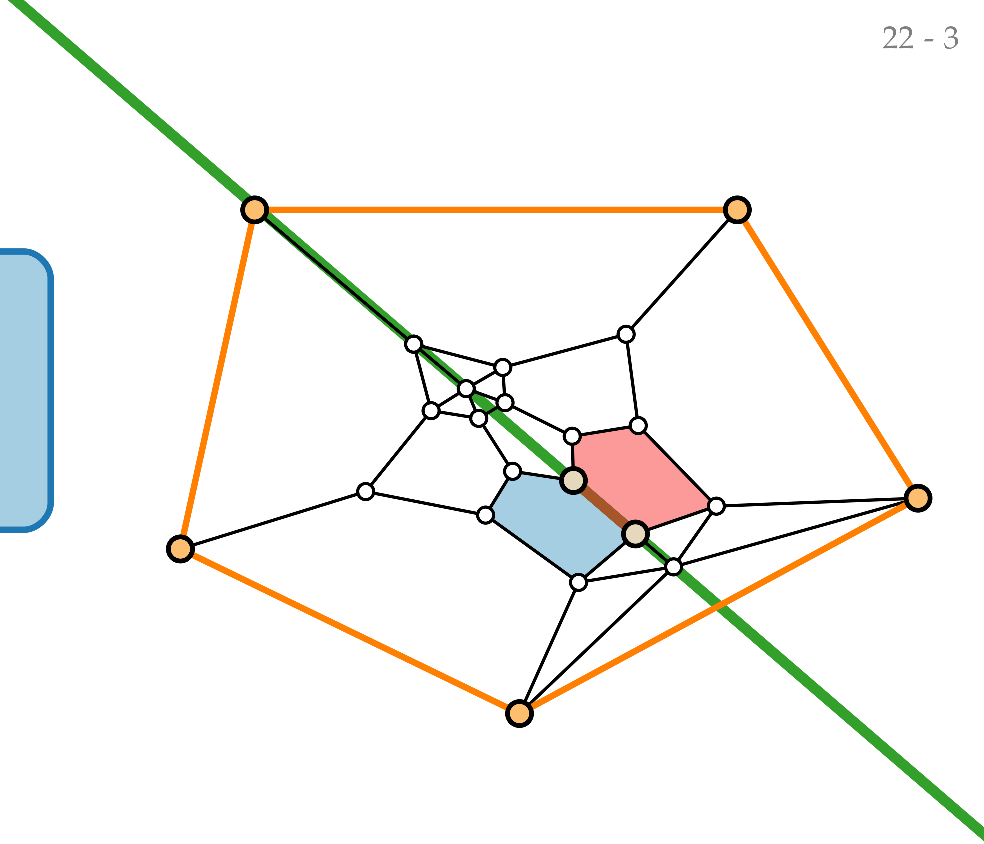

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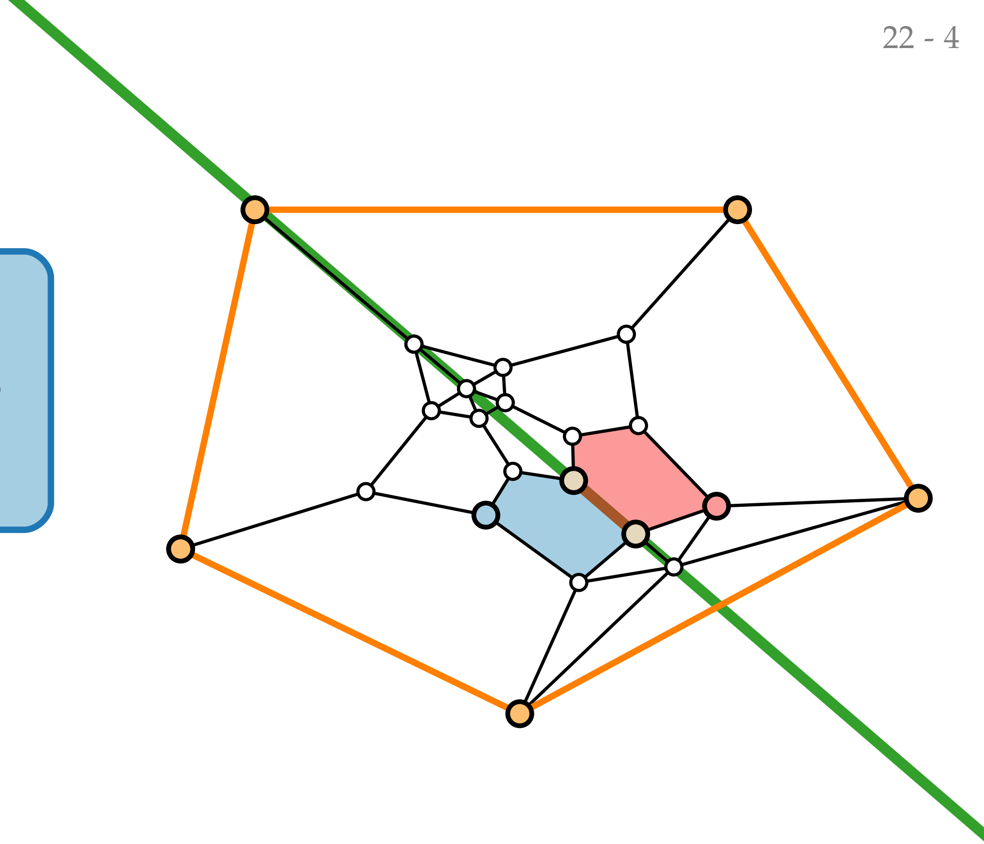

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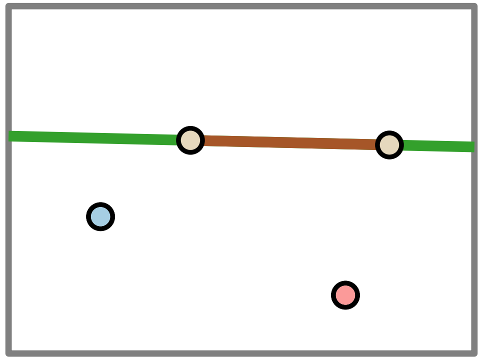

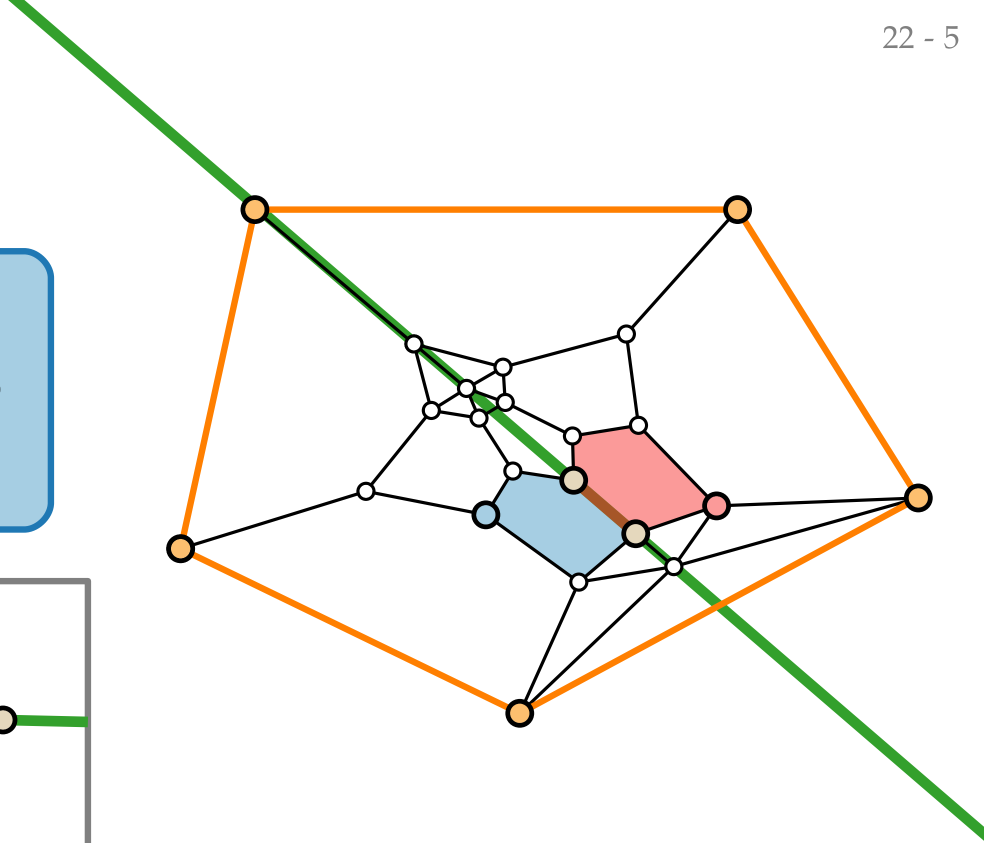

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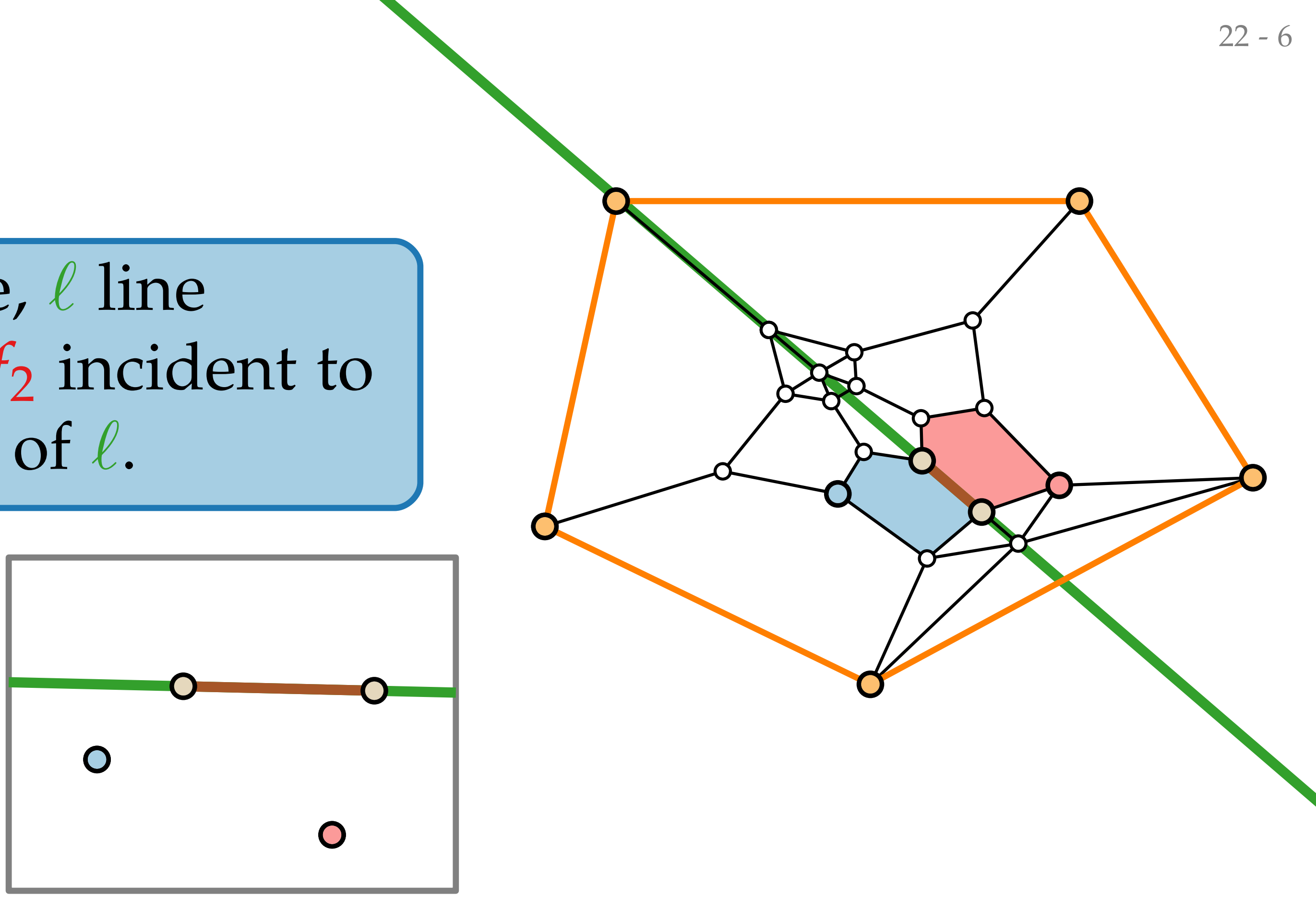

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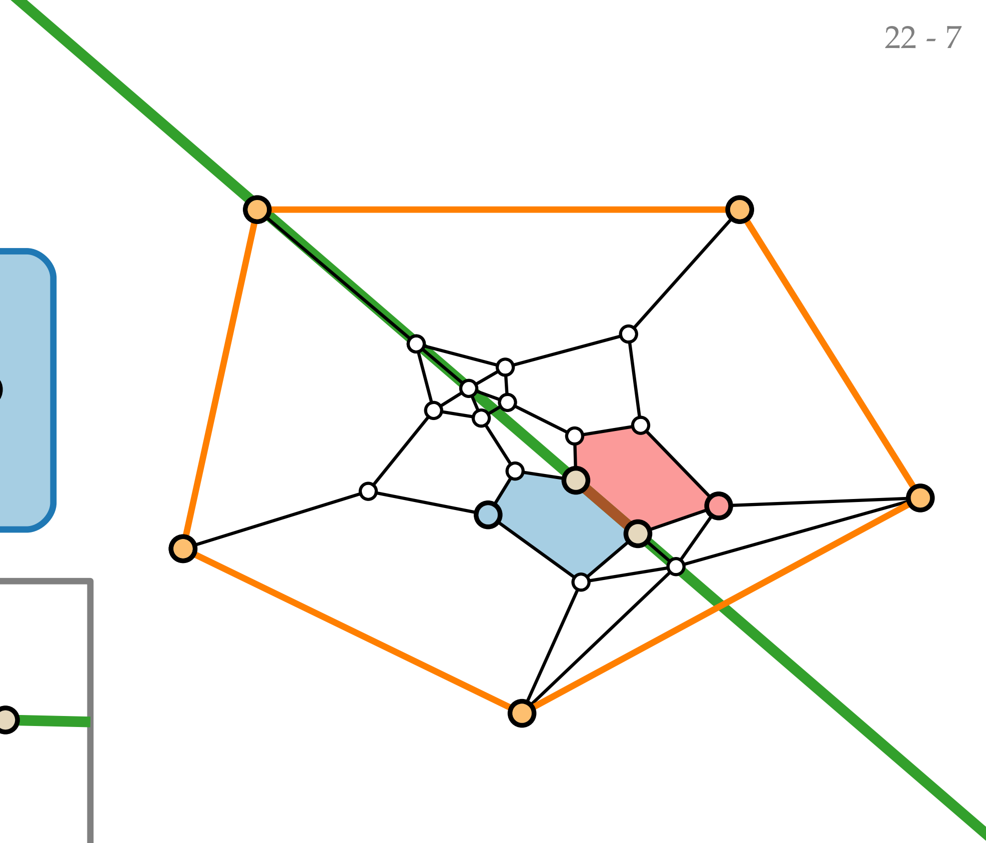

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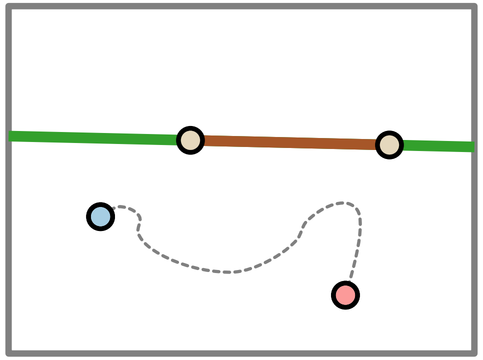

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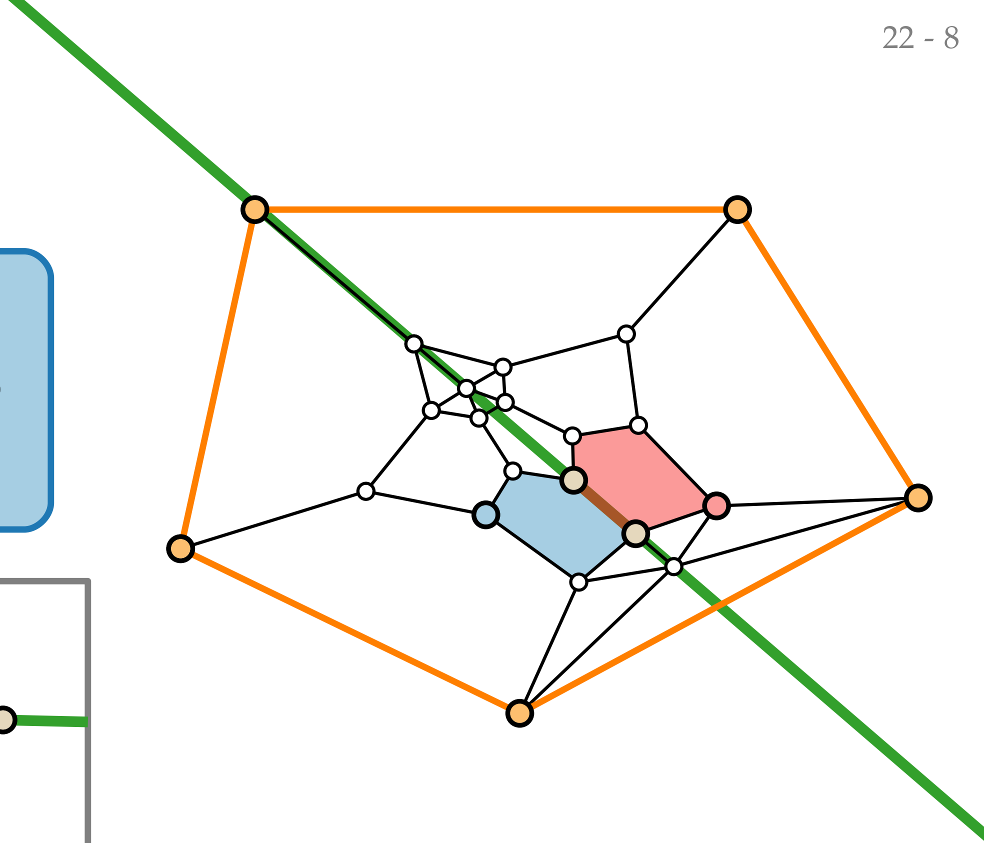

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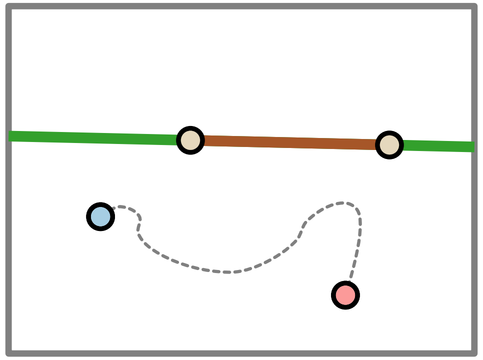

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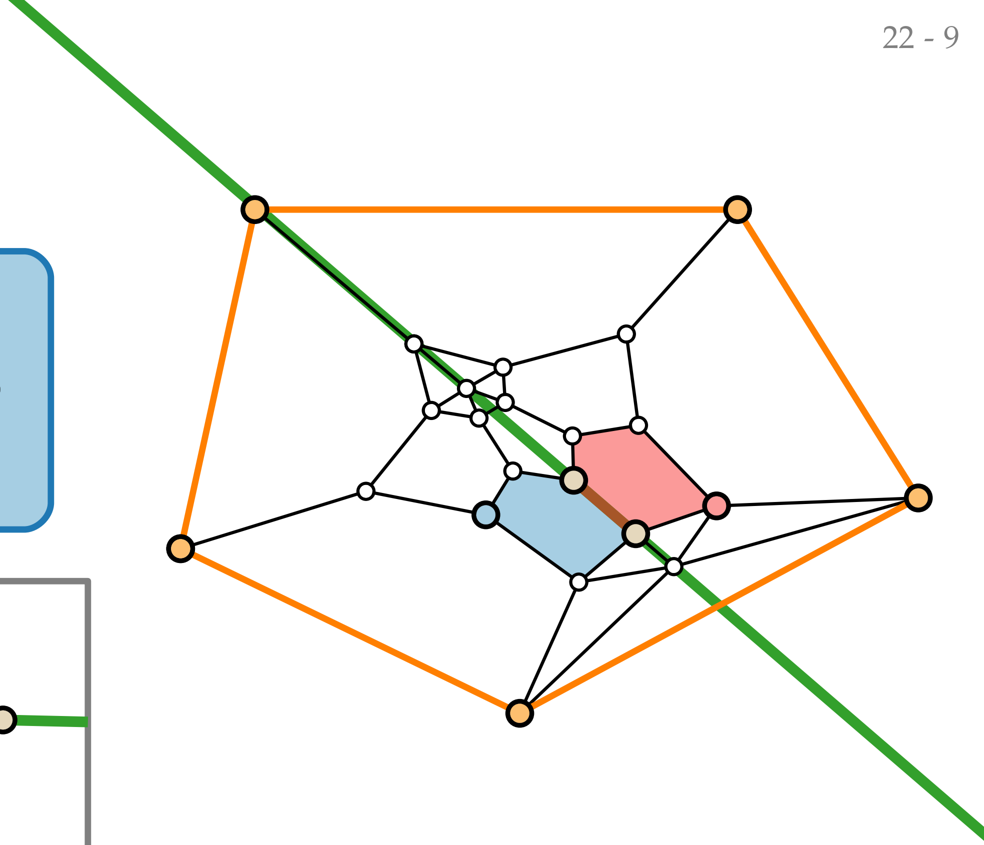

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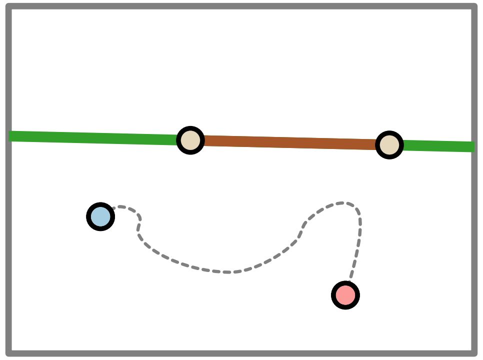

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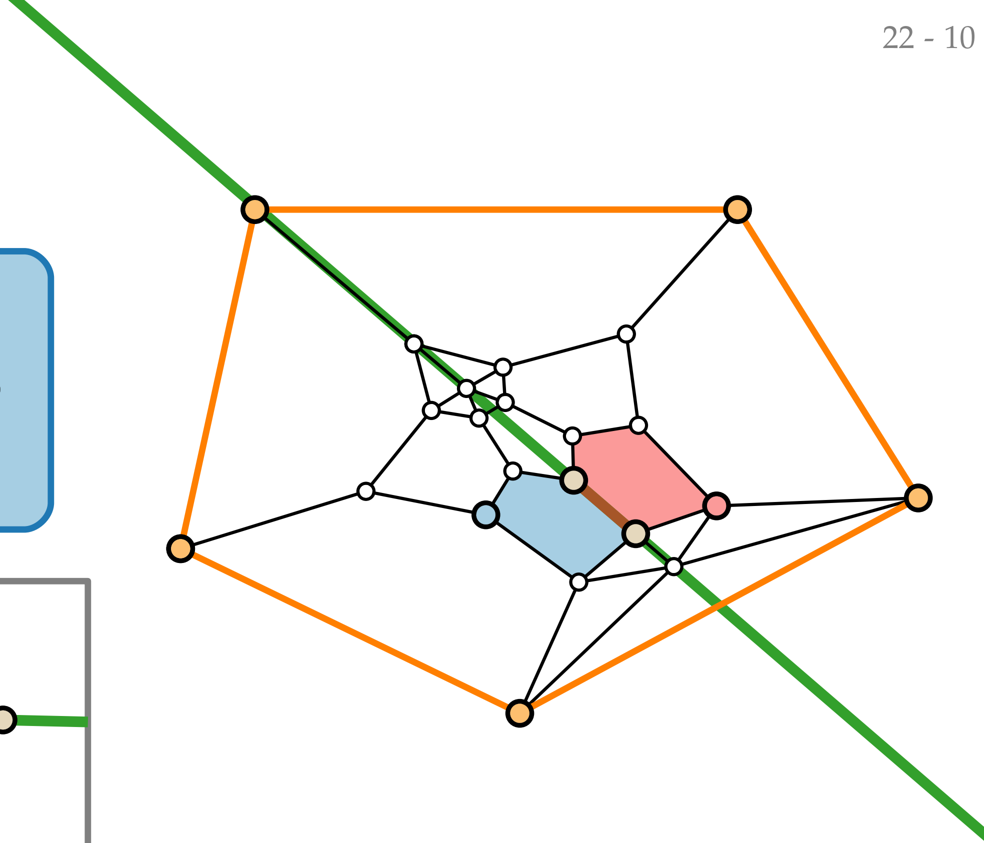

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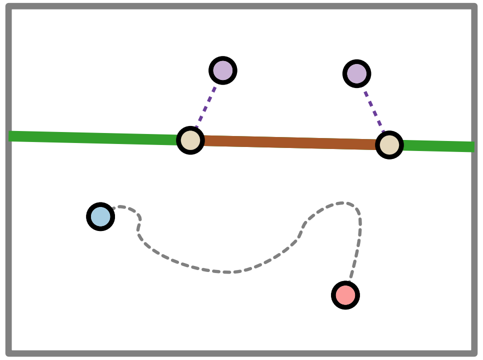

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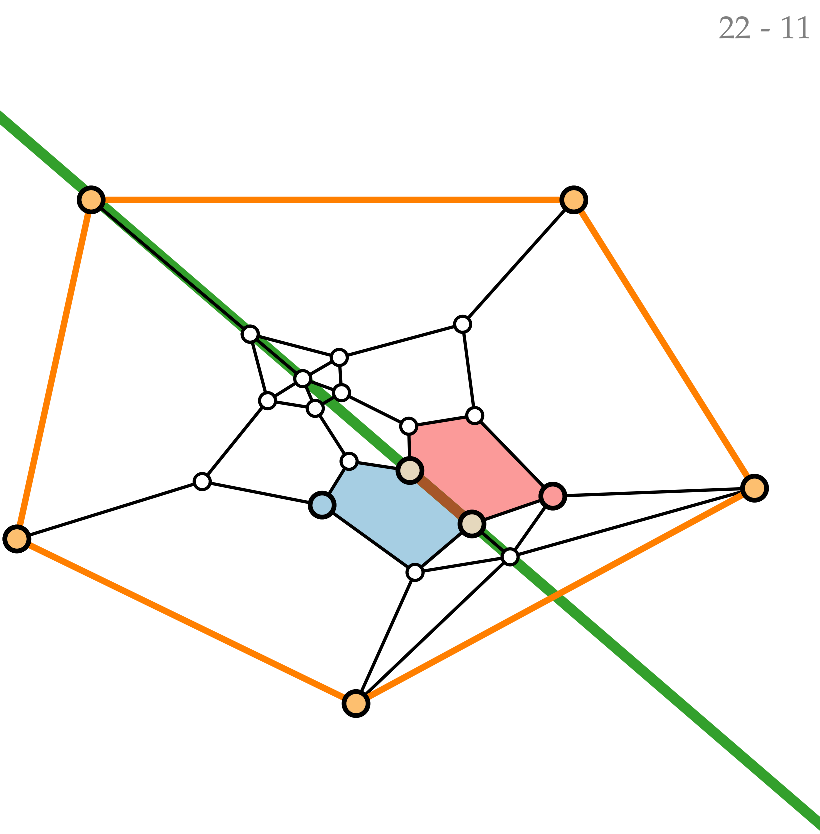
Proof of Tutte’s Theorem



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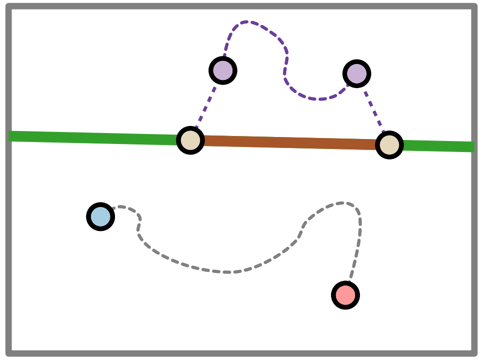

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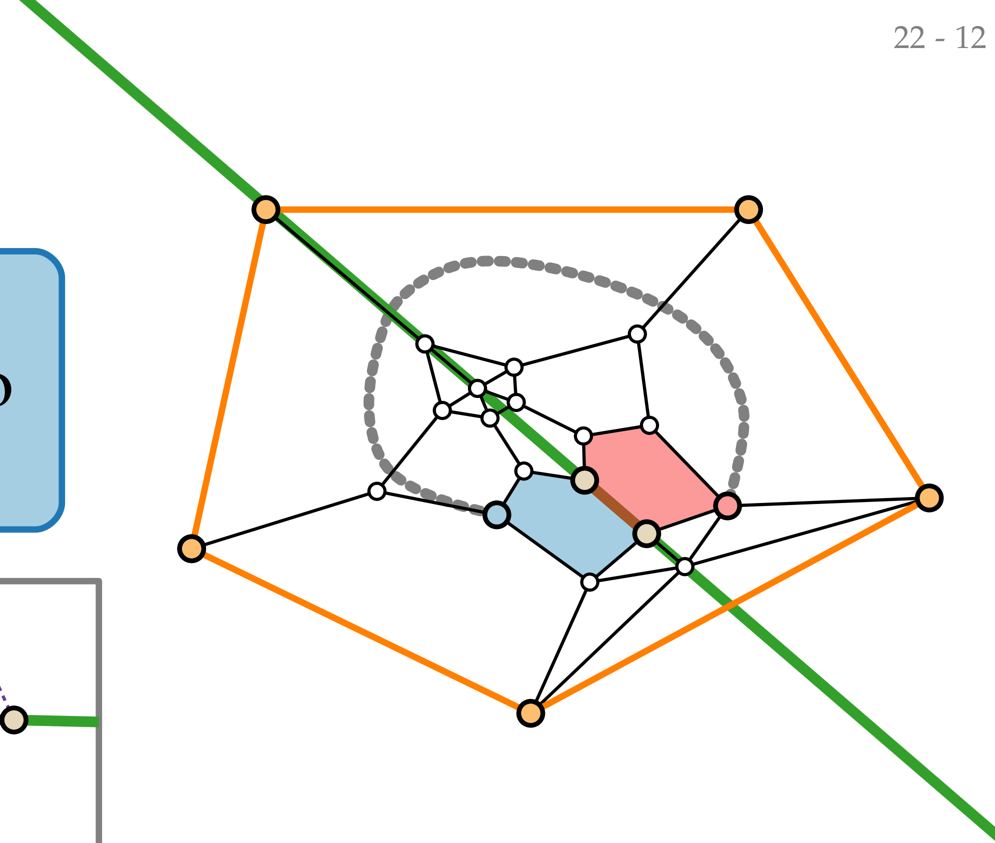

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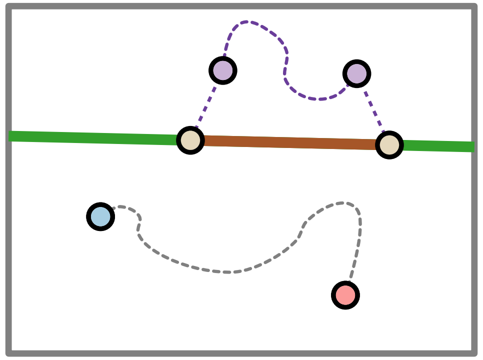

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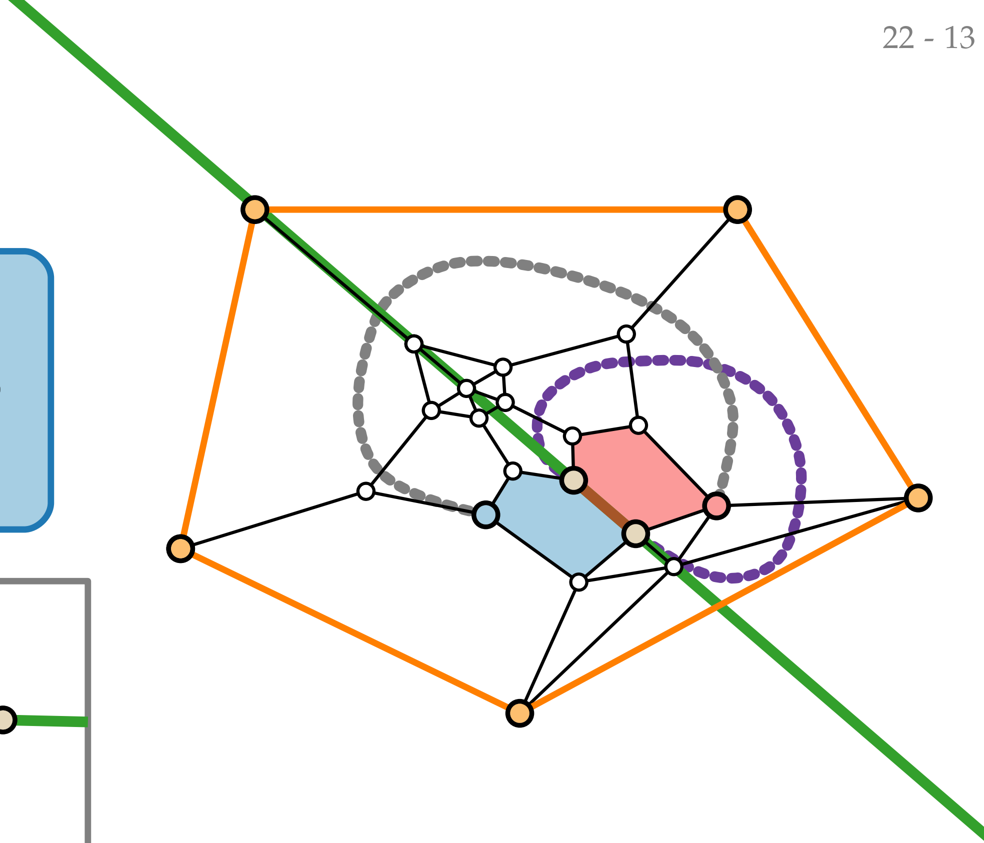

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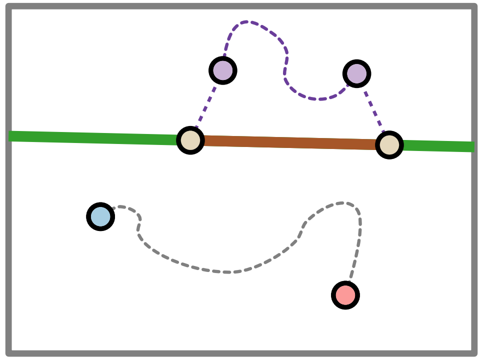

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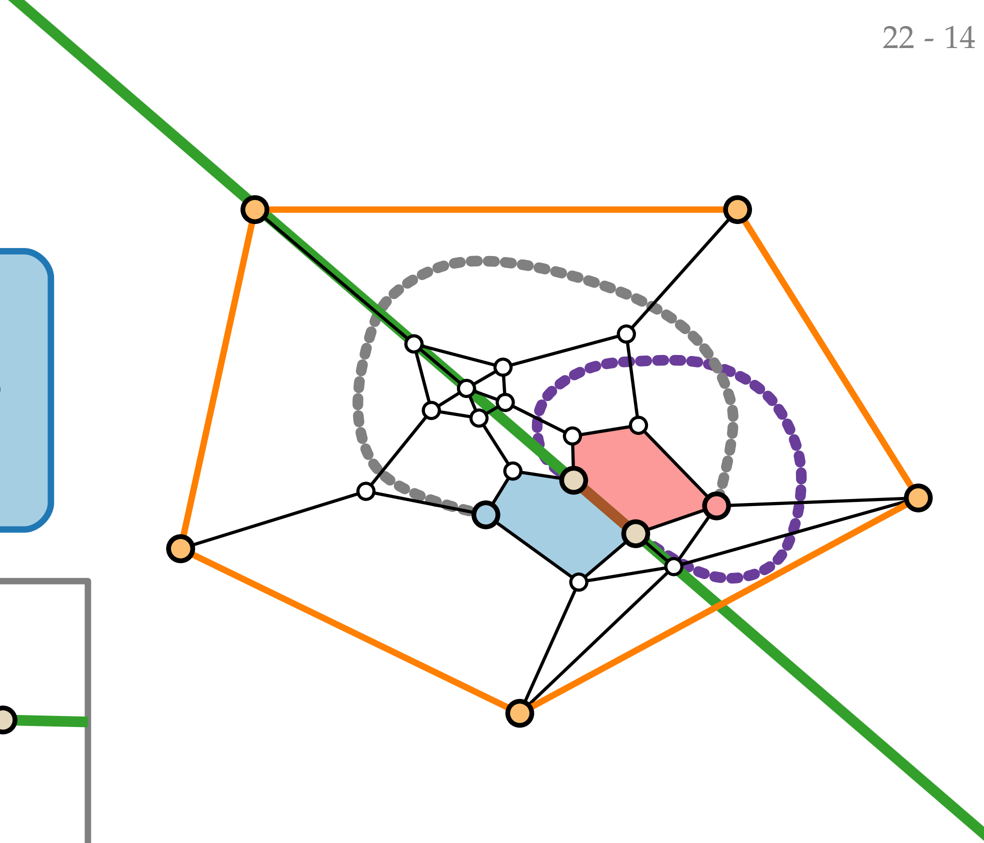

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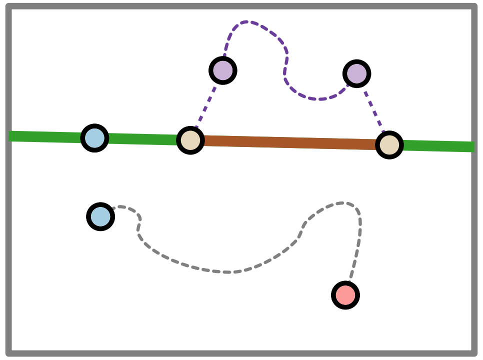

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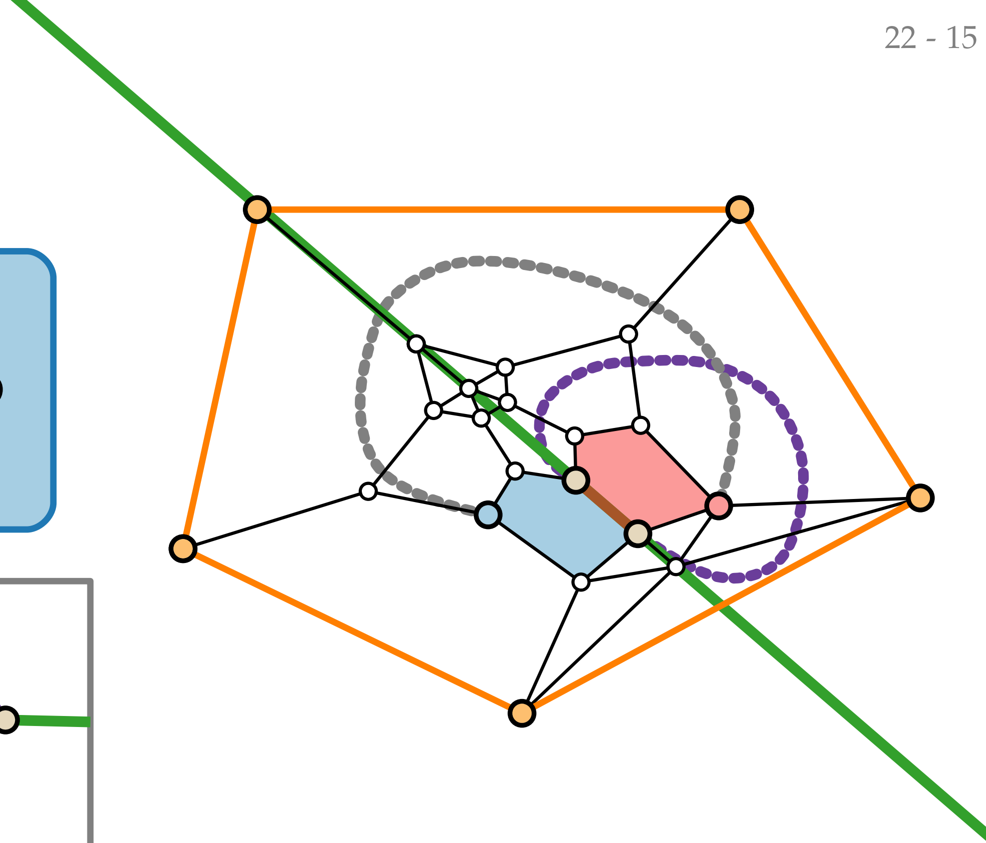

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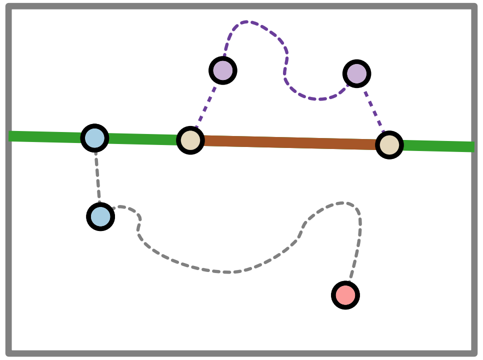

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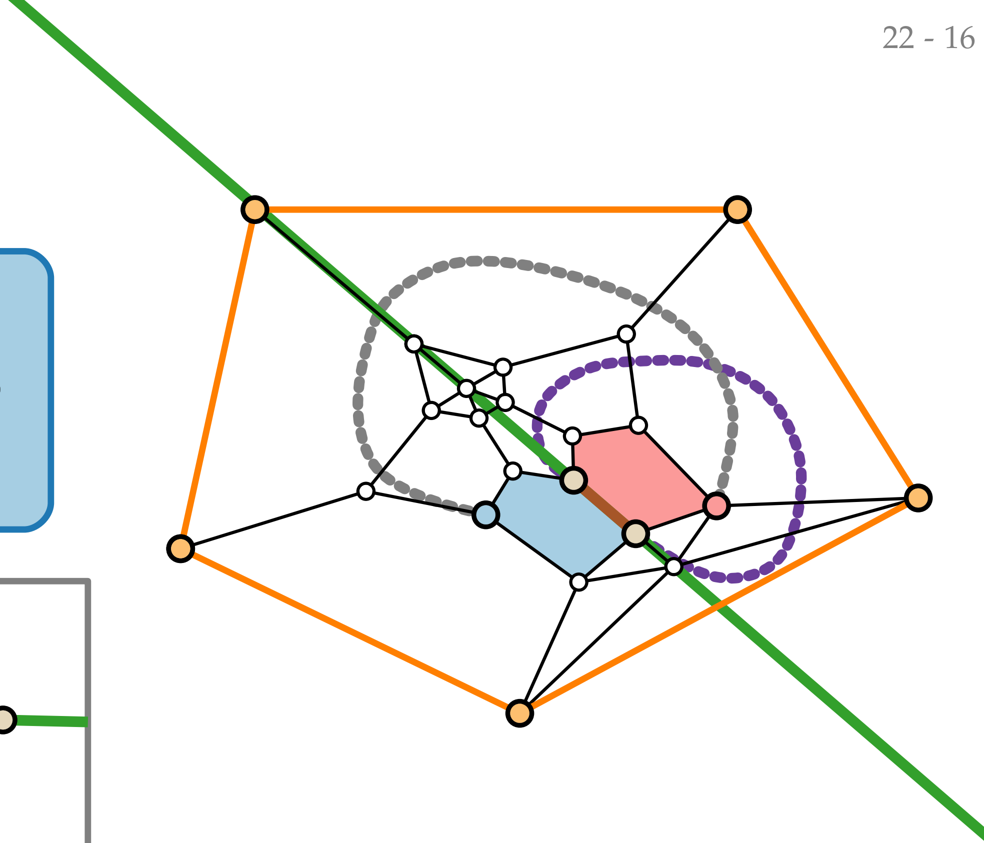

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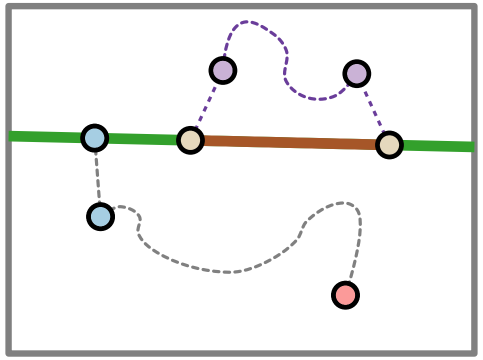

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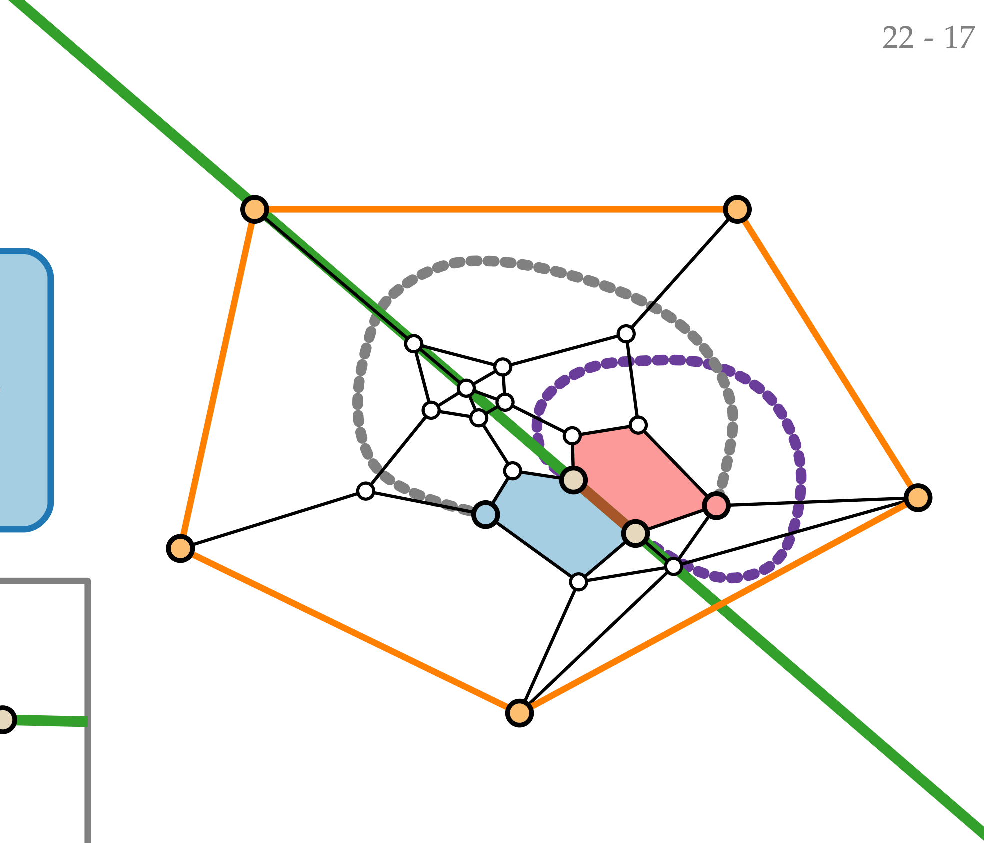

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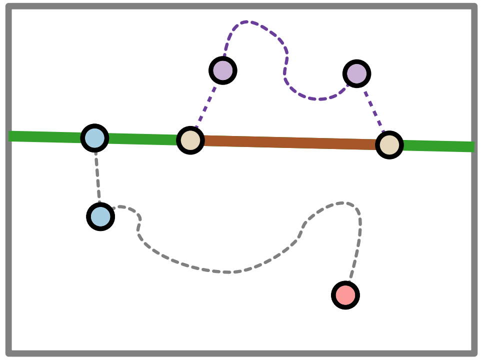

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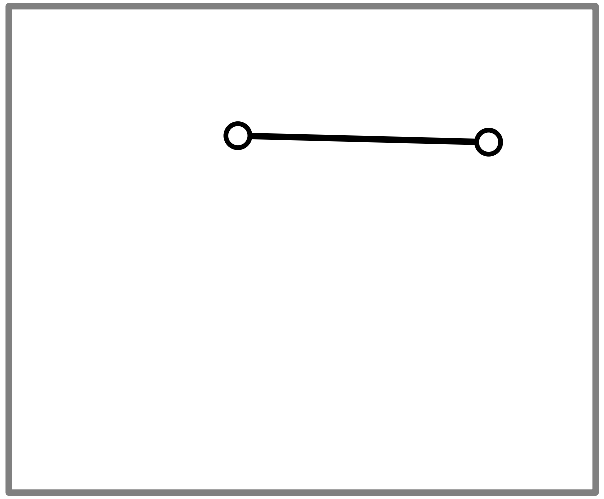

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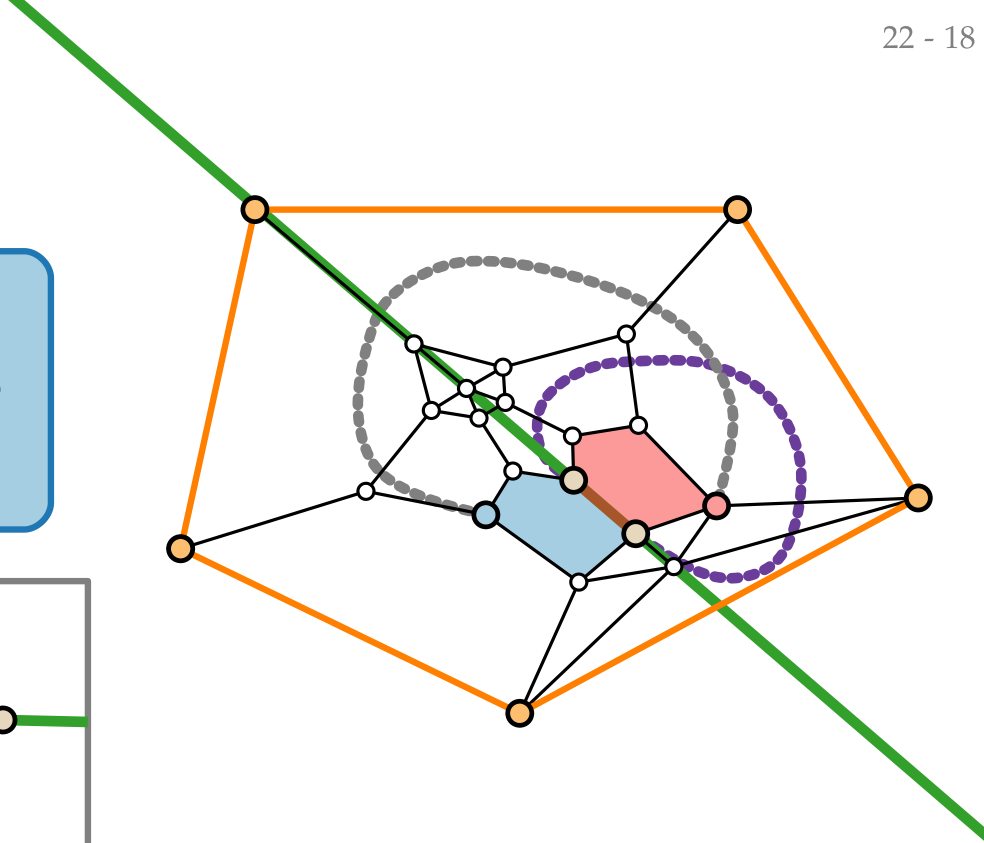

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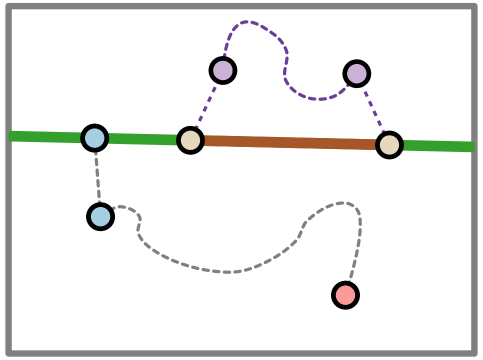

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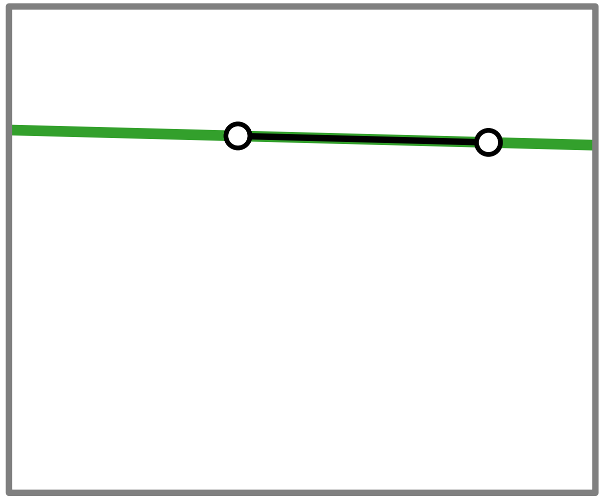

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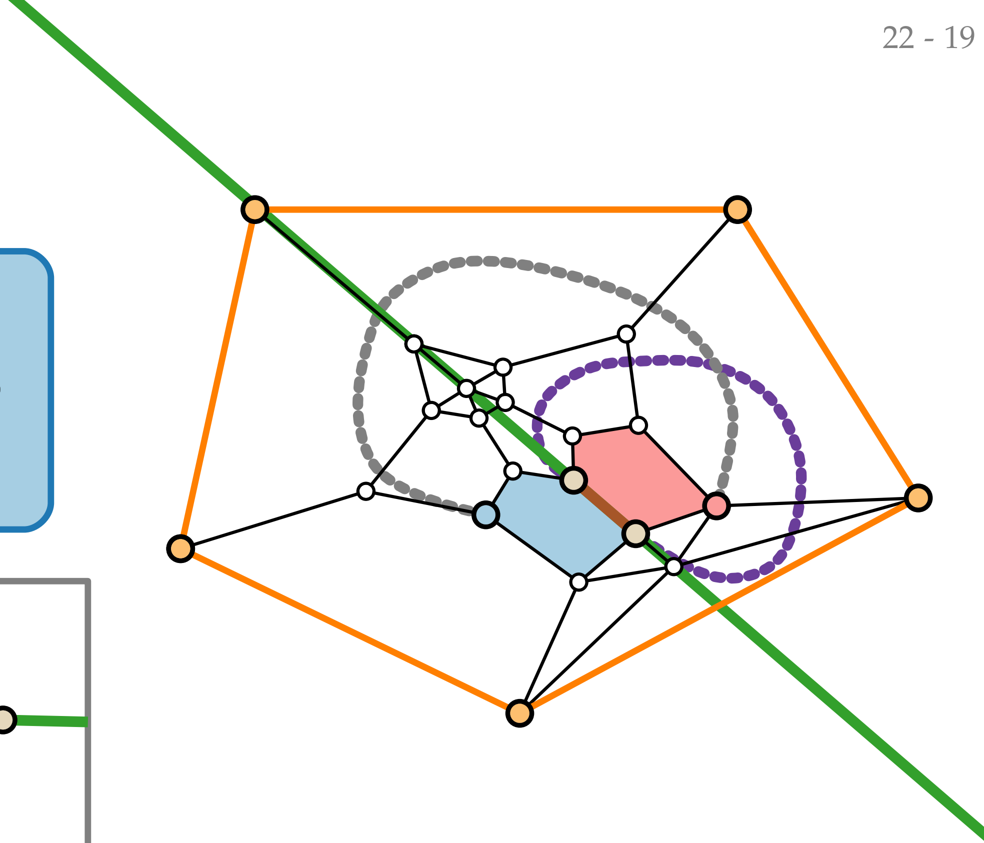

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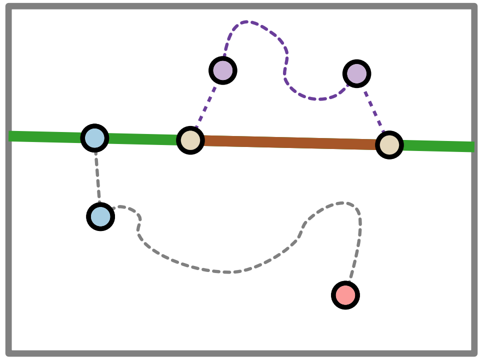

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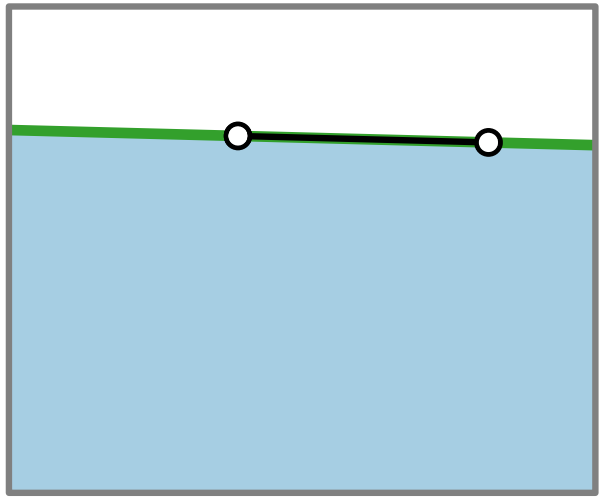

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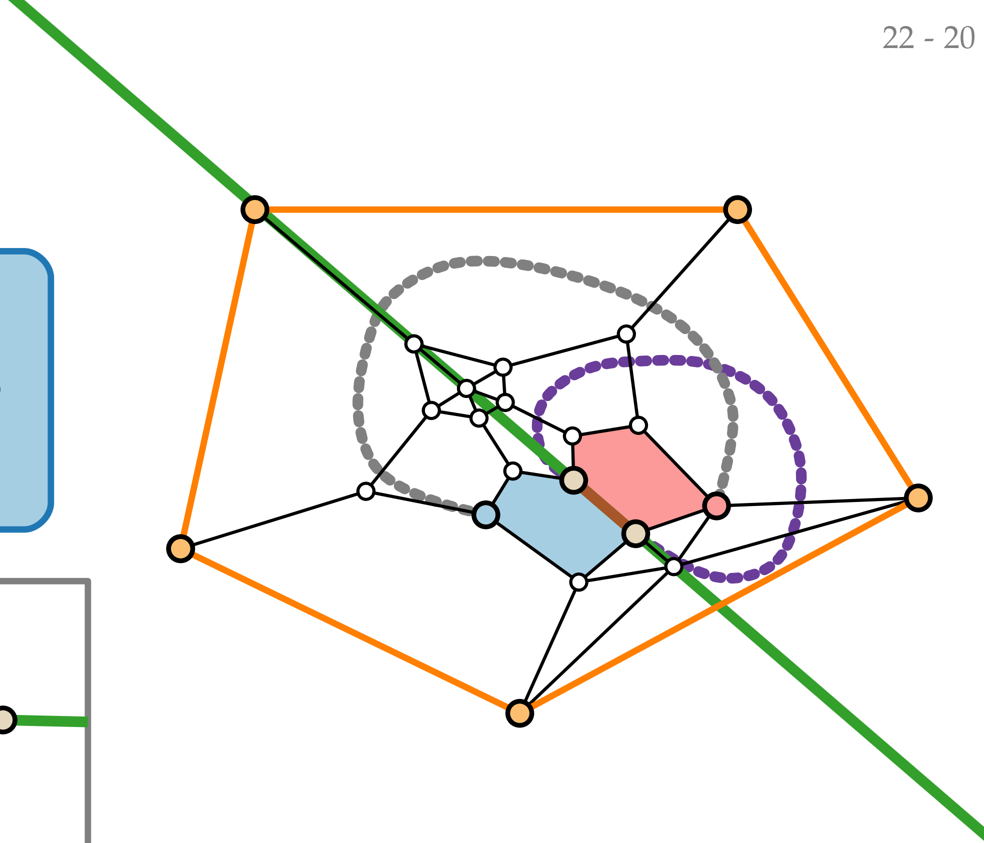

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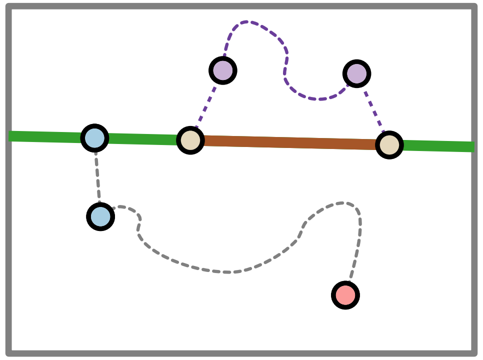

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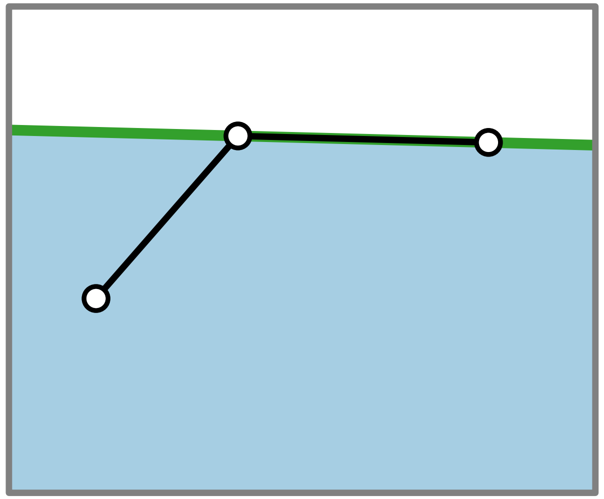

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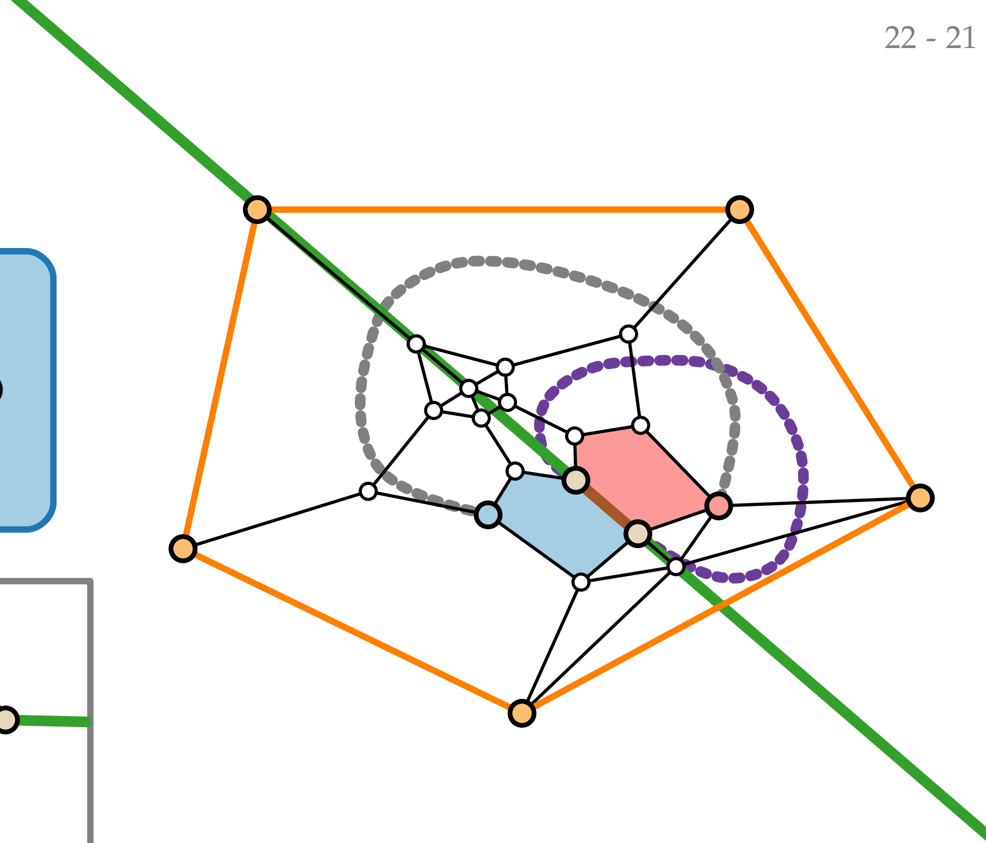

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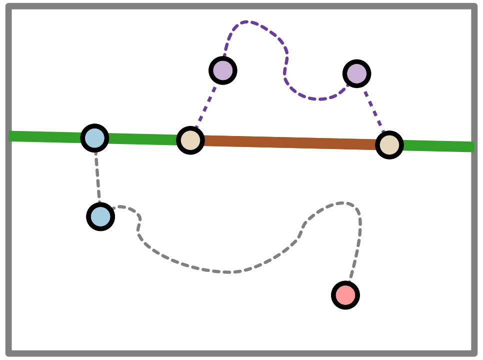

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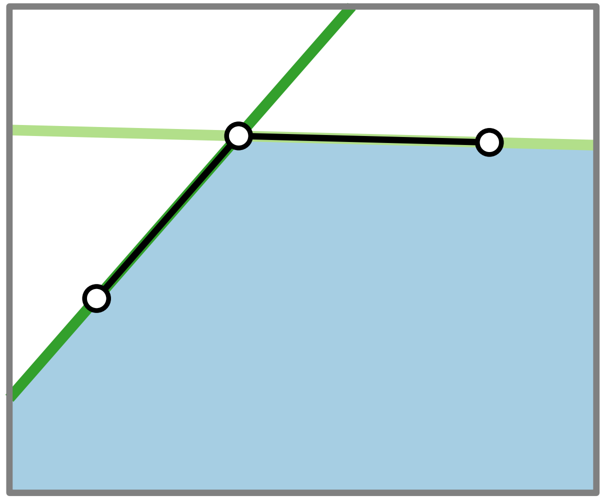

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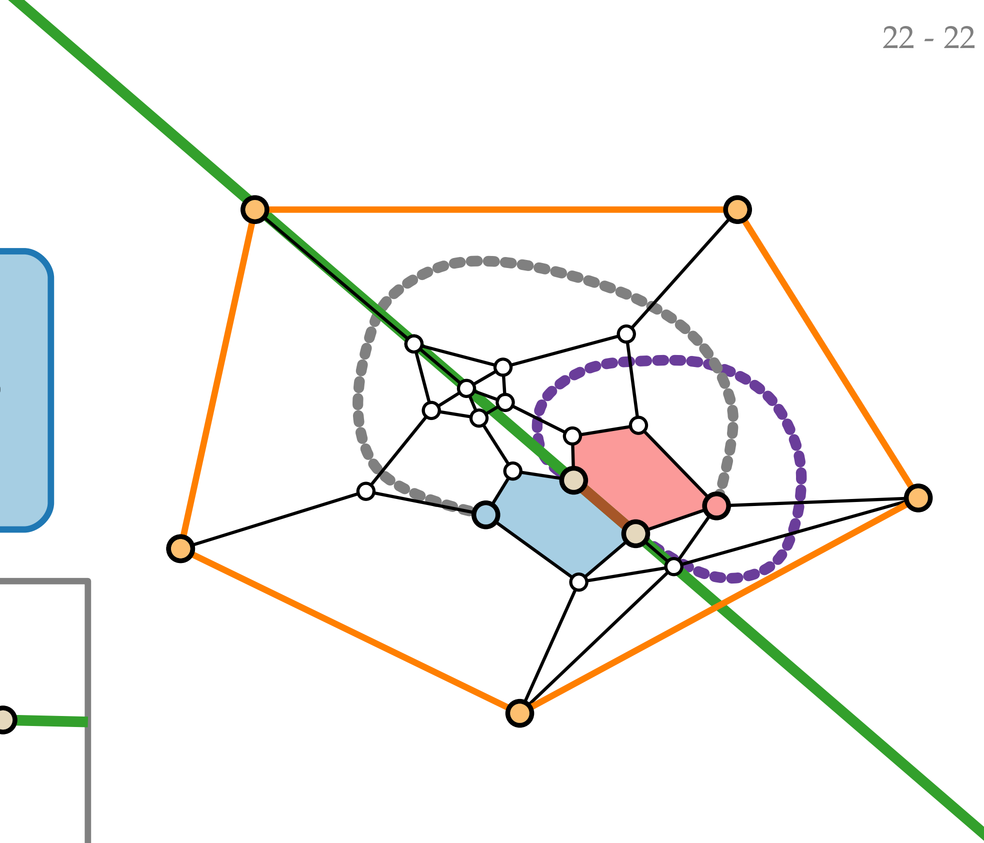

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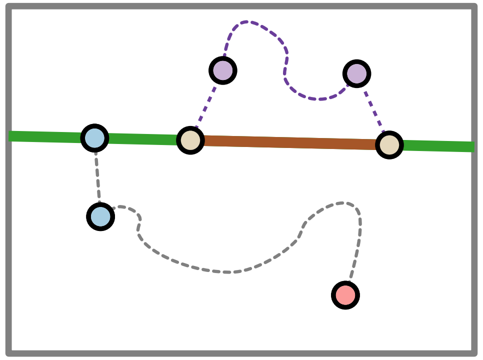

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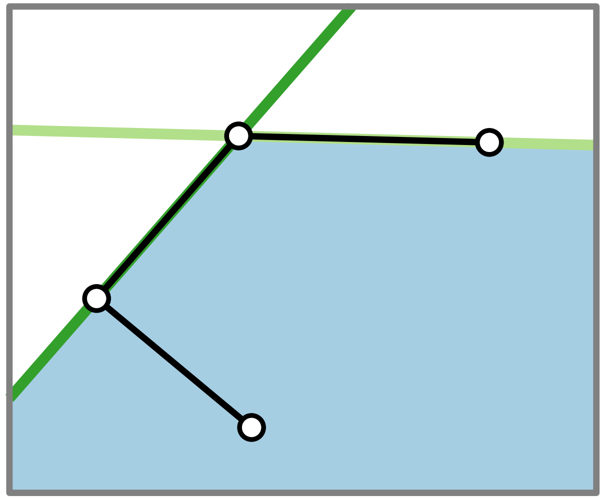

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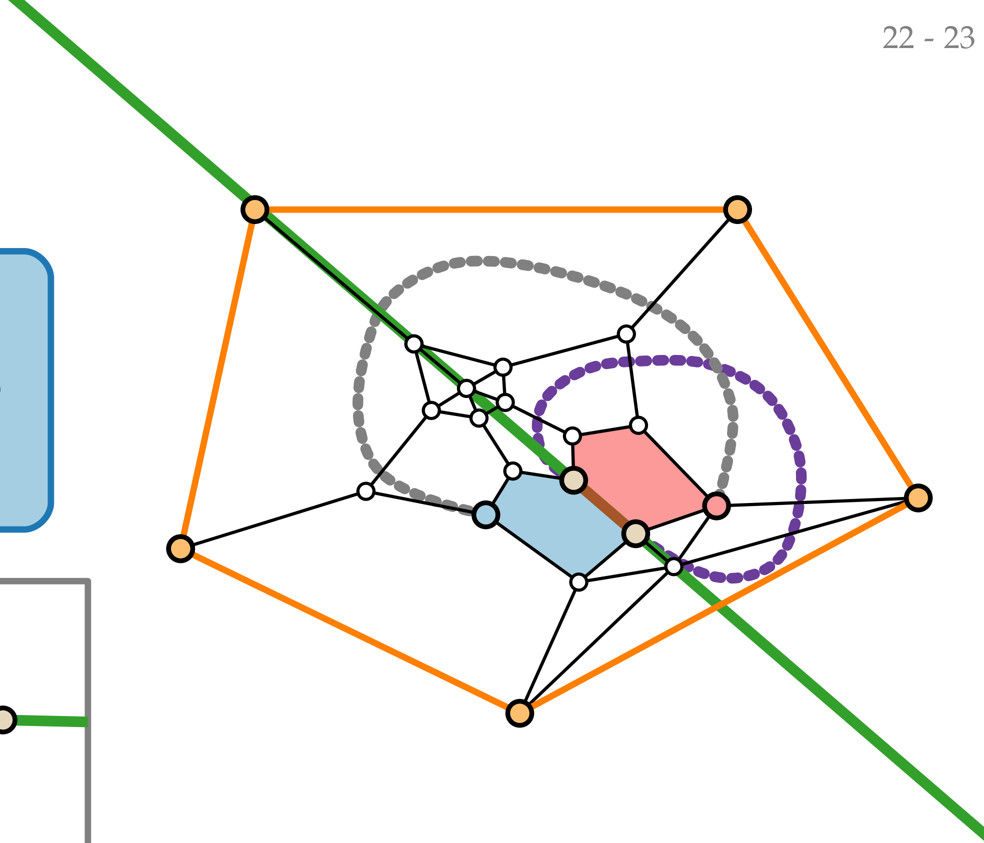

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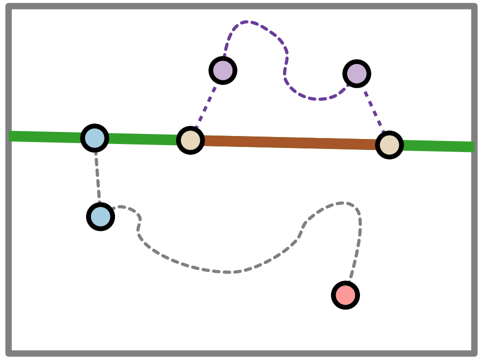

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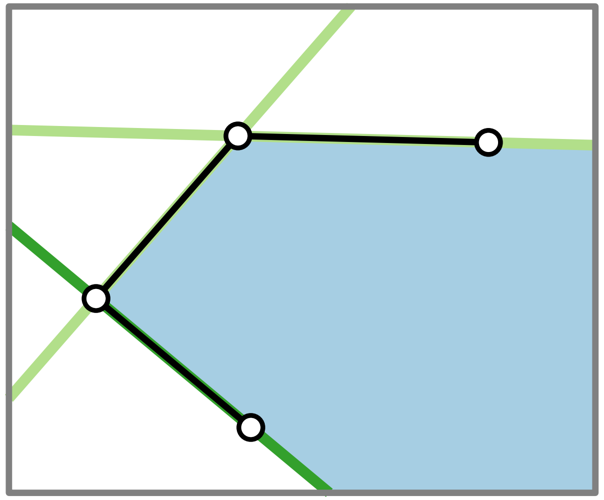

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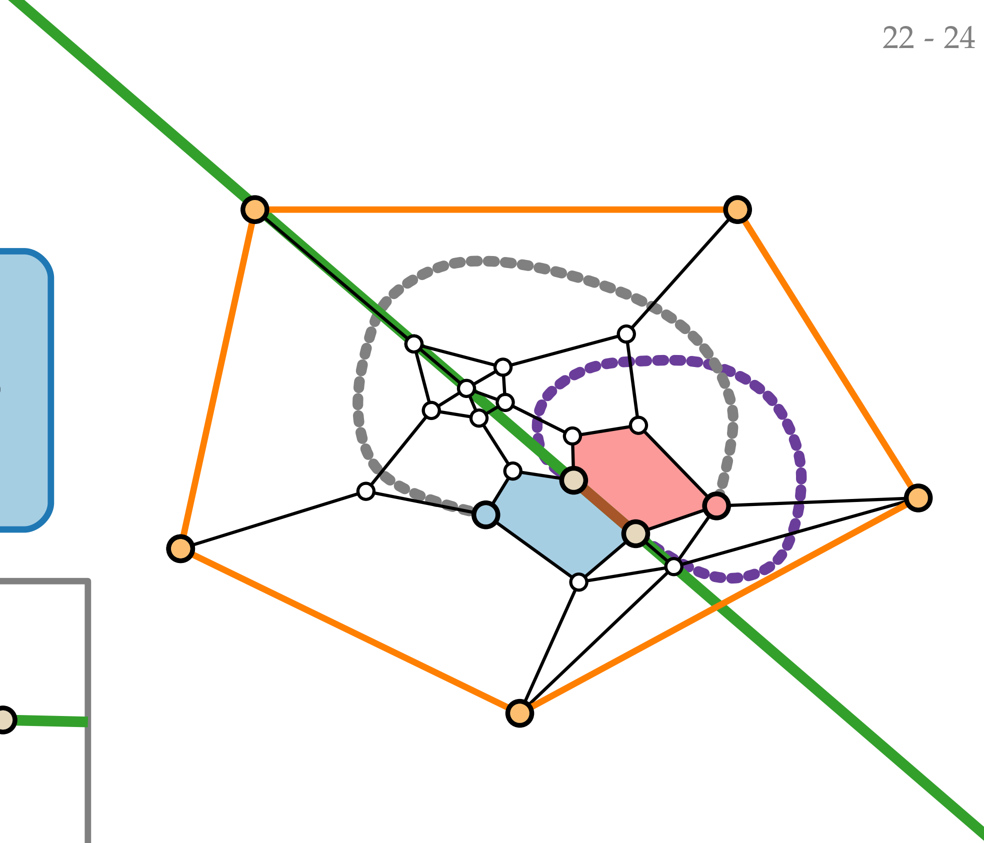

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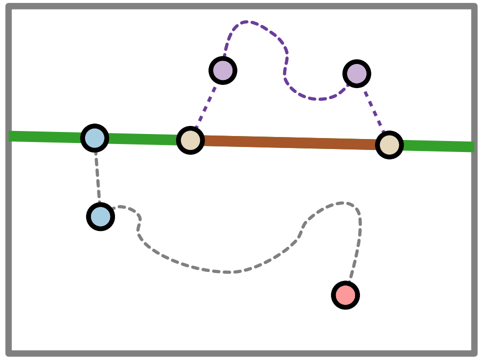

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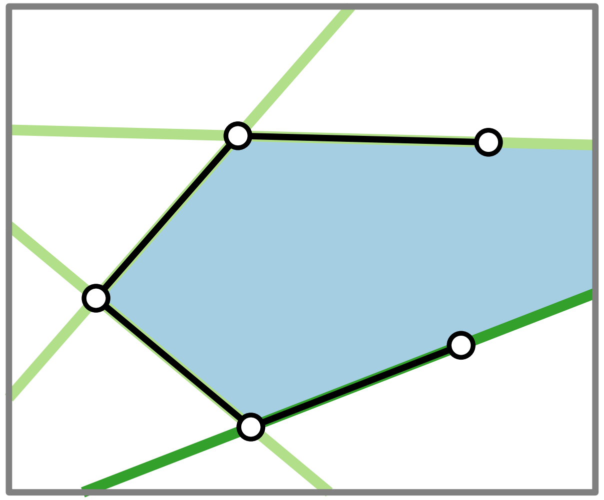

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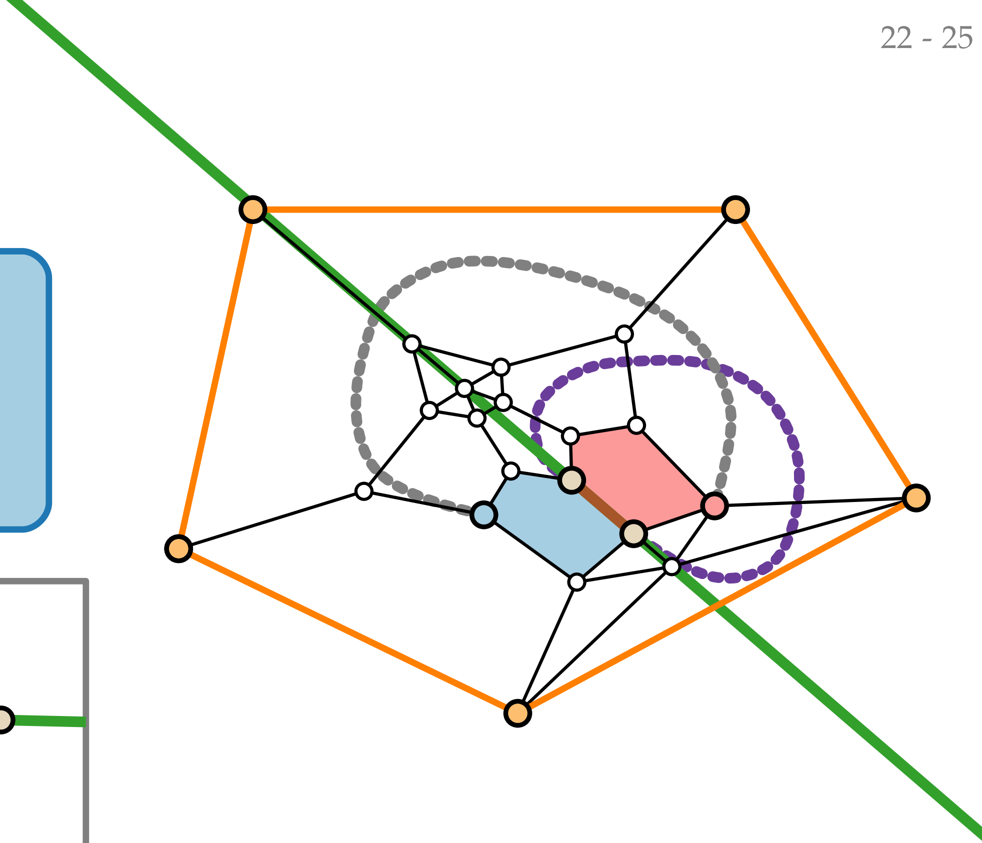

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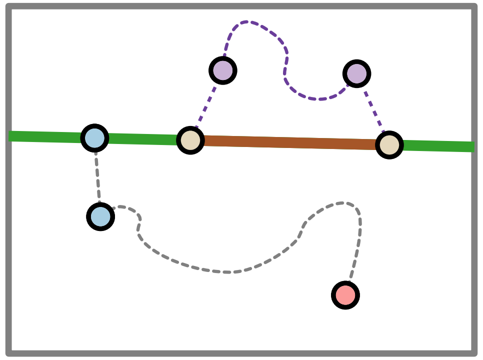

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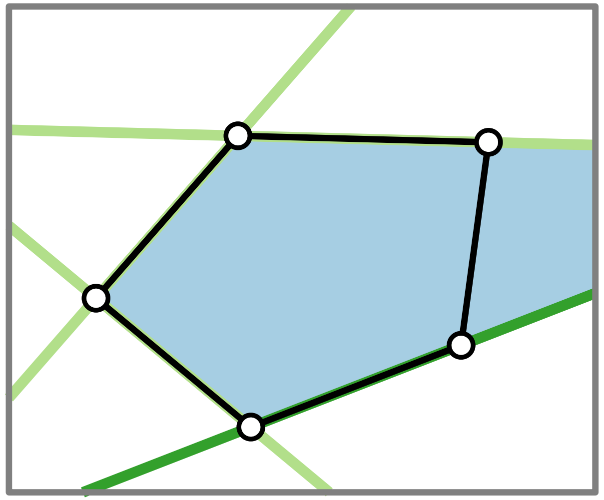

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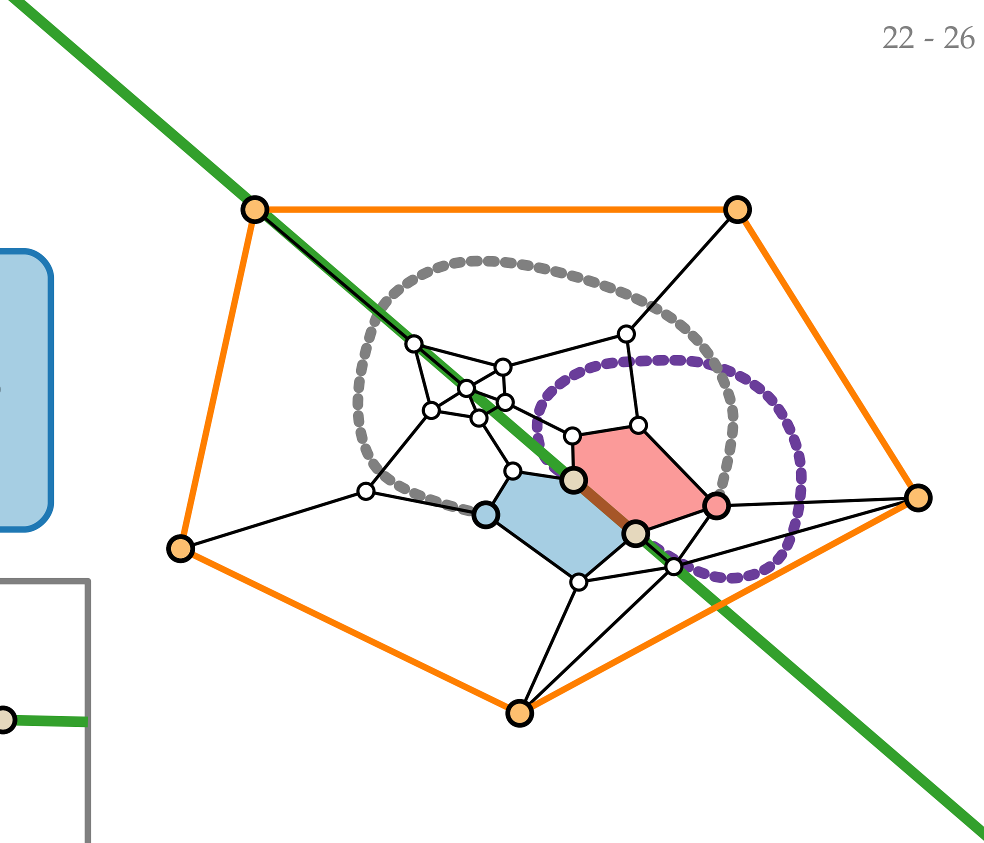

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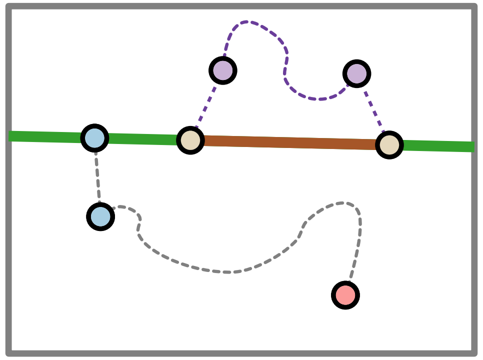

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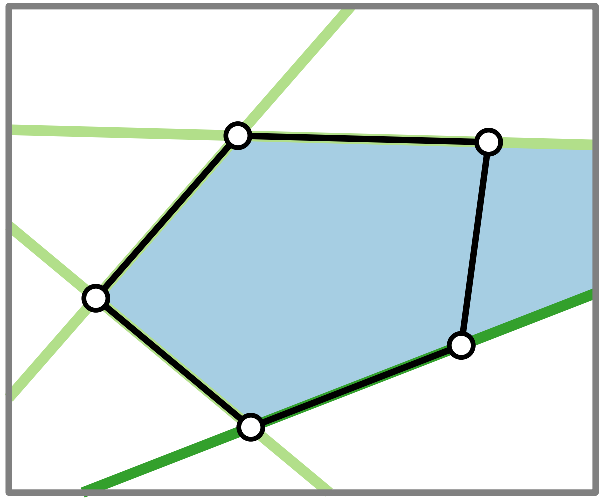

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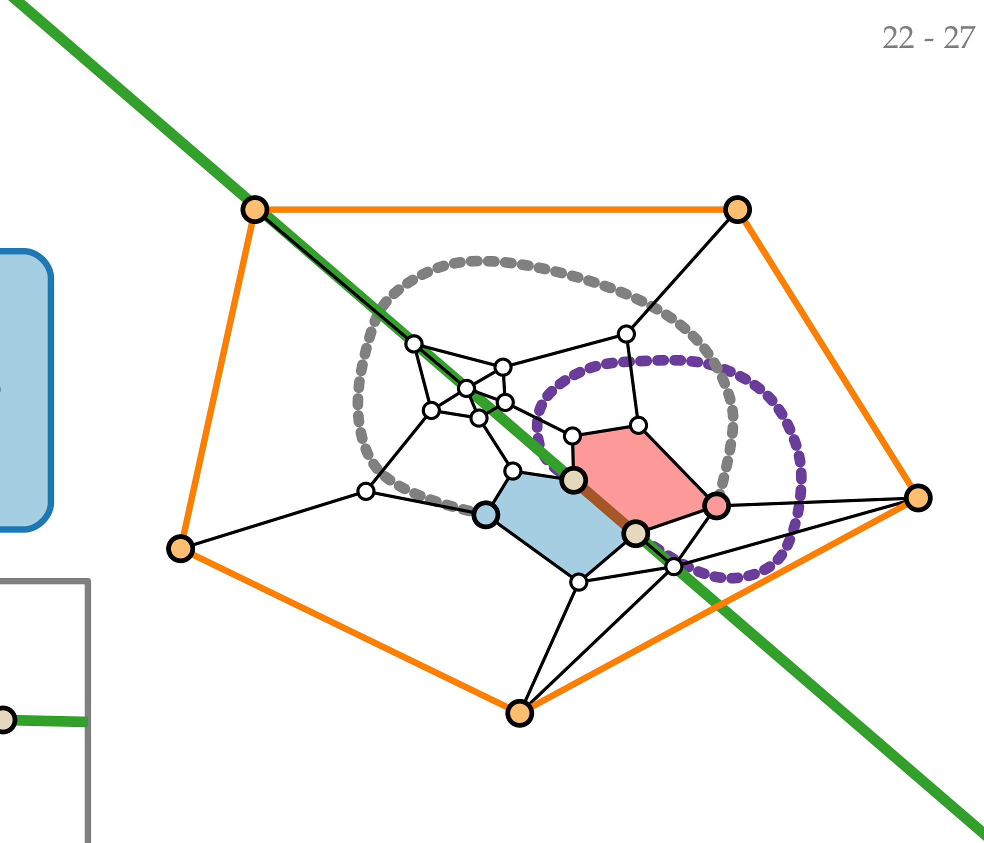

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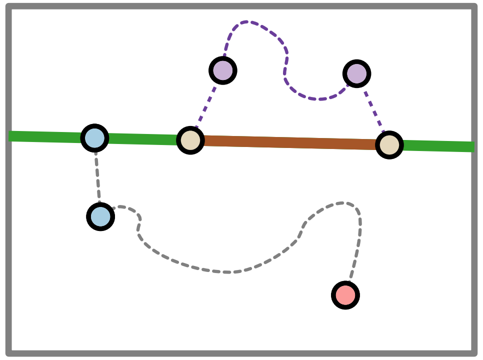

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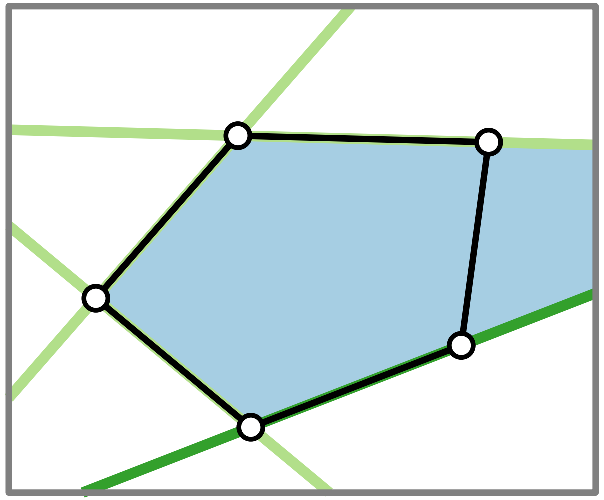

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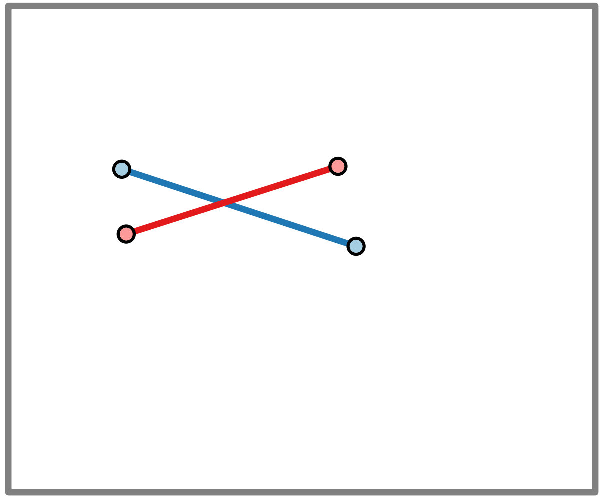

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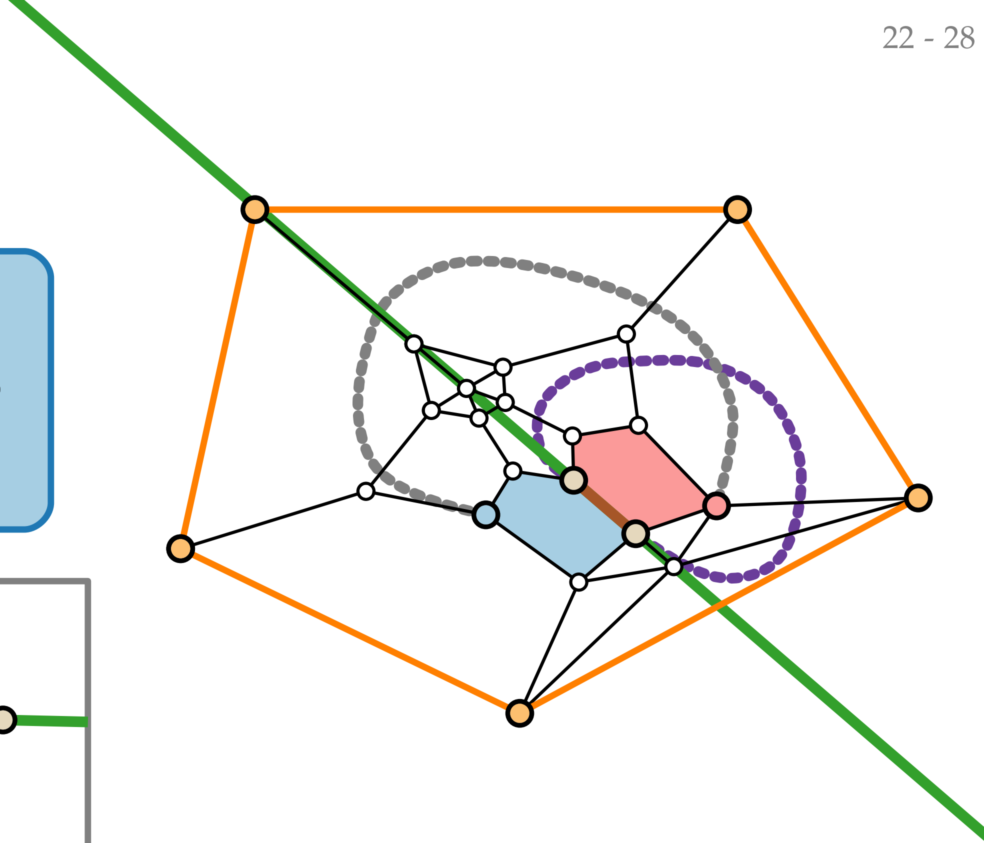

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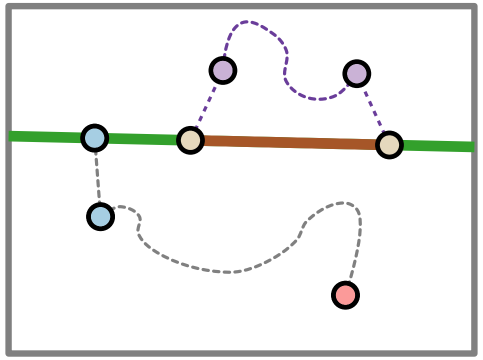
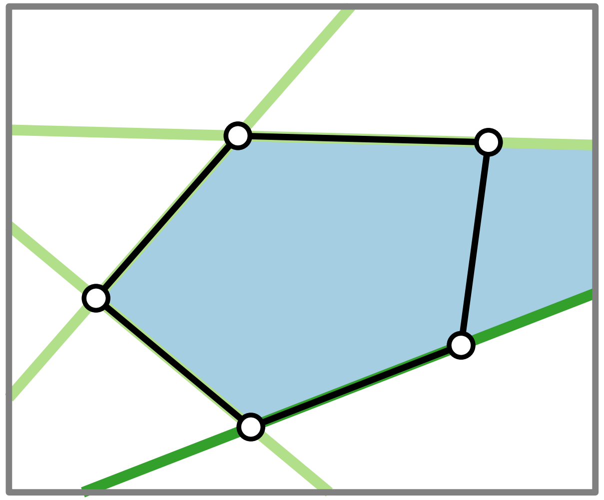
Property 4. No vertex is collinear with all of its neighbors.



Lemma. All faces are strictly convex.



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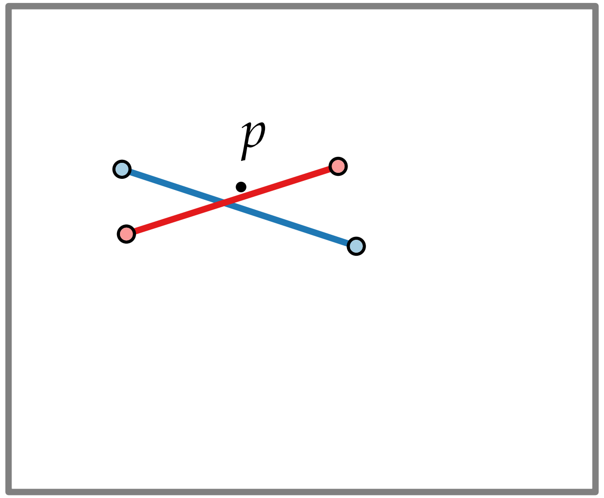
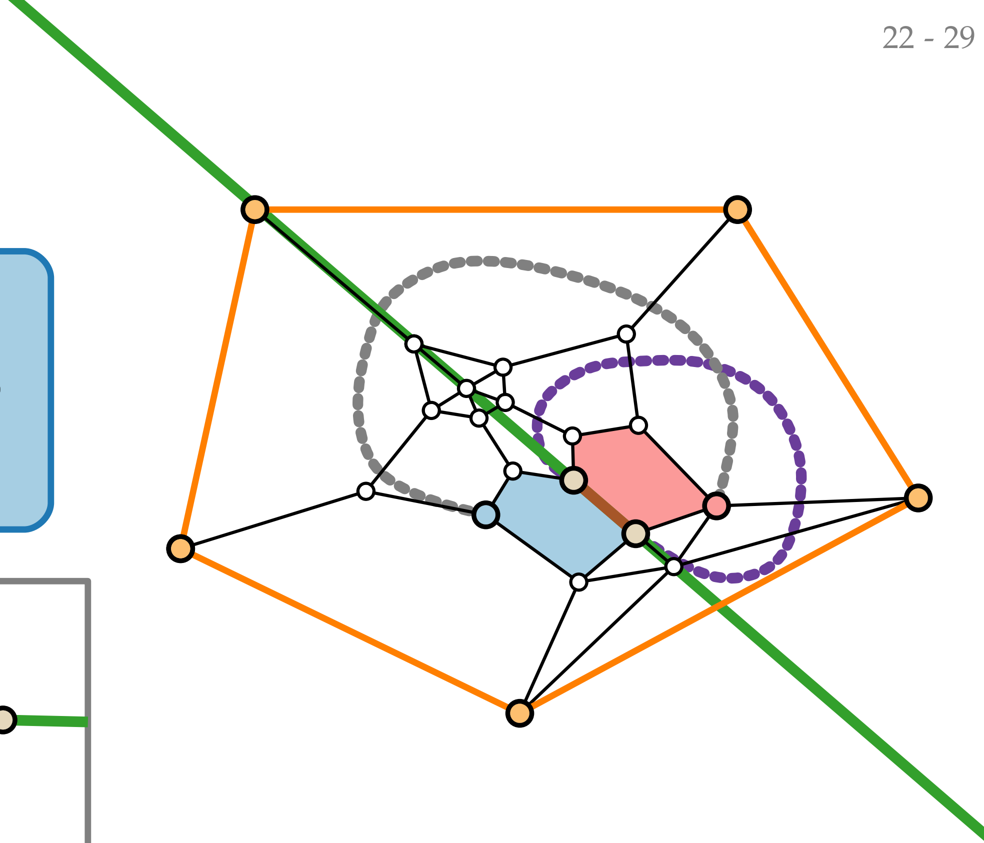


Proof of Tutte's Theorem



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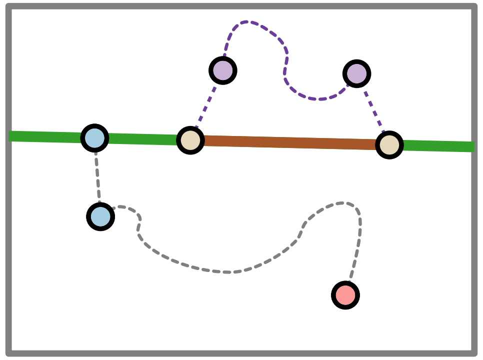

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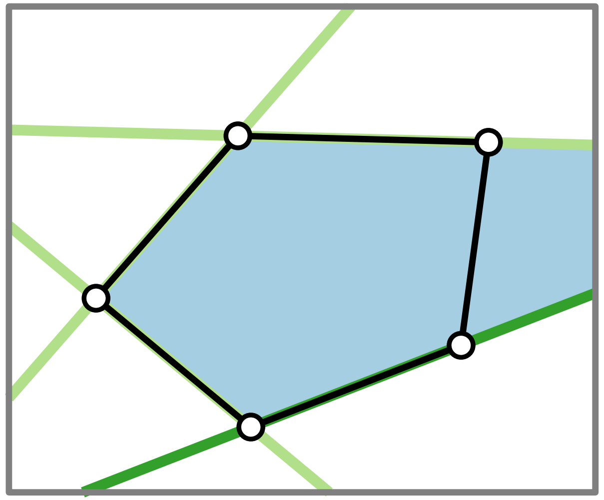

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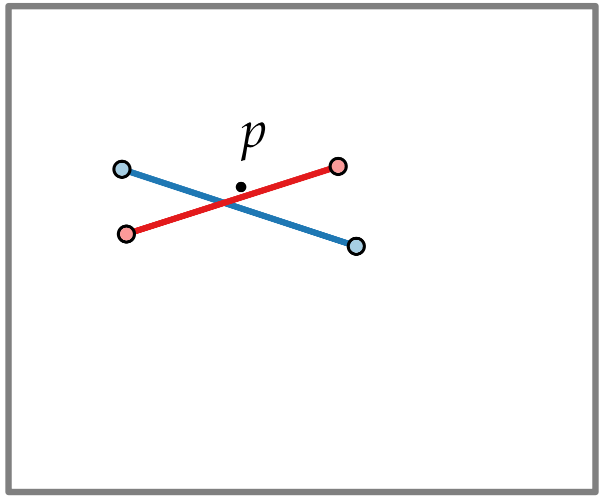
p inside two faces



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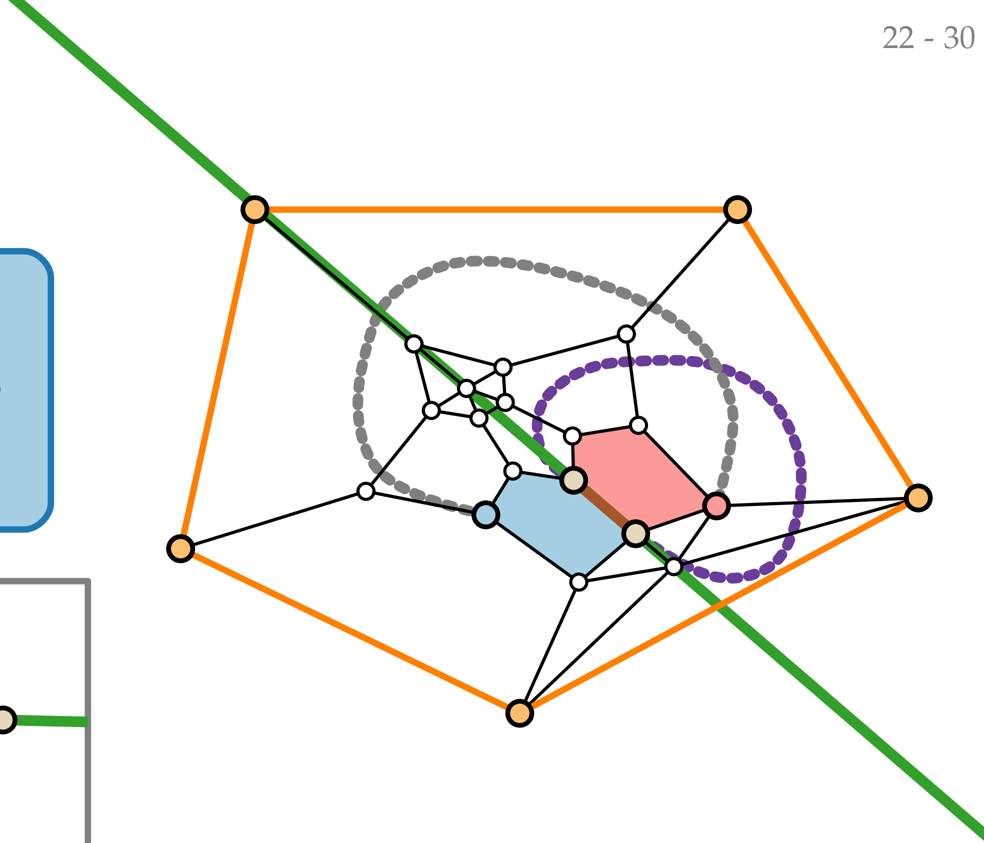

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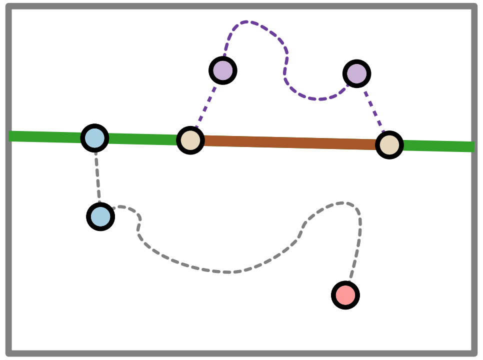

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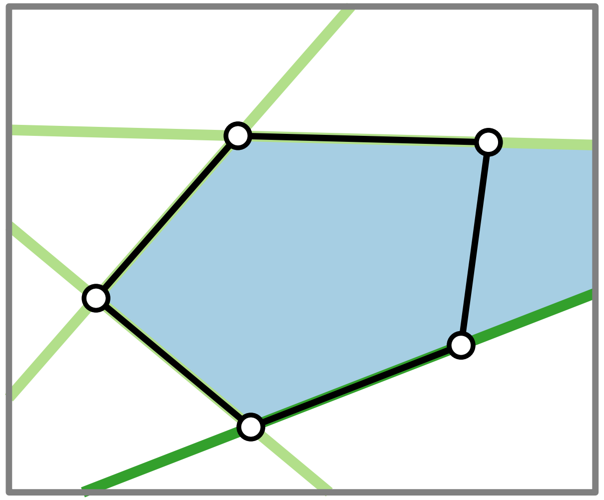

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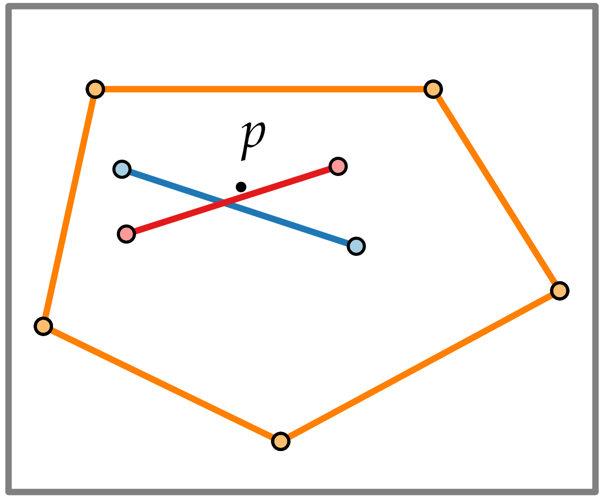

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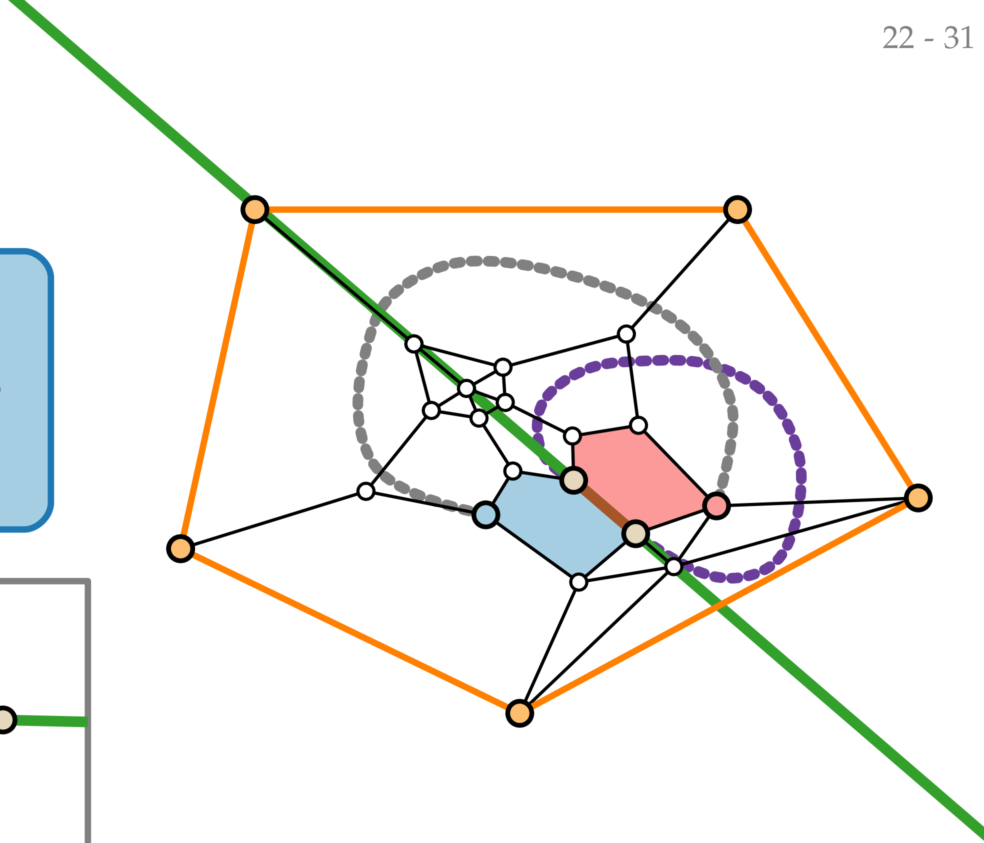

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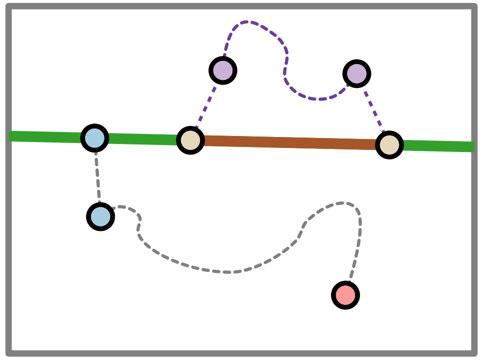

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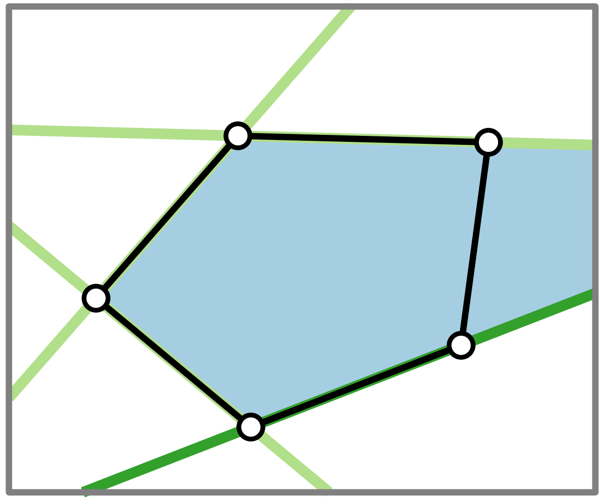

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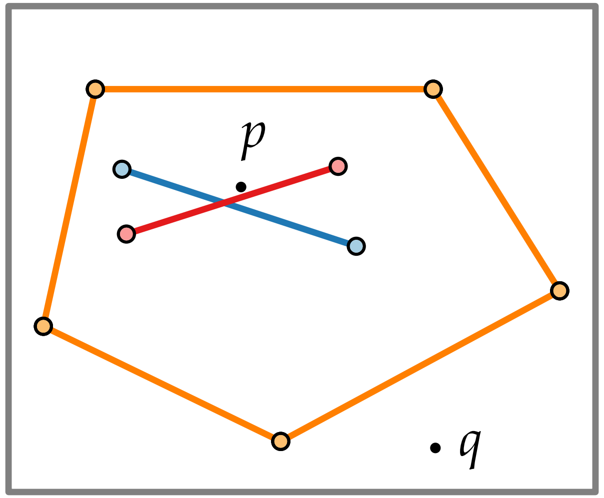

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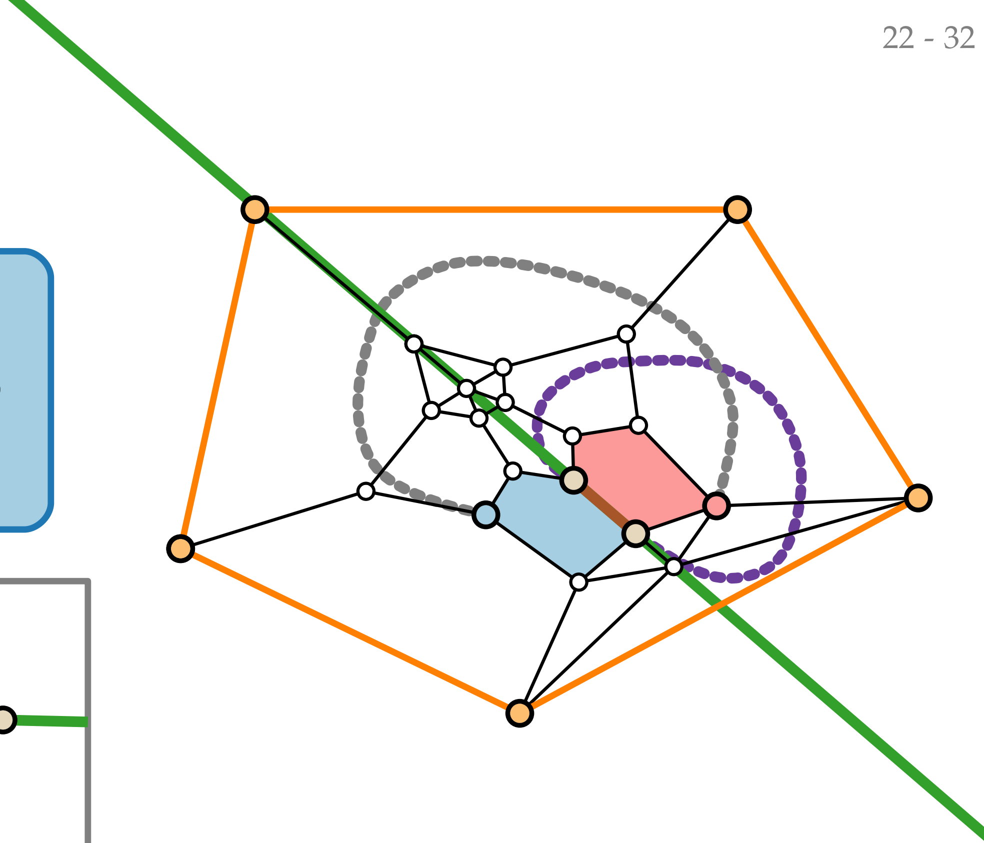

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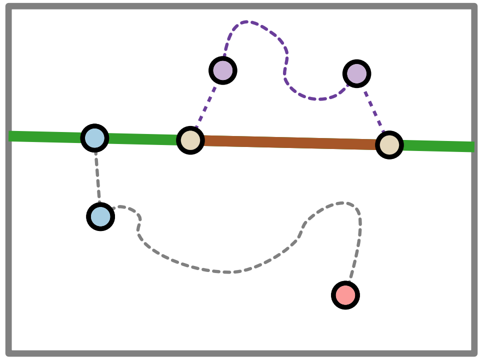

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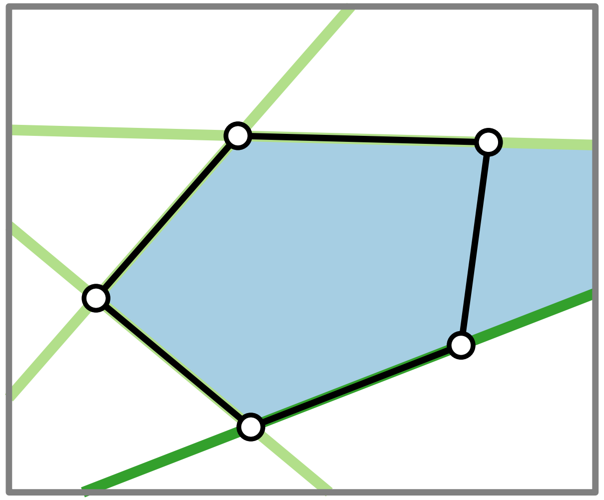

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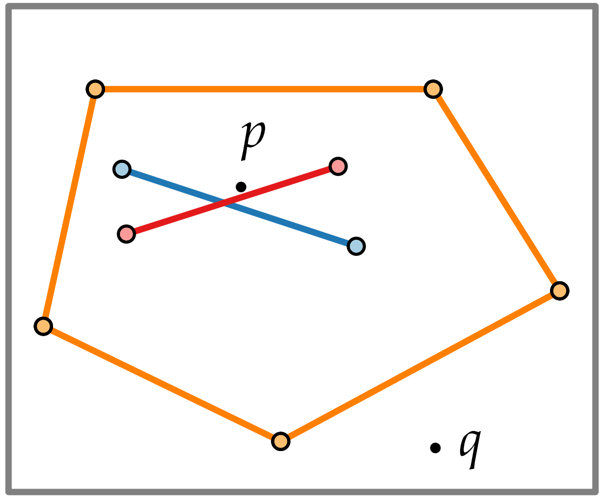

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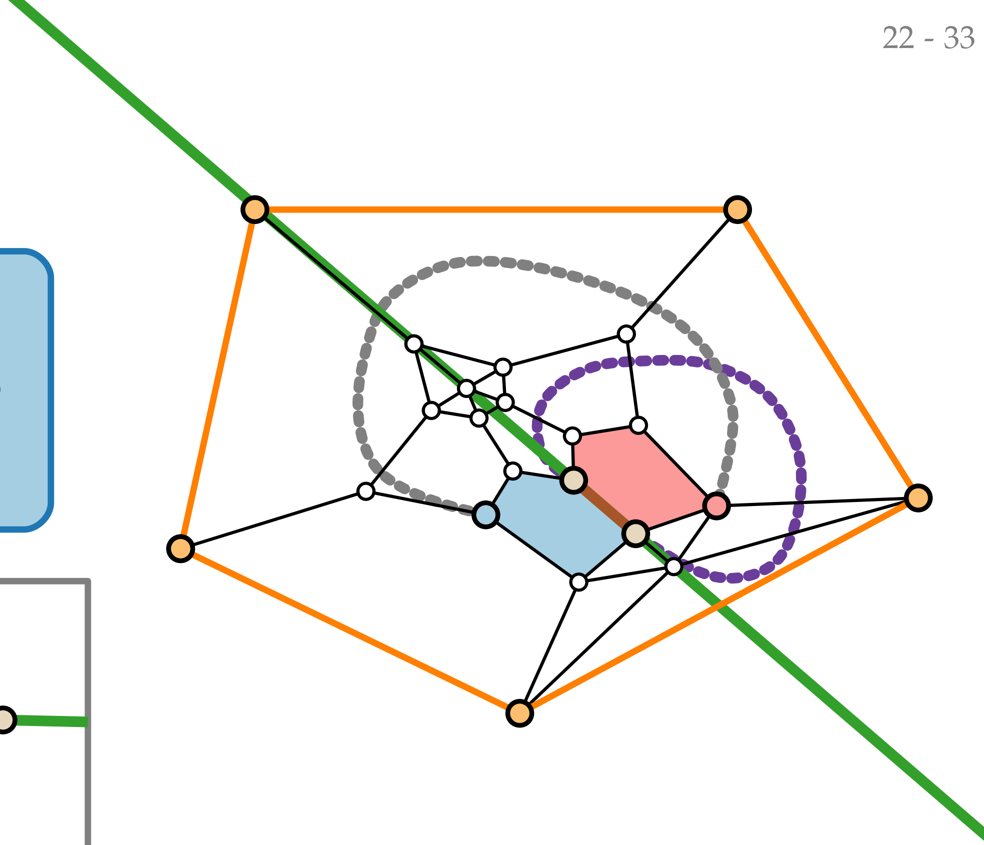

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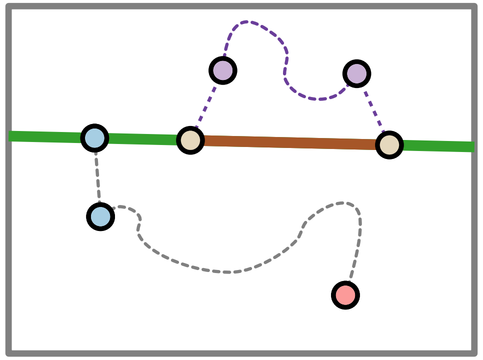

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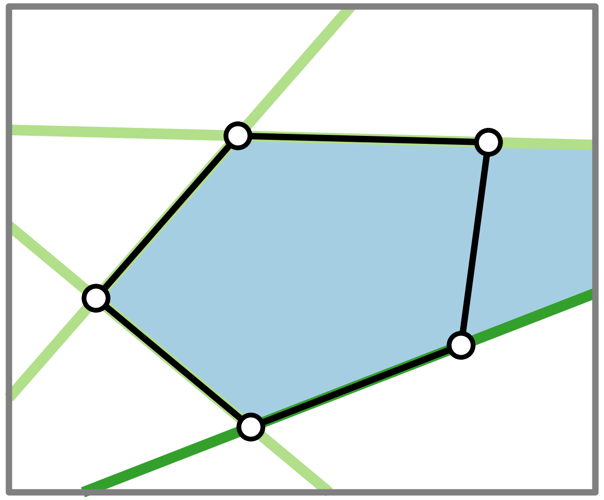

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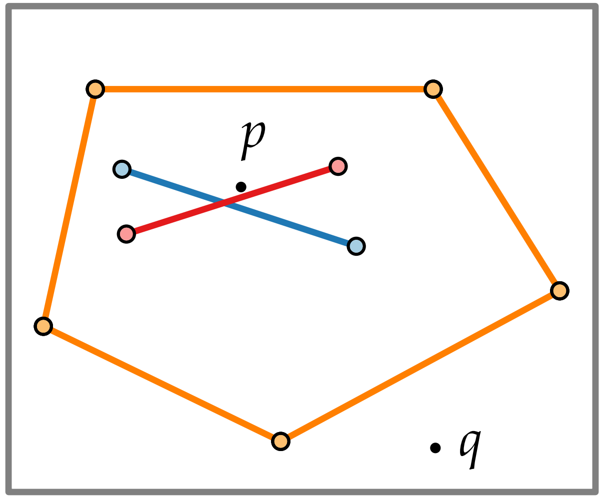

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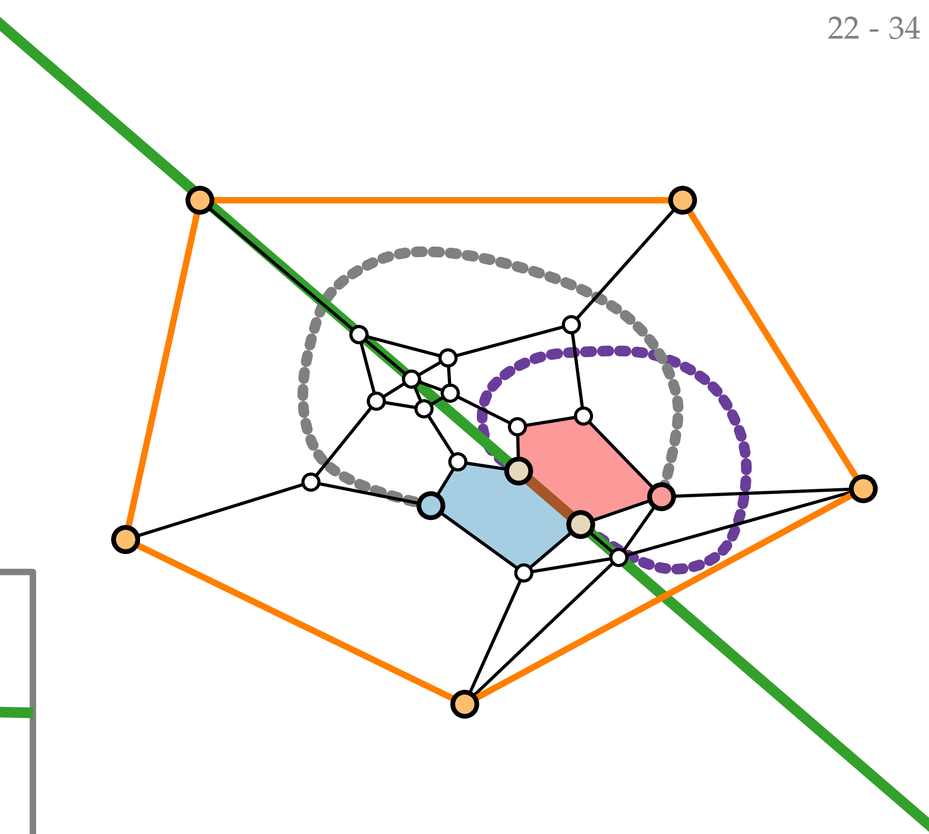

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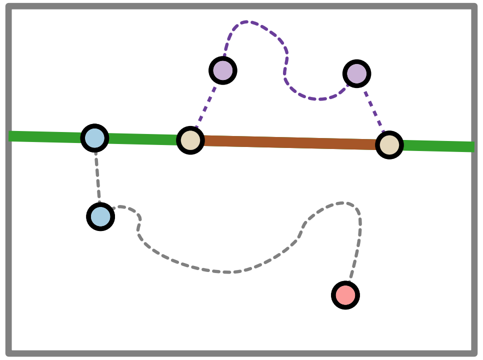

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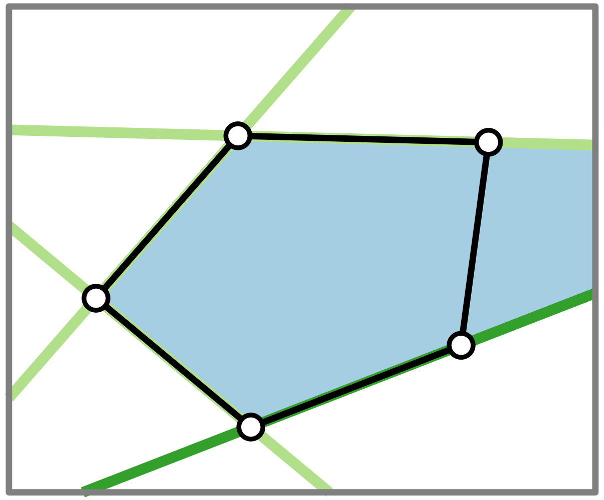

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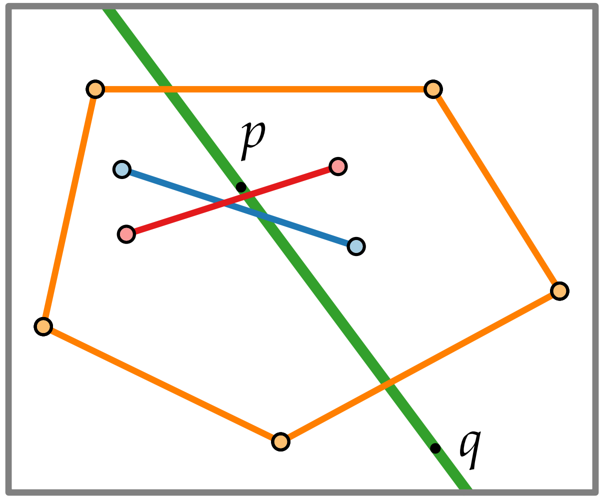

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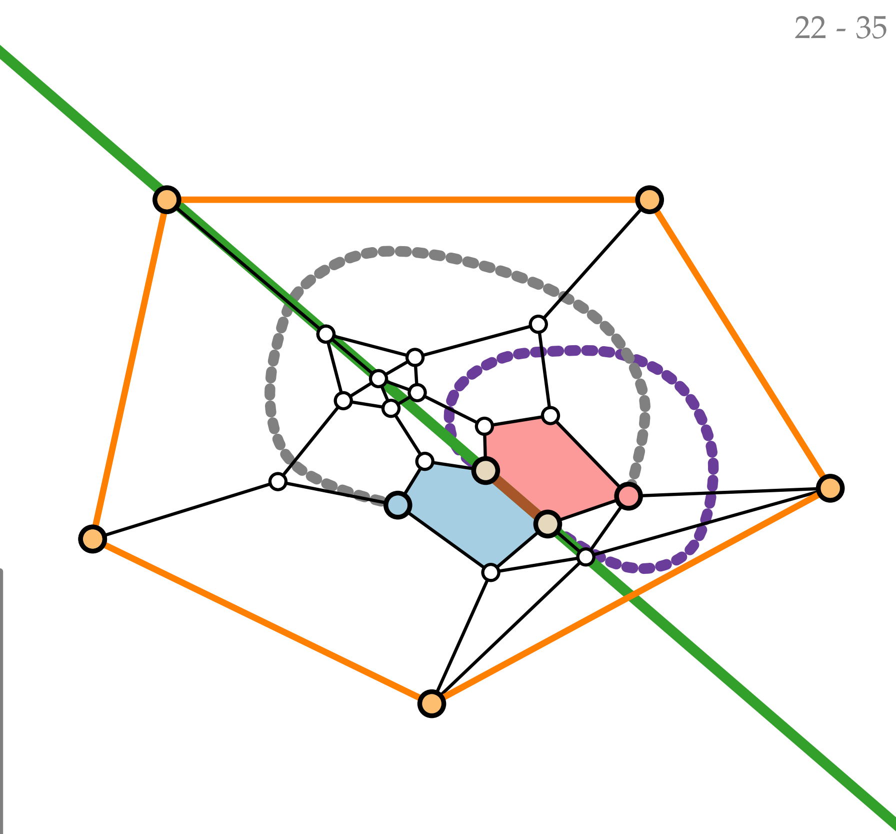

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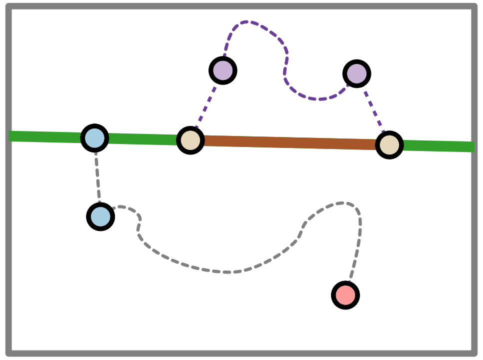

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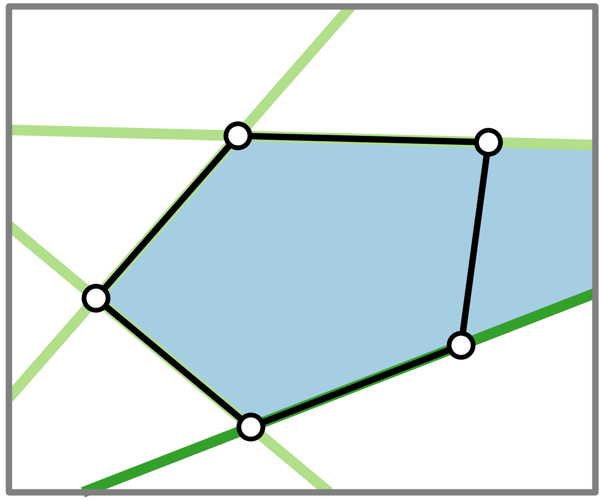

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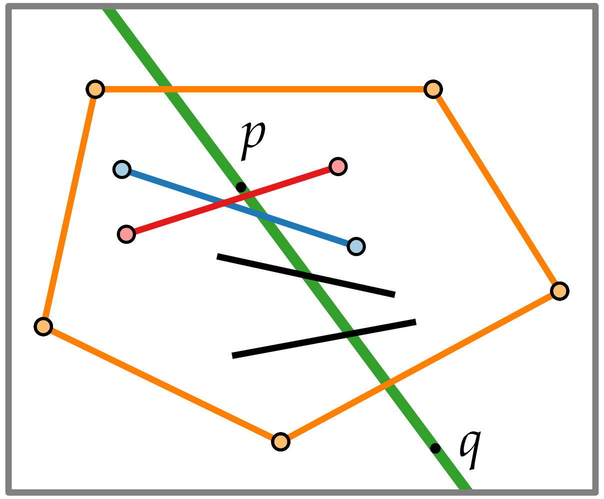

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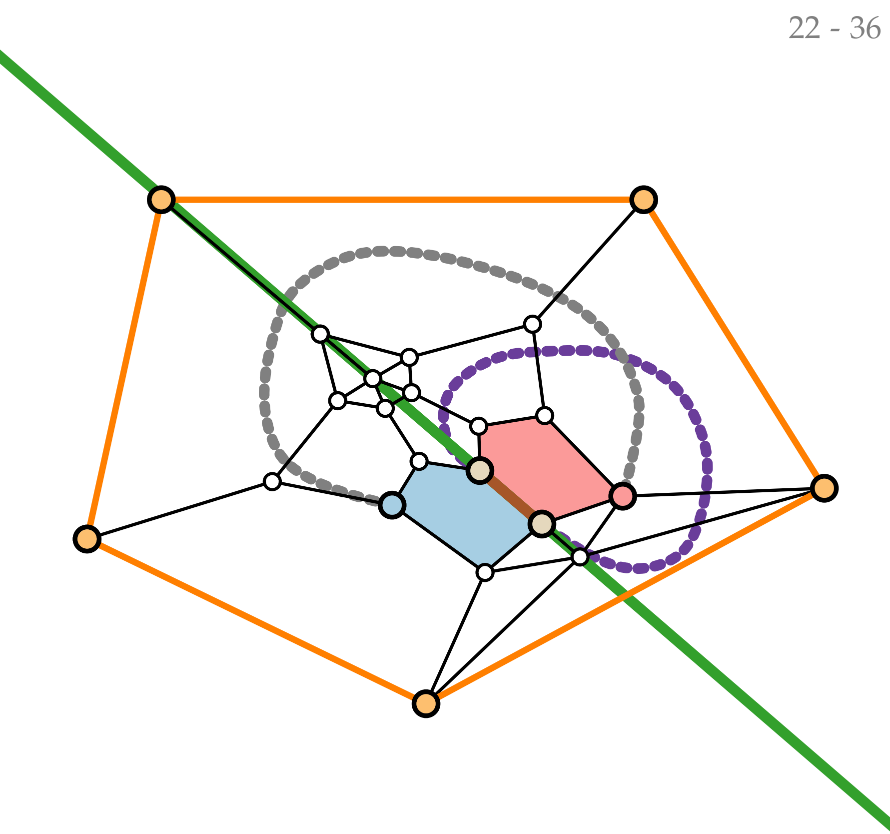

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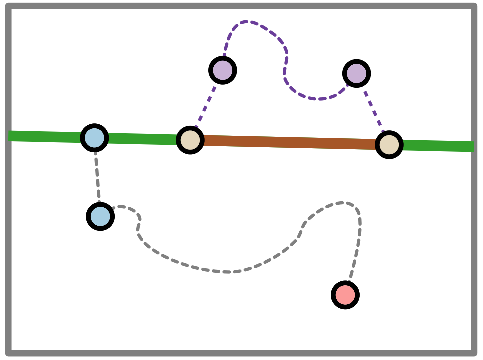

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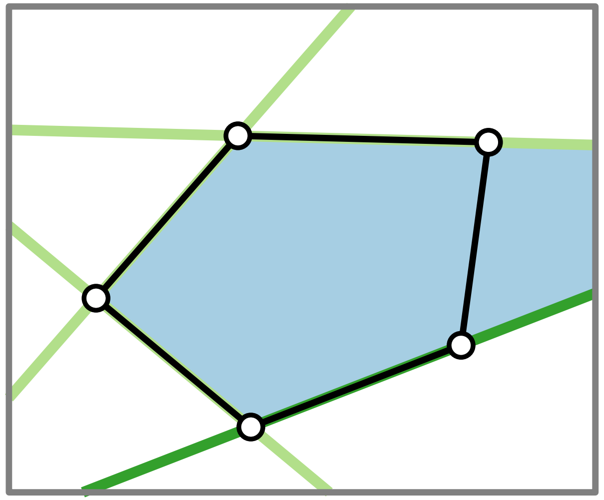

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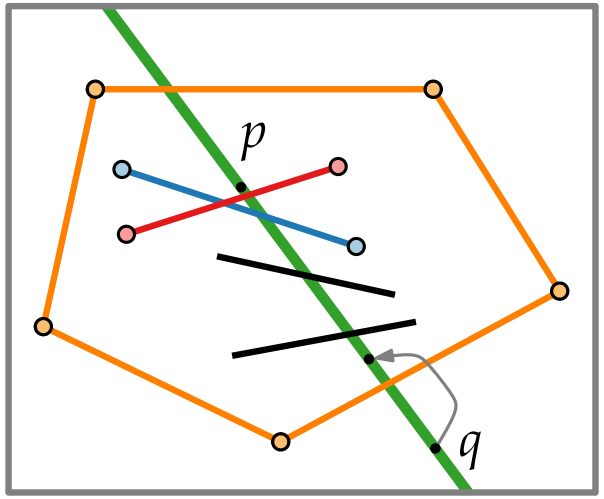

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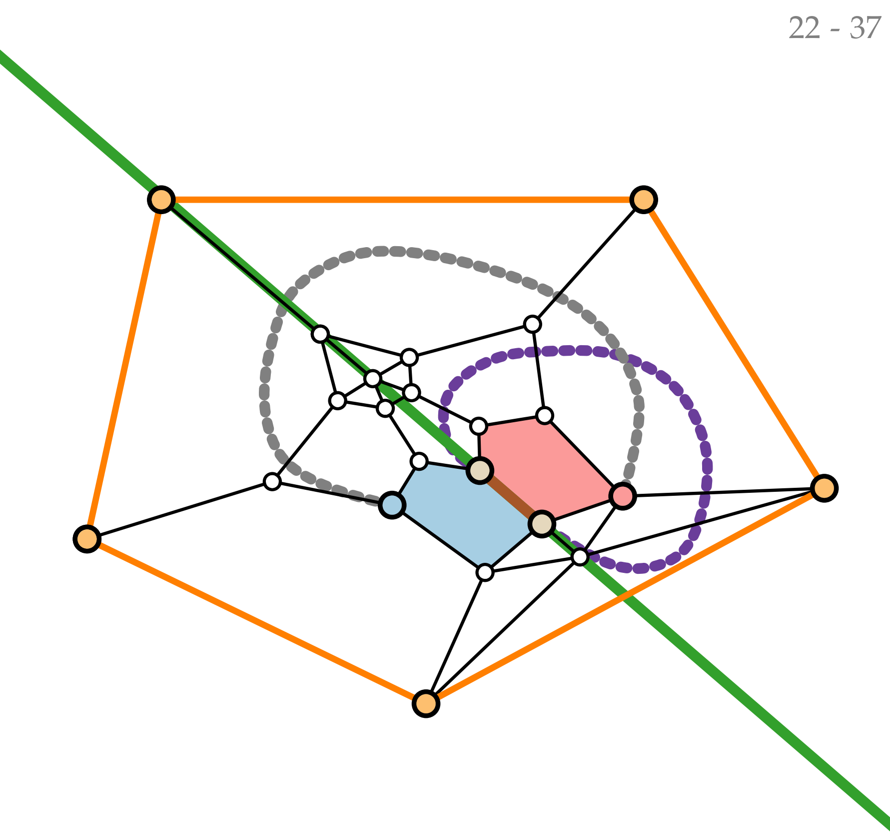

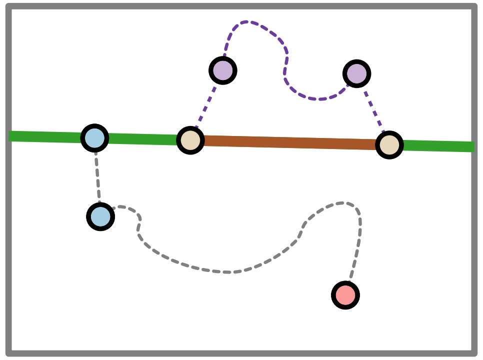

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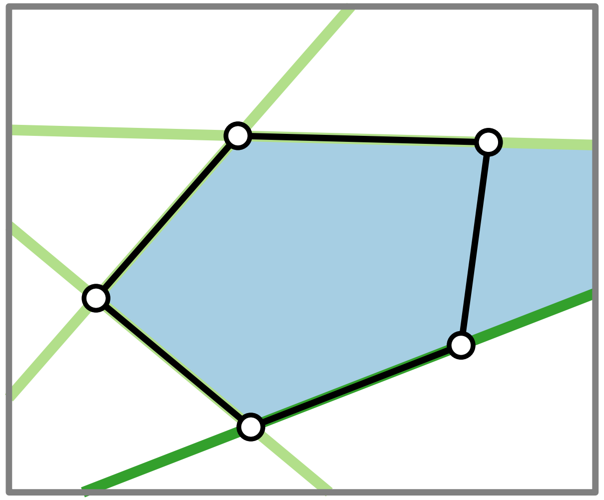

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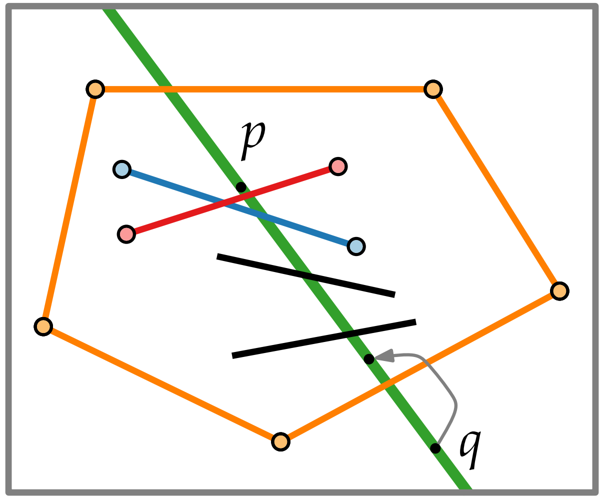


p inside two faces

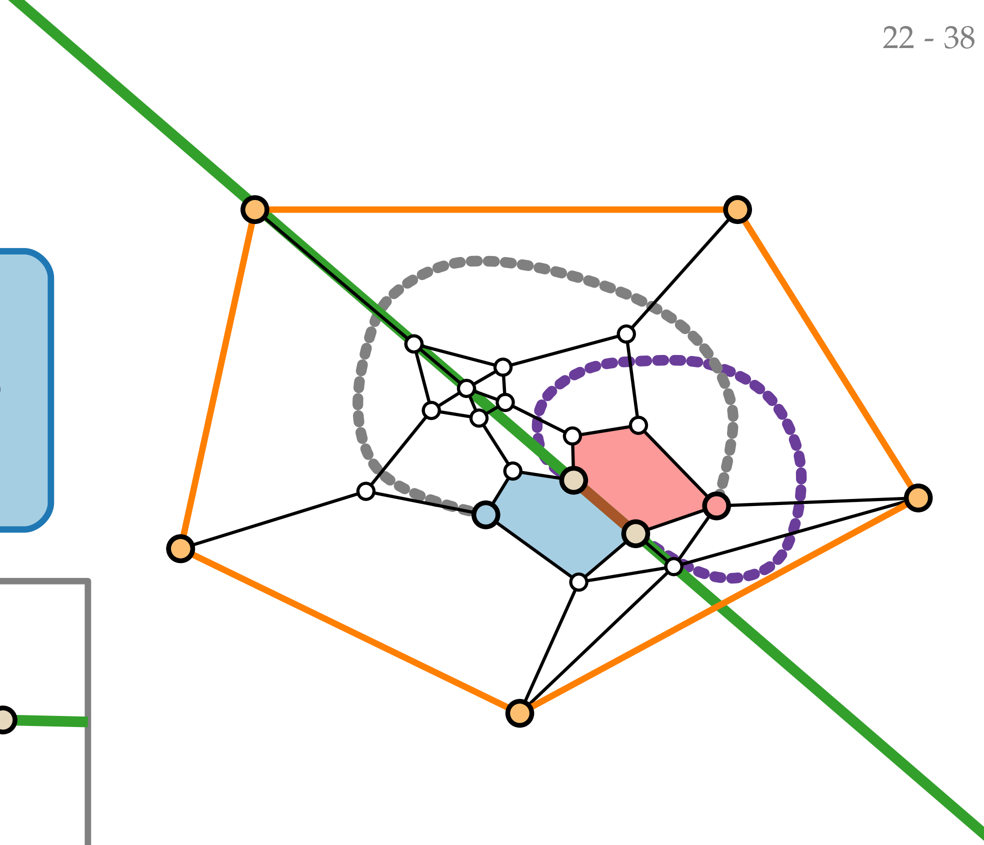
Property 2. All free vertices lie inside C.

$\Rightarrow q$ in one face

jumping over edge

$\rightarrow$ #faces the same

Proof of Tutte's Theorem



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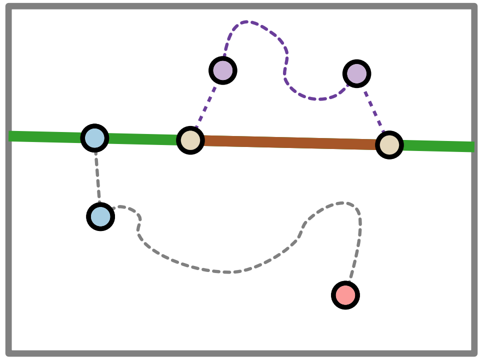

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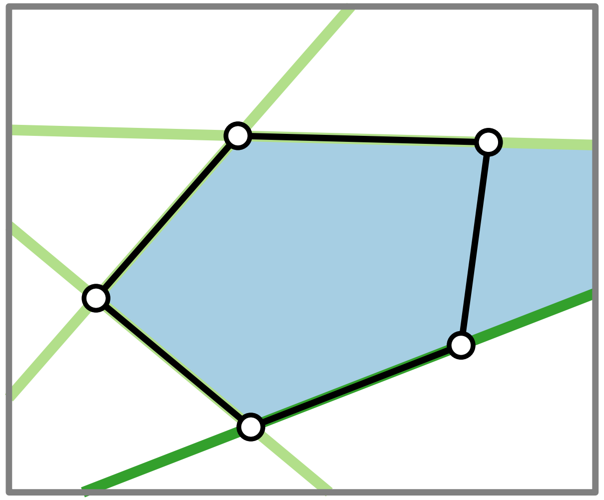

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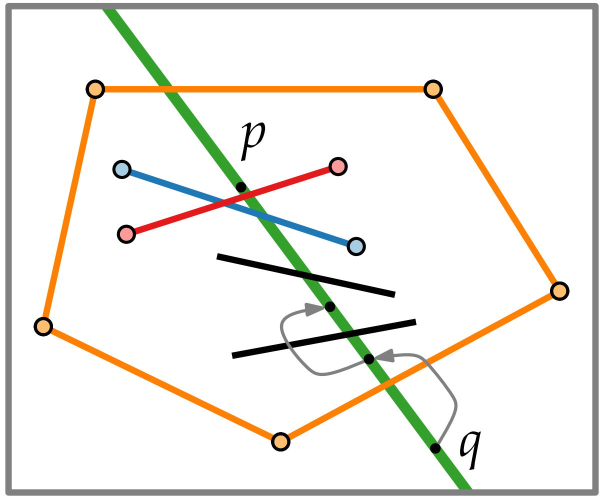

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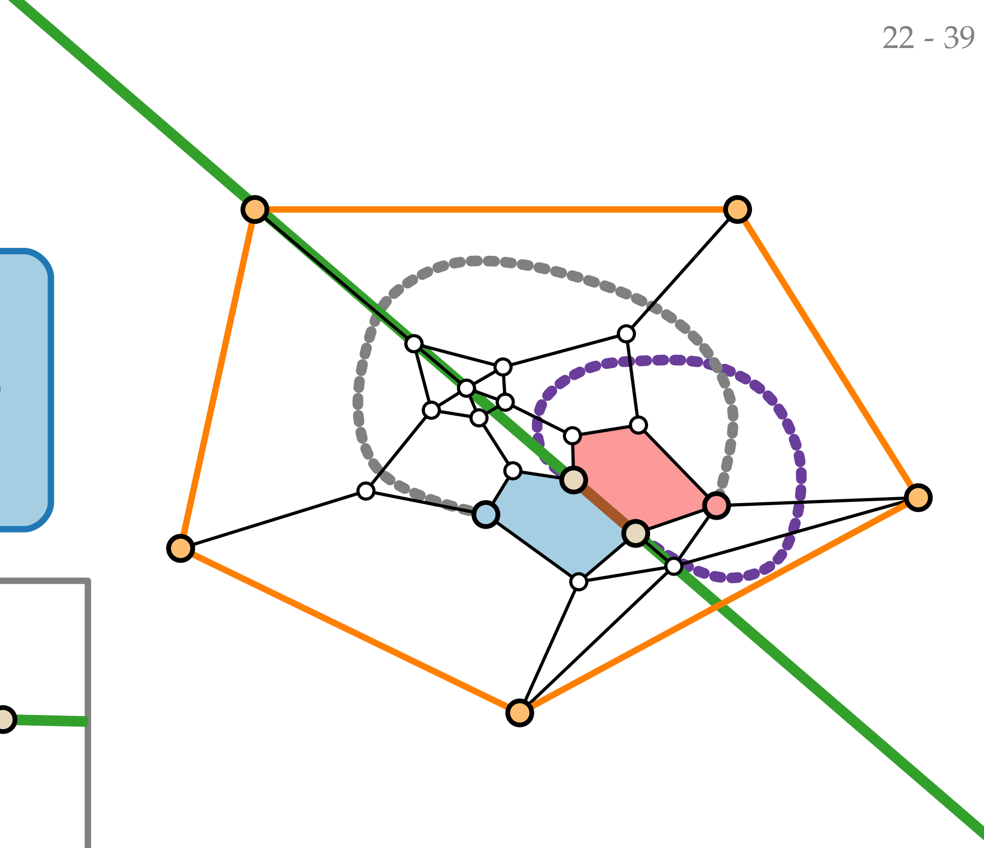

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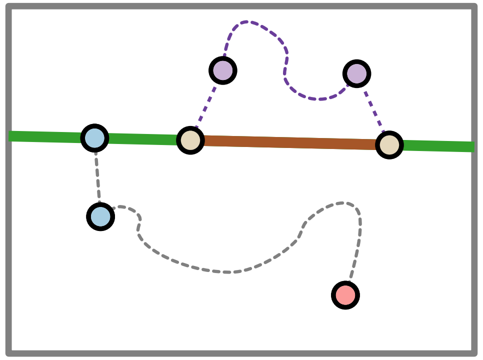

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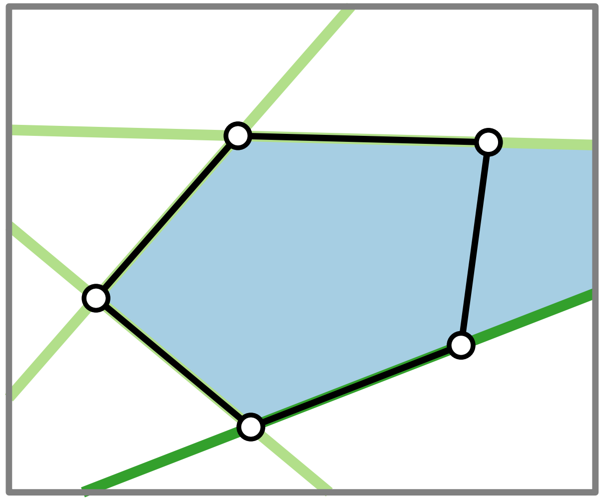

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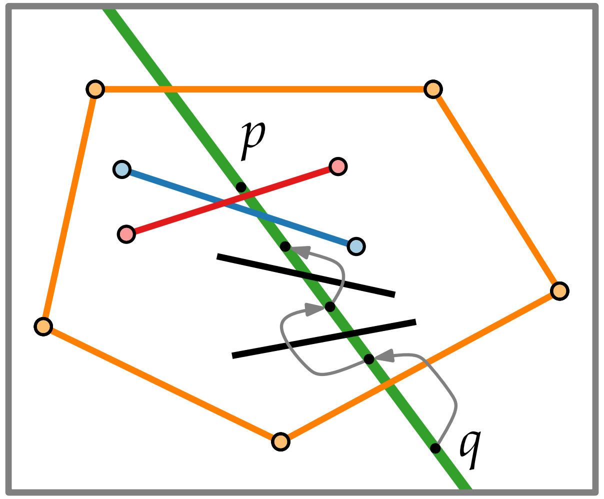

p inside two faces

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jumping over edge

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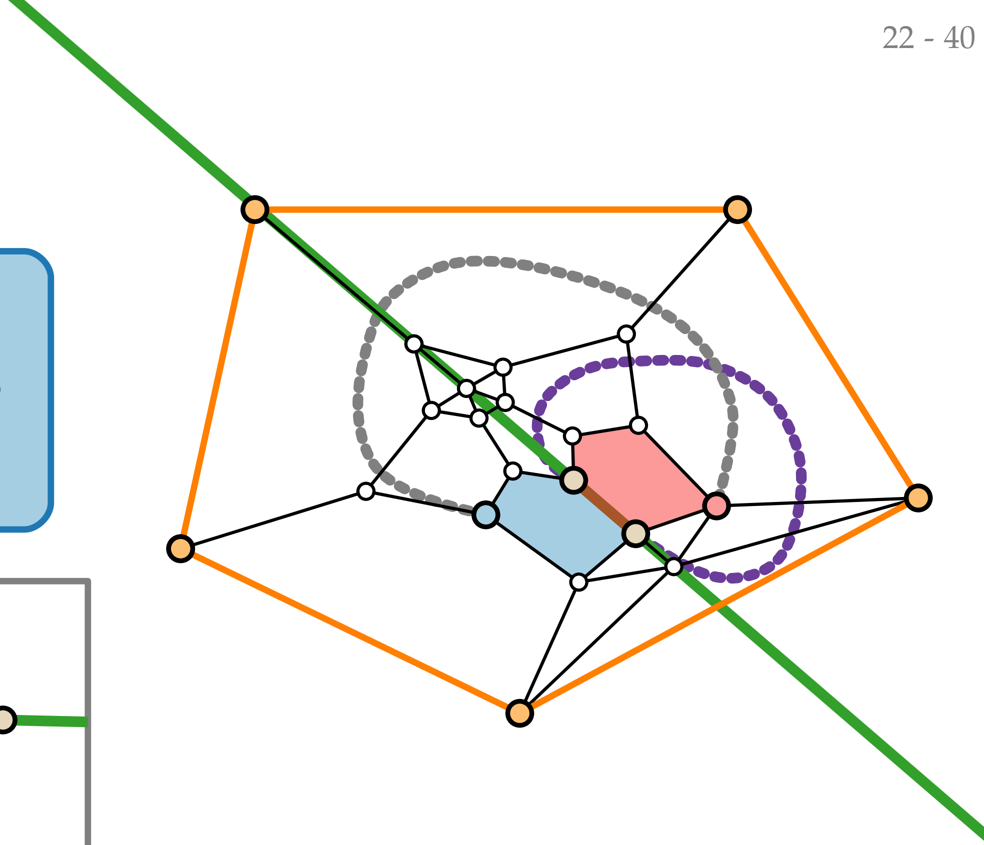

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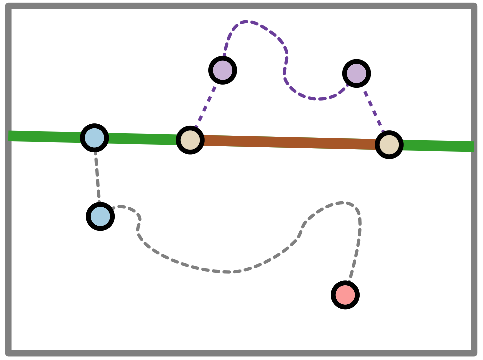

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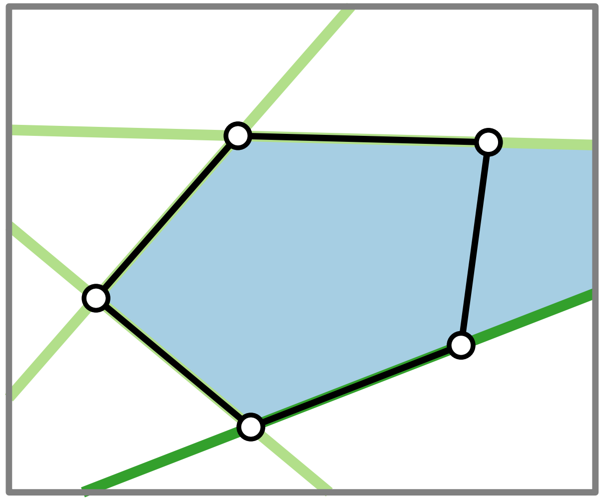

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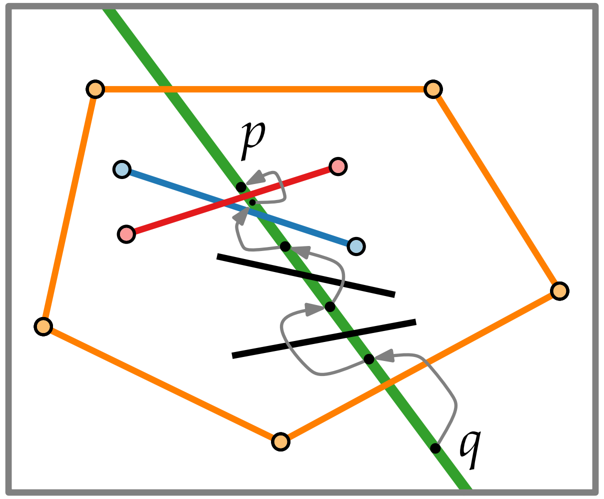

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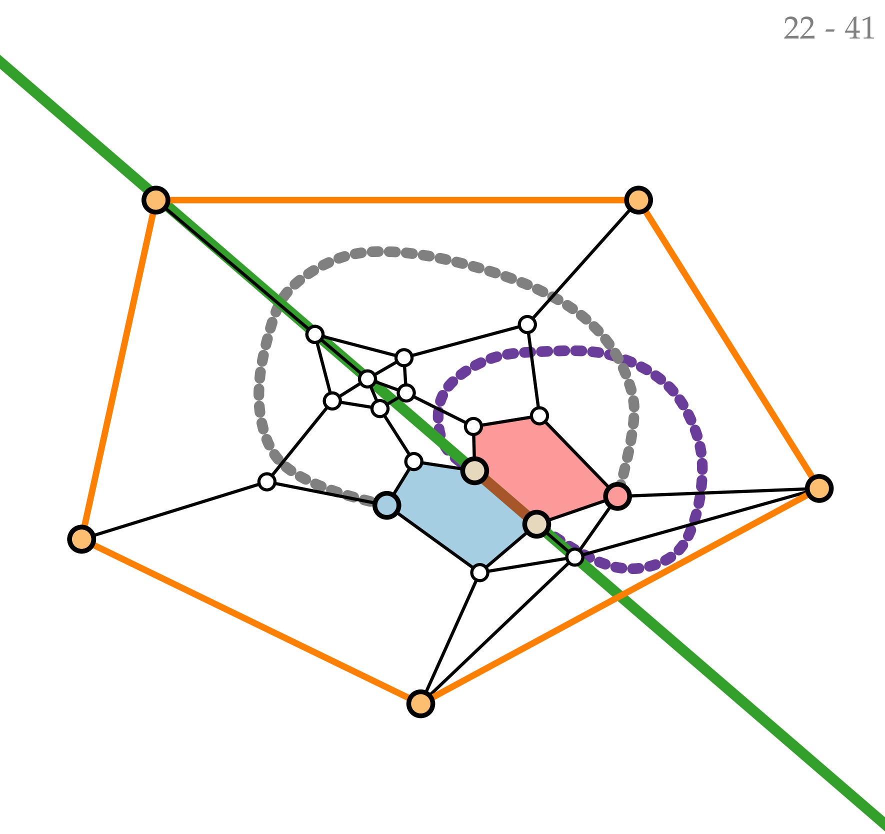

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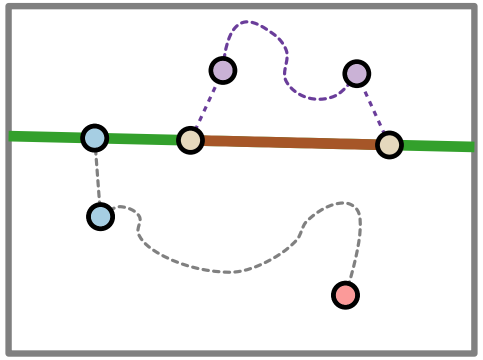
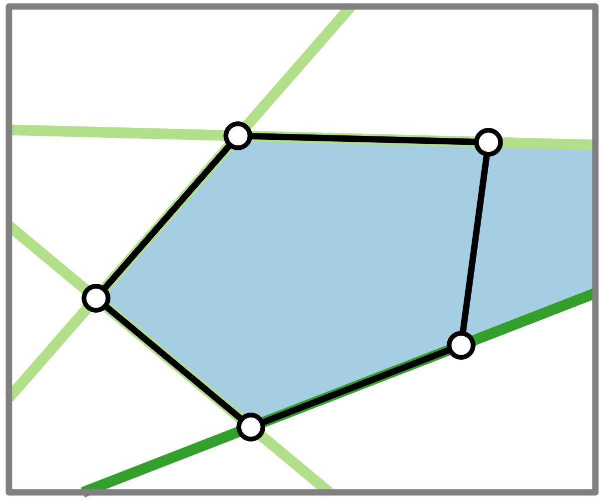
$\exists uv \in E$ on one side of ℓ $\Rightarrow$ $\exists vw \in E$ on other side

Property 3. Let ℓ be any line.
Let V_ℓ be all vertices on one side of ℓ.
Then $G[V_\ell]$ is connected.

Property 4. No vertex is collinear with all of its neighbors.



Lemma. All faces are strictly convex.



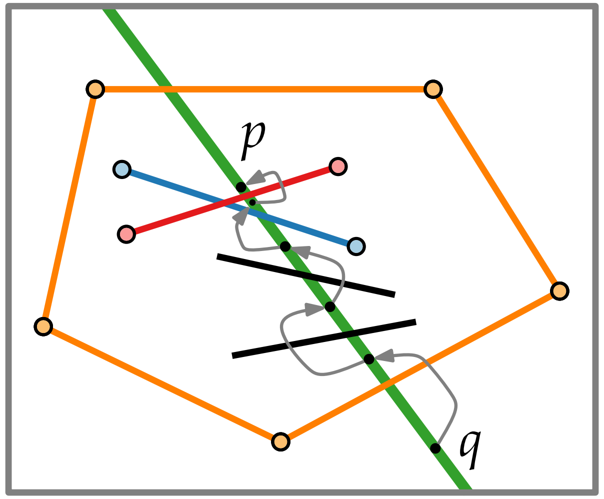
p inside two faces

Property 2. All free vertices lie inside C.

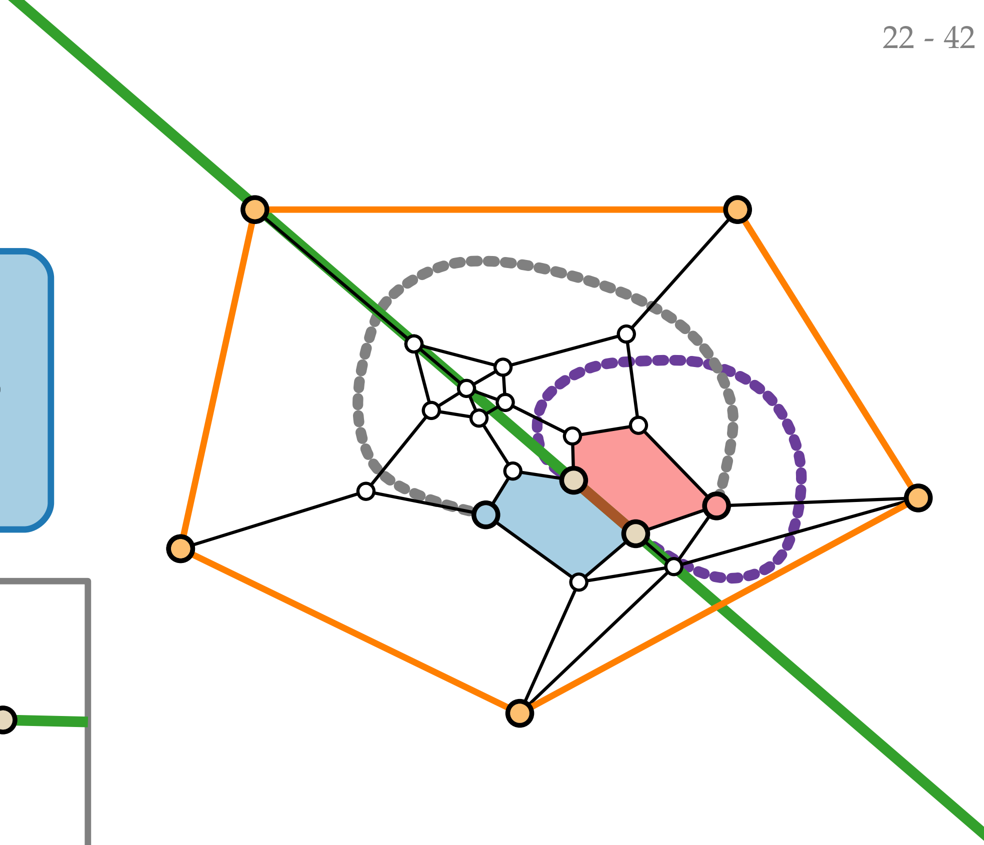
$\Rightarrow$ q in one face

jumping over edge
$\rightarrow$ #faces the same
$\Rightarrow$ p inside one face

Lemma. The drawing is planar.



Proof of Tutte's Theorem



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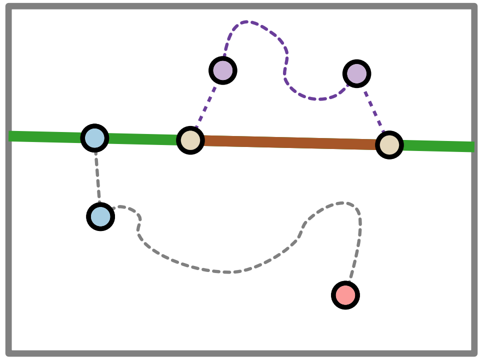

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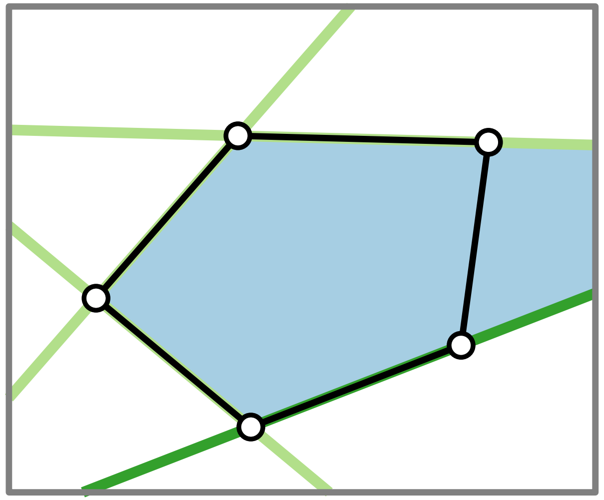
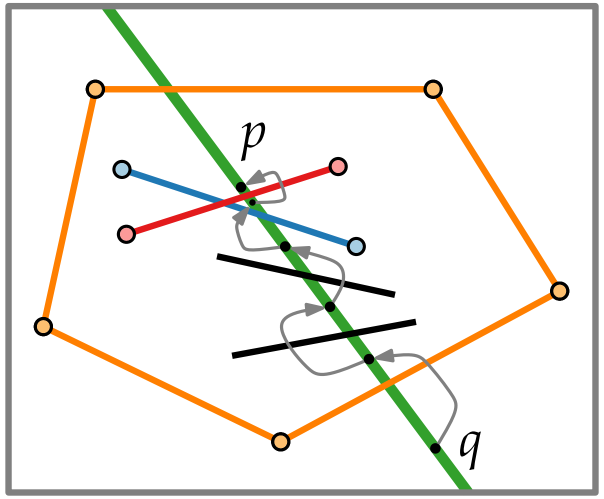


Lemma. The drawing is planar.

Property 2. All free vertices lie inside C.

p inside two faces
$\Rightarrow q$ in one face
jumping over edge
$\rightarrow$ #faces the same
$\Rightarrow p$ inside one face ϟ



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Referenced papers:

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- [Eades, Wormald 1990] Fixed edge-length graph drawing is NP-hard
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