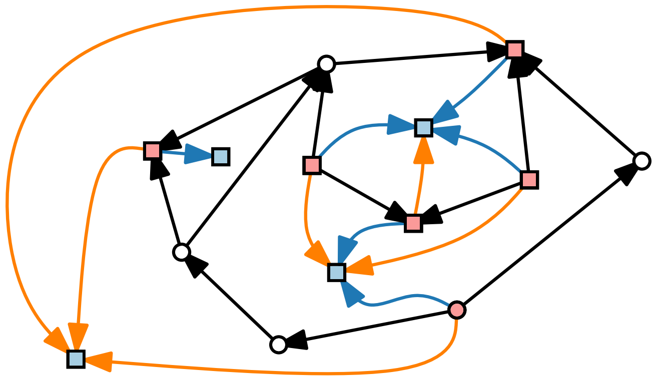
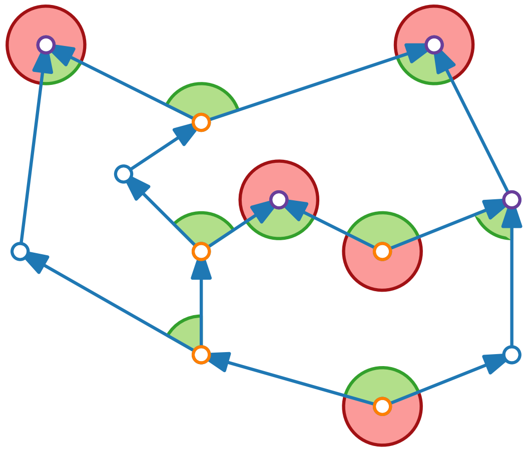
Universität Trier

Visualization of Graphs



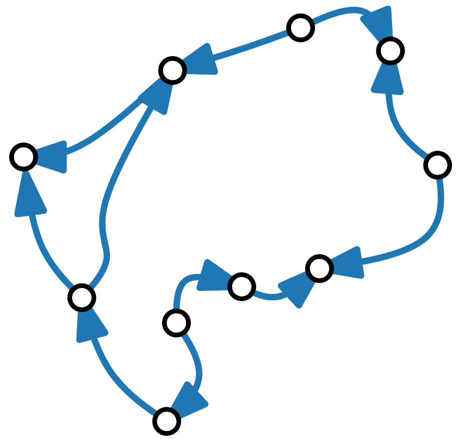
Lecture 7:
Upward Planar Drawings



Part I:
Characterization

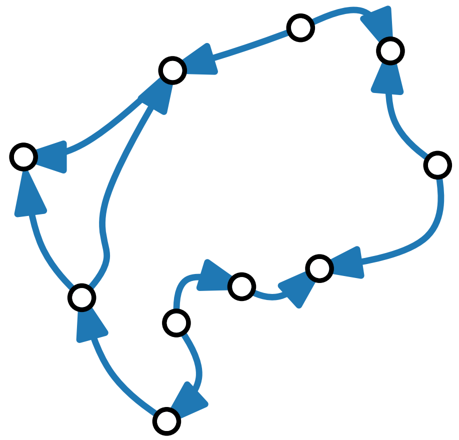
Philipp Kindermann

Upward Planar Drawings – Motivation



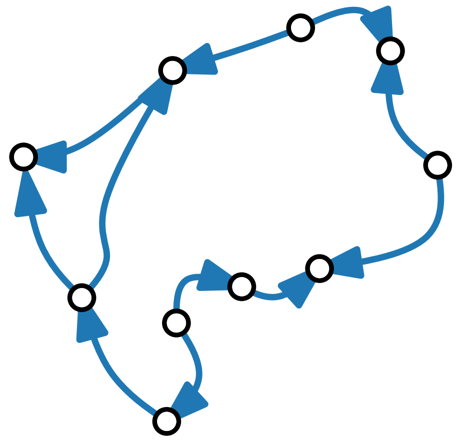
Upward Planar Drawings – Motivation

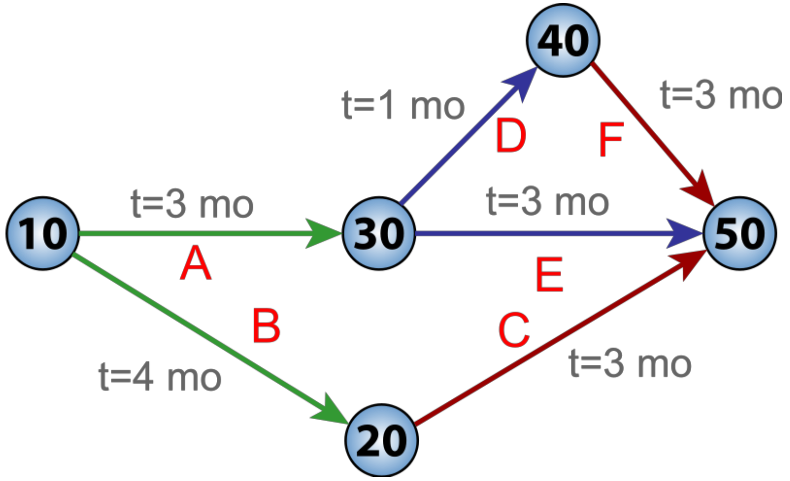
- What may the direction of edges in a digraph represent?



Upward Planar Drawings – Motivation

- What may the direction of edges in a digraph represent?
  - Time

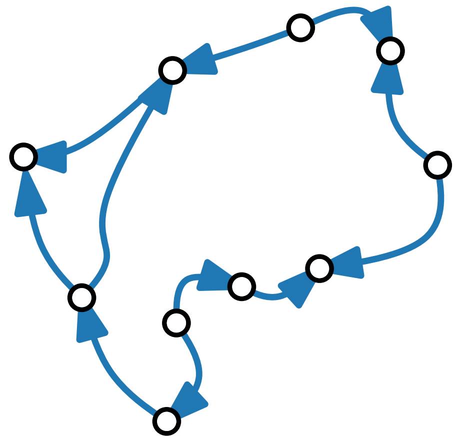


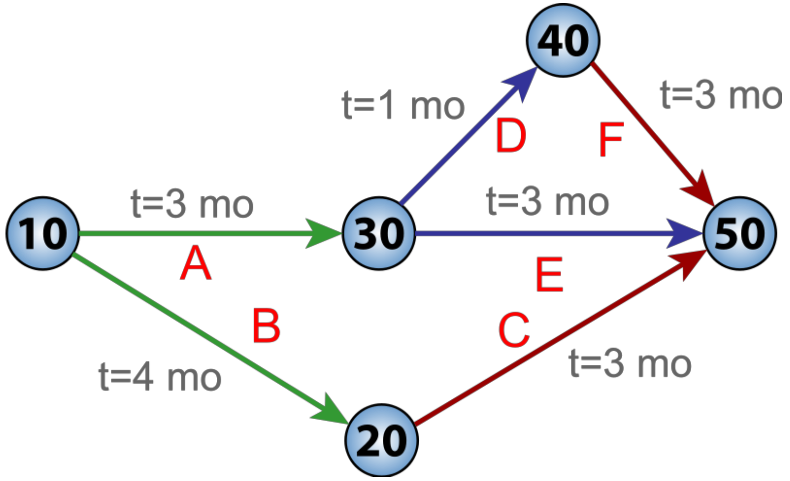


PERT diagram

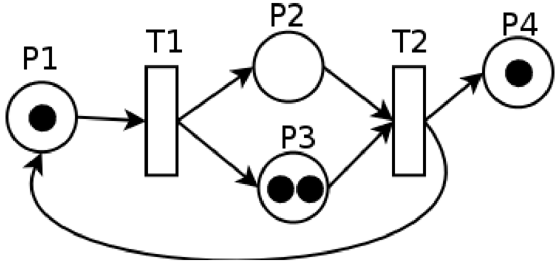
Upward Planar Drawings – Motivation

- What may the direction of edges in a digraph represent?
  - Time
  - Flow





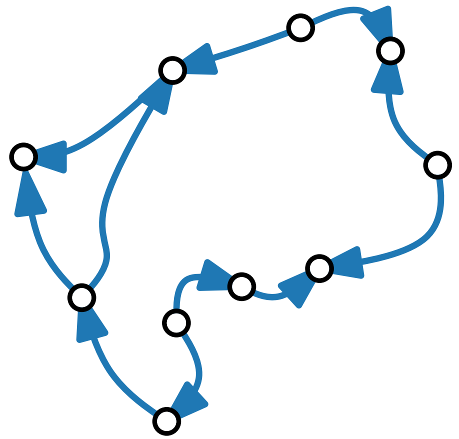
PERT diagram

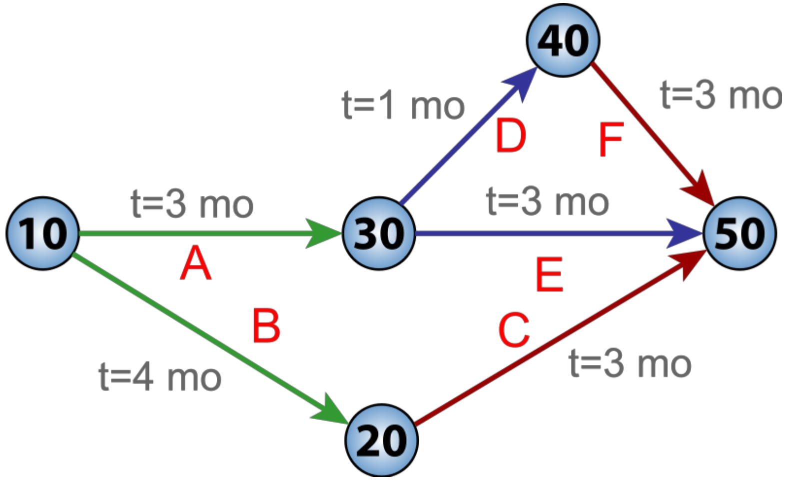


Petri net

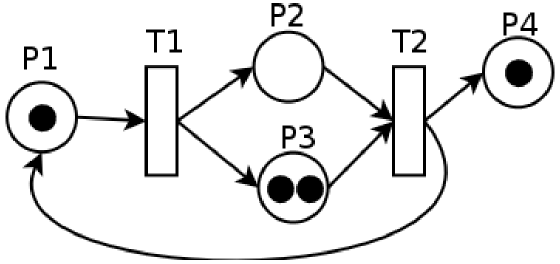
Upward Planar Drawings – Motivation

- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchy

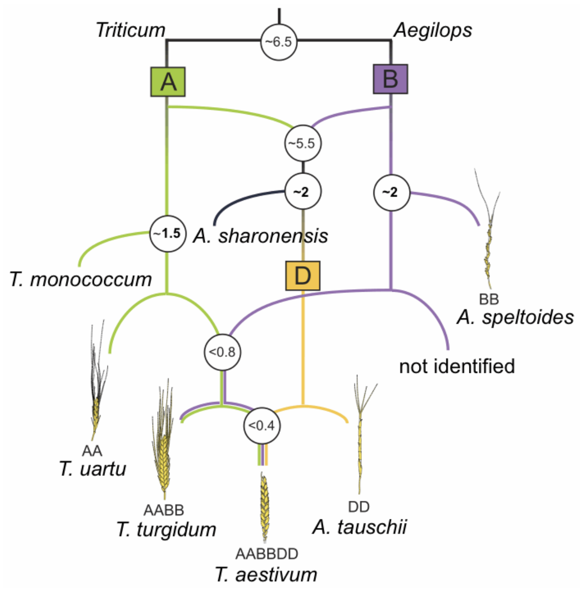




PERT diagram



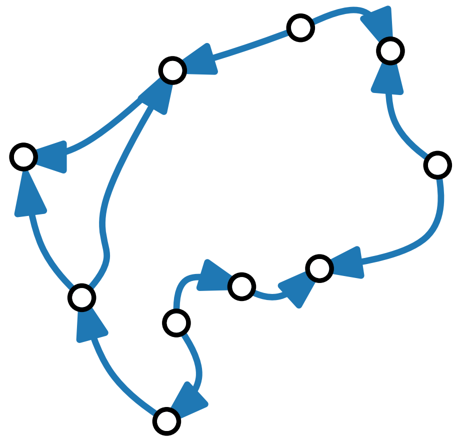
Petri net

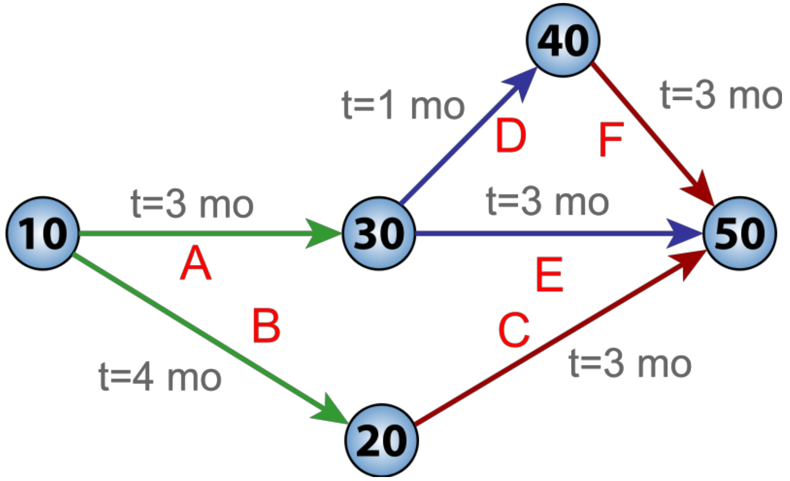


Phylogenetic network

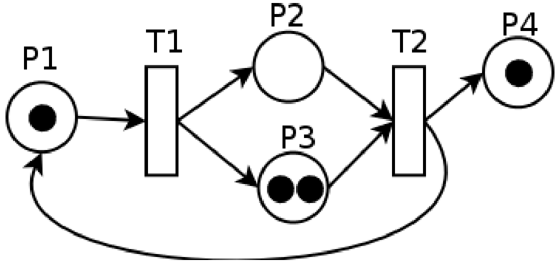
Upward Planar Drawings – Motivation

- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchy
  - ...

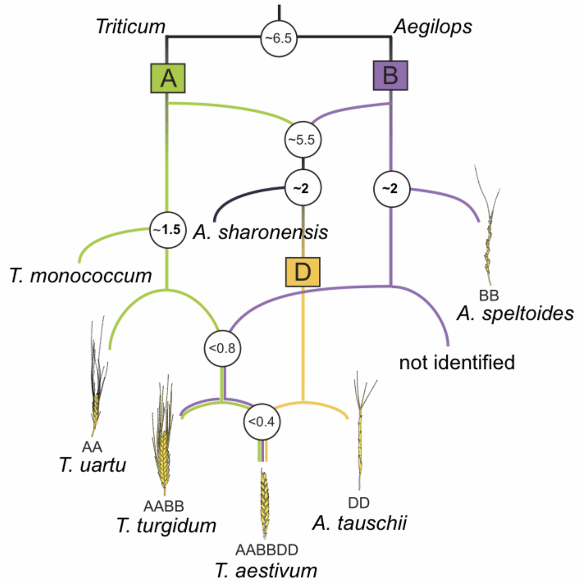




PERT diagram

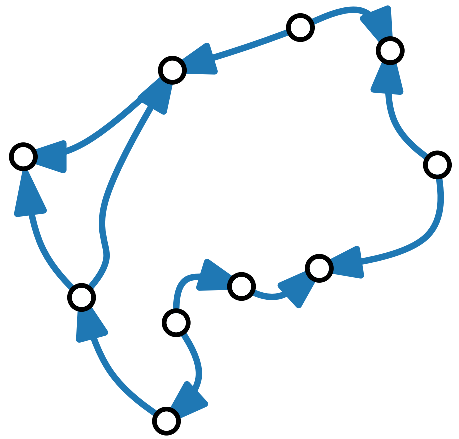


Petri net

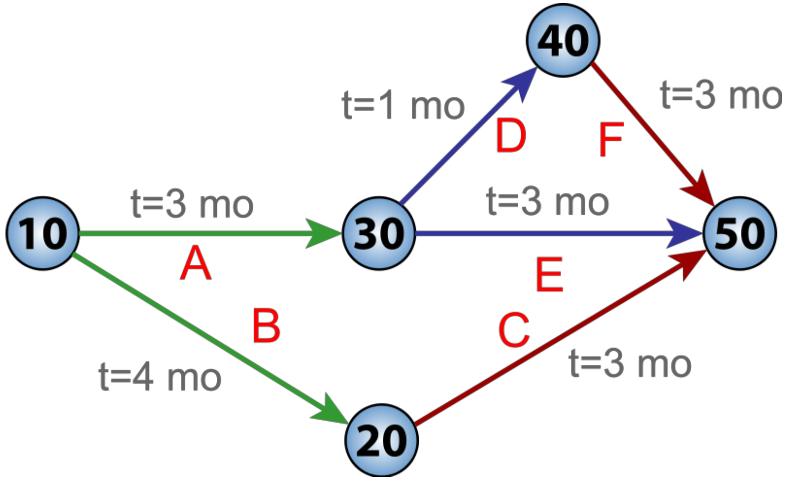


Phylogenetic network

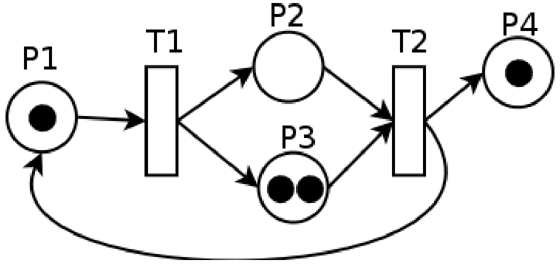
Upward Planar Drawings – Motivation



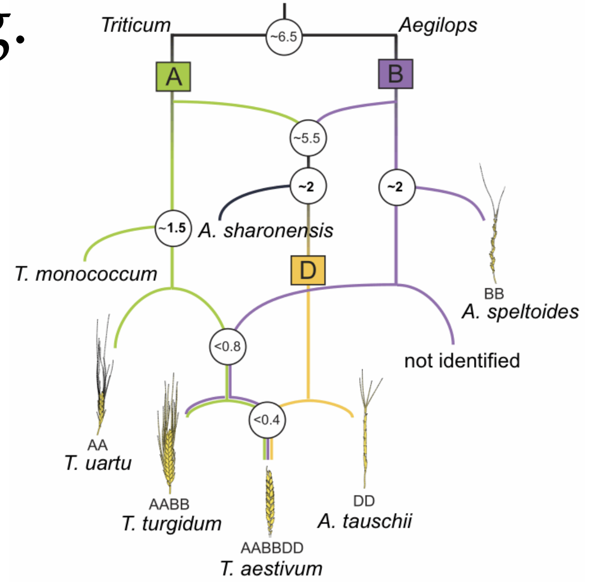
- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchy
  - ...
- Would be nice to have general direction preserved in drawing.



PERT diagram



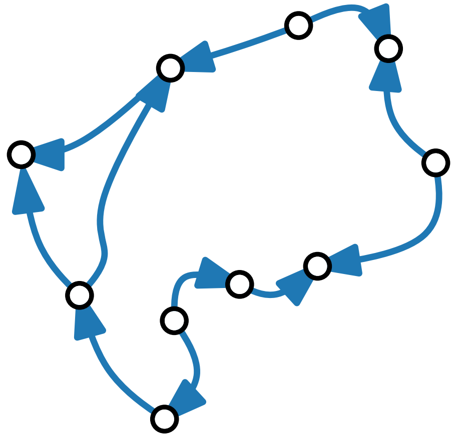
Petri net



Phylogenetic network

Upward Planar Drawings – Definition

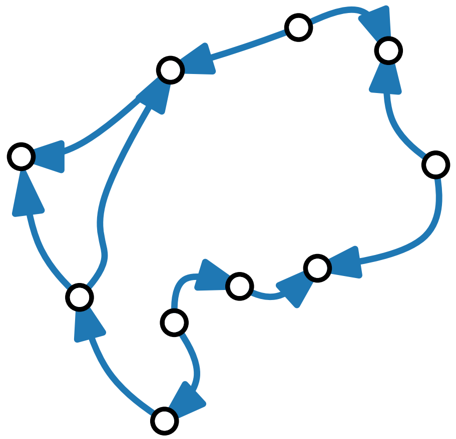
A directed graph $G = (V, E)$ is **upward planar** when it admits a drawing Γ that is



Upward Planar Drawings – Definition

A directed graph $G = (V, E)$ is **upward planar** when it admits a drawing Γ that is

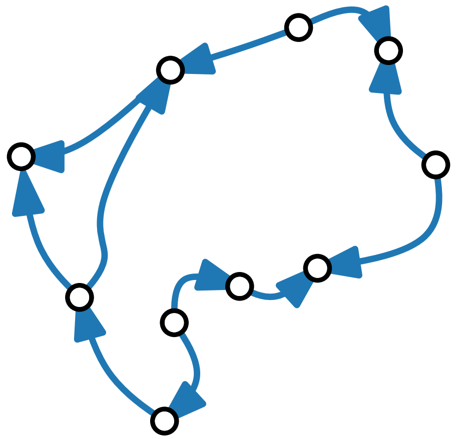
- planar and



Upward Planar Drawings – Definition

A directed graph $G = (V, E)$ is **upward planar** when it admits a drawing Γ that is

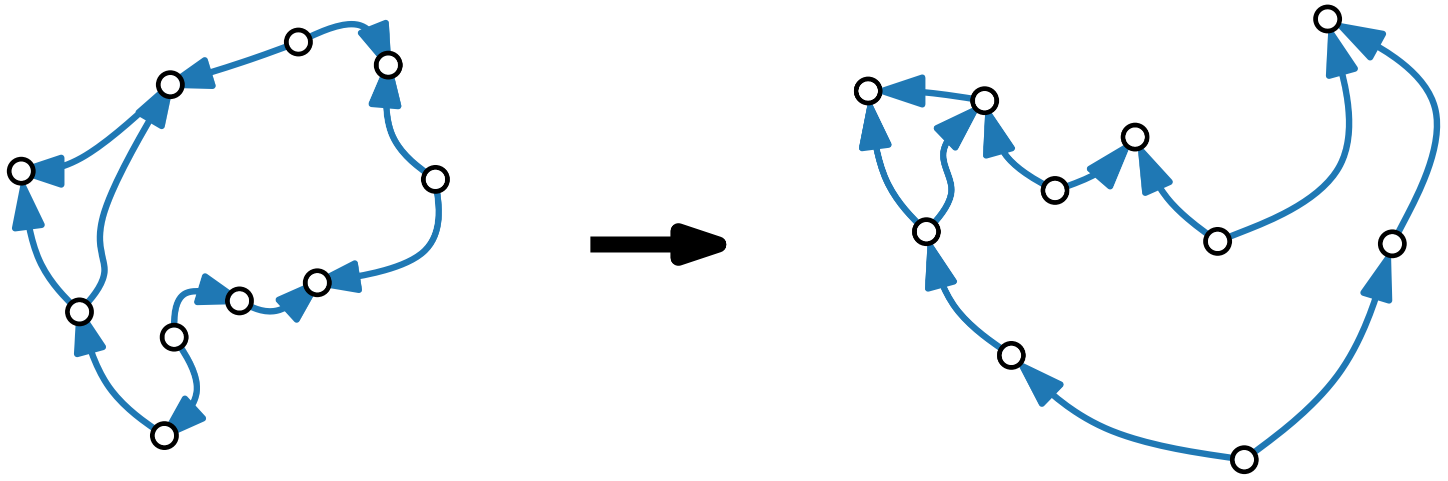
- planar and
- where each edge is drawn as an upward, y-monotone curve.



Upward Planar Drawings – Definition

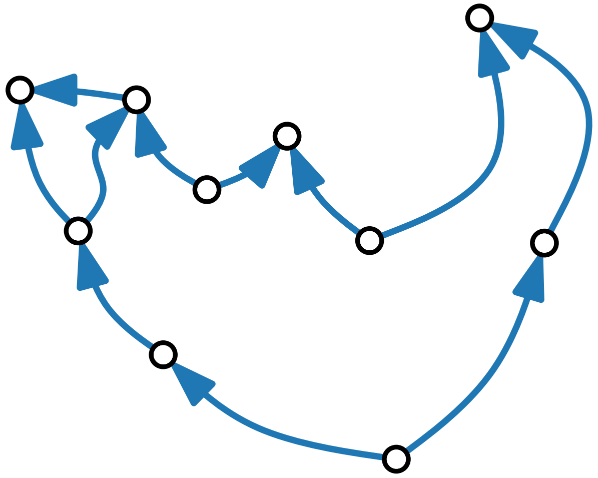
A directed graph $G = (V, E)$ is **upward planar** when it admits a drawing Γ that is

- planar and
- where each edge is drawn as an upward, y-monotone curve.



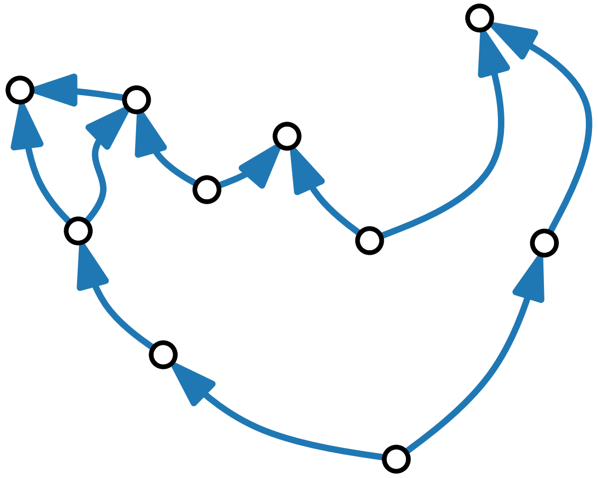
Upward Planarity – Necessary Conditions

- For a digraph G to be upward planar, it has to be:



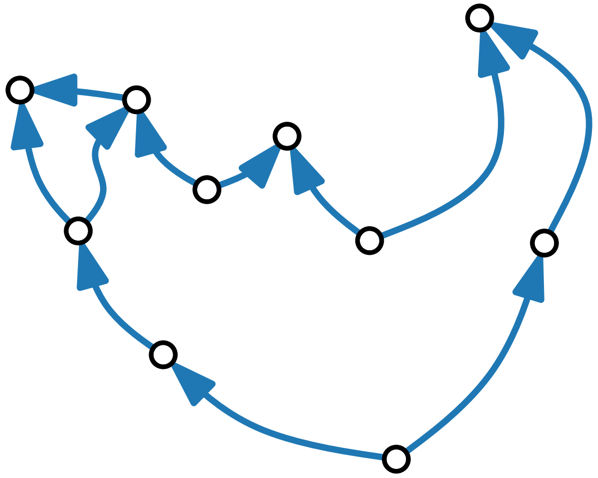
Upward Planarity – Necessary Conditions

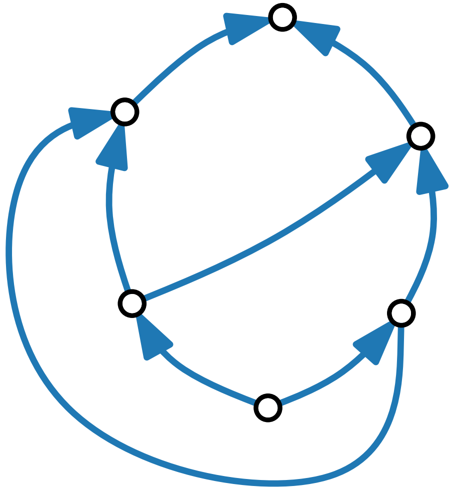
- For a digraph G to be upward planar, it has to be:
  - planar



Upward Planarity – Necessary Conditions

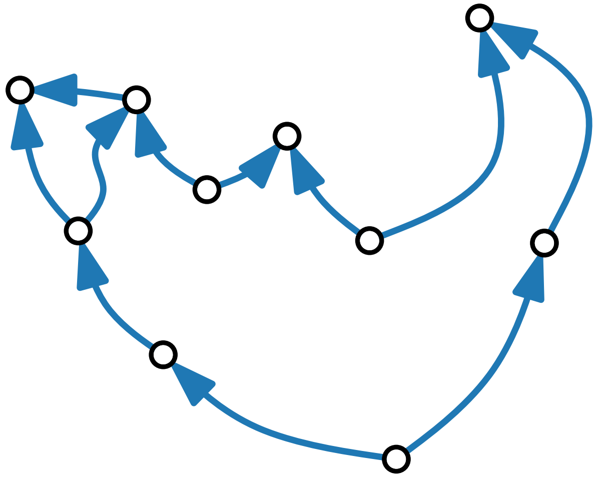
- For a digraph G to be upward planar, it has to be:
  - planar
  - acyclic

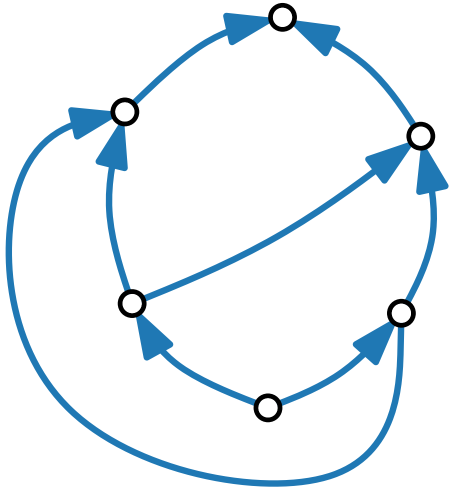


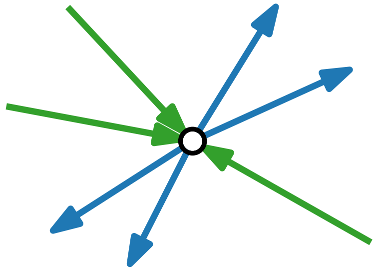


Upward Planarity – Necessary Conditions

- For a digraph G to be upward planar, it has to be:
  - planar
  - acyclic

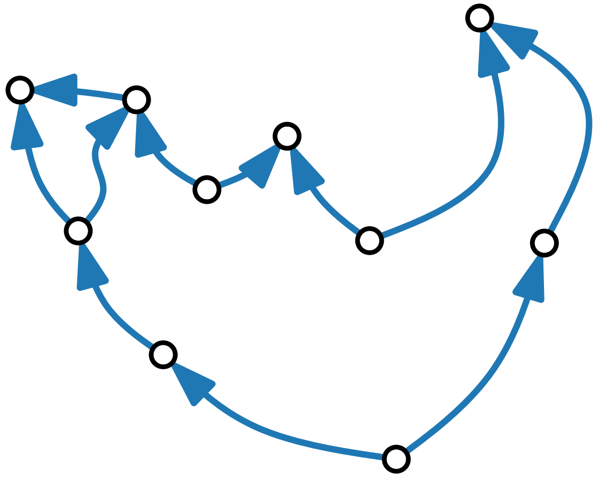
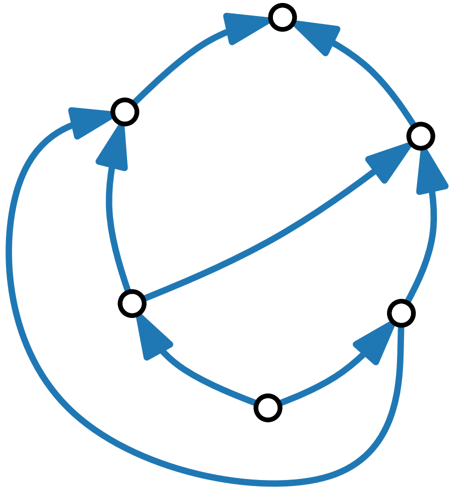
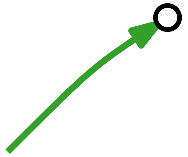
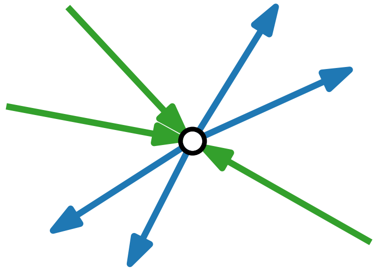






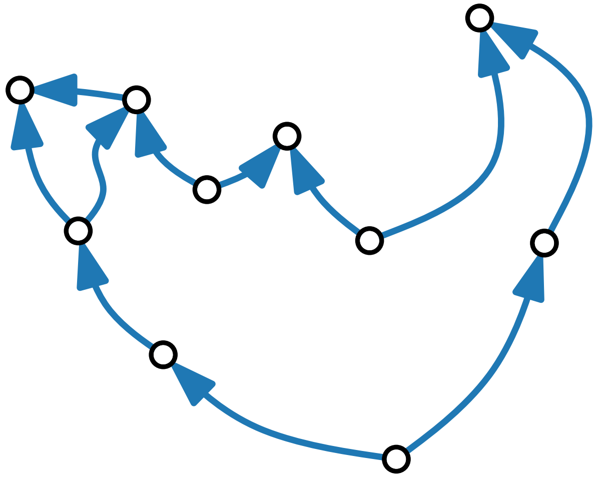
Upward Planarity – Necessary Conditions

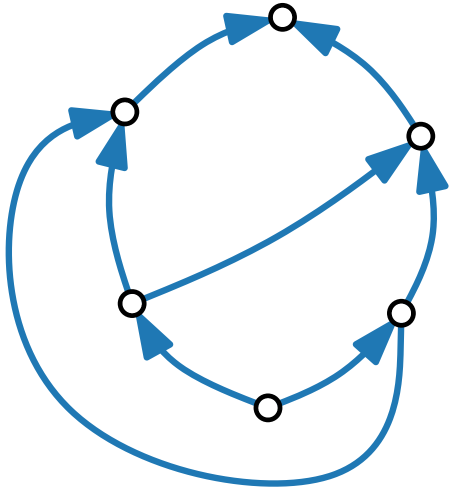
- For a digraph G to be upward planar, it has to be:
  - planar
  - acyclic

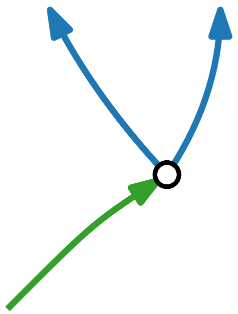





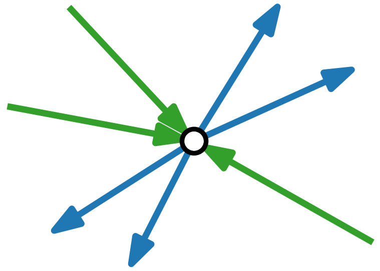
Upward Planarity – Necessary Conditions

- For a digraph G to be upward planar, it has to be:
  - planar
  - acyclic



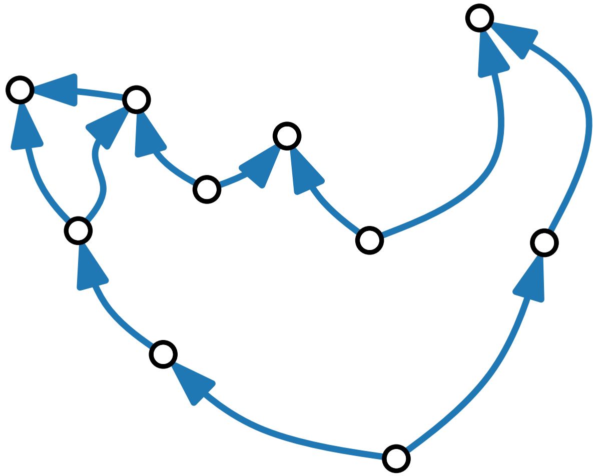


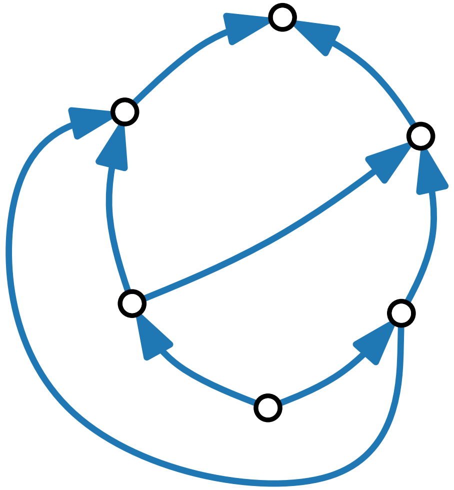


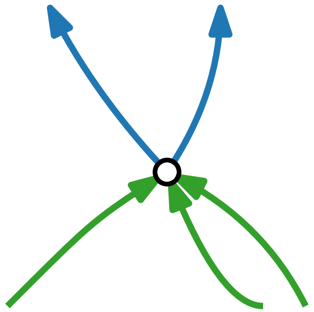


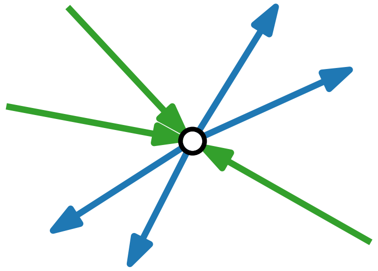
Upward Planarity – Necessary Conditions

- For a digraph G to be upward planar, it has to be:
  - planar
  - acyclic



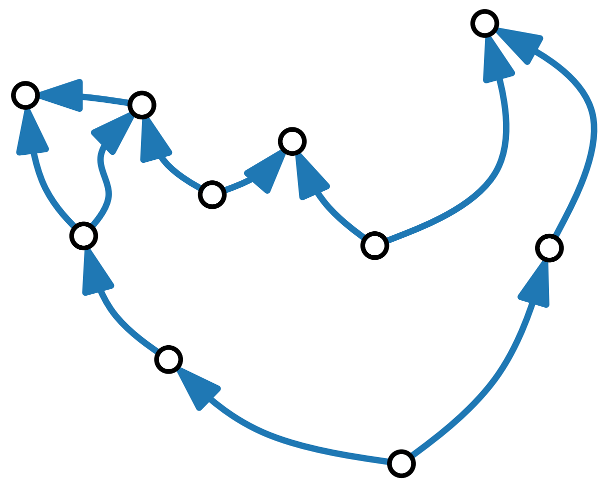


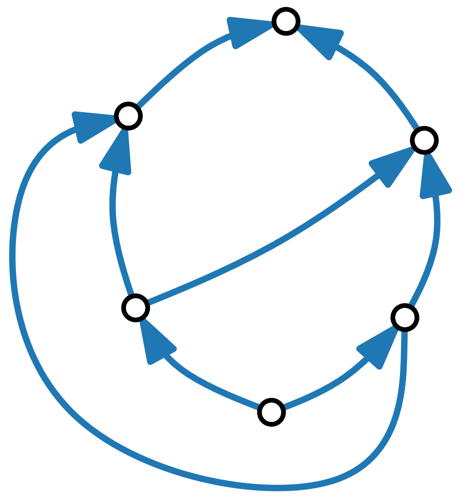


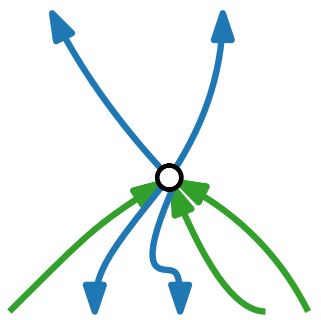


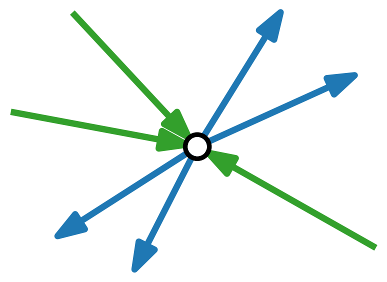
Upward Planarity – Necessary Conditions

- For a digraph G to be upward planar, it has to be:
  - planar
  - acyclic



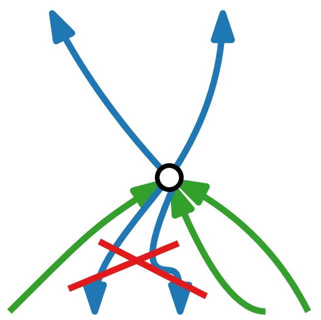


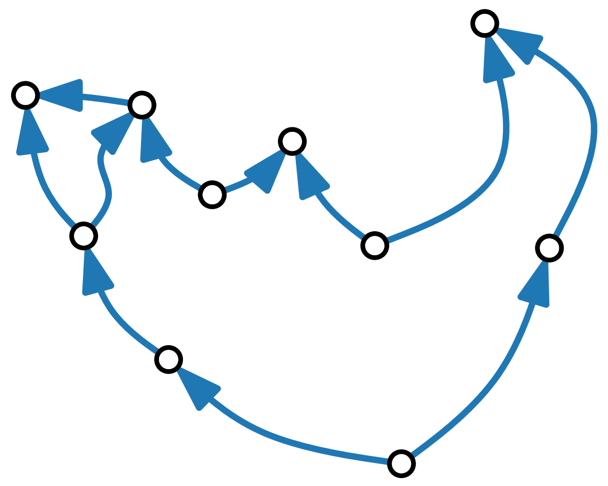


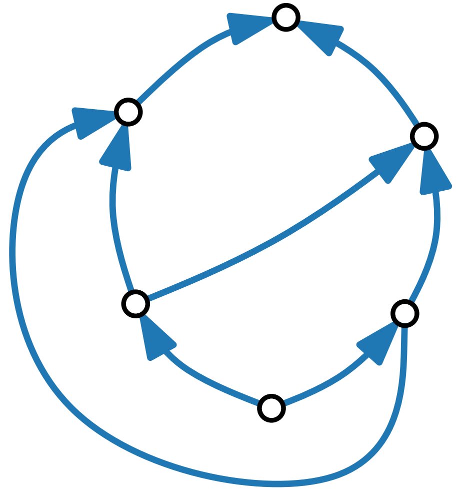


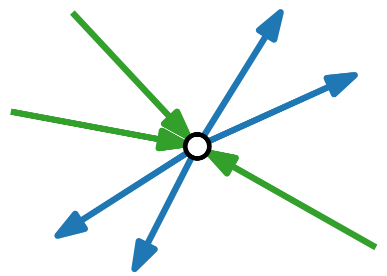
Upward Planarity – Necessary Conditions

- For a digraph G to be upward planar, it has to be:
  - planar
  - acyclic



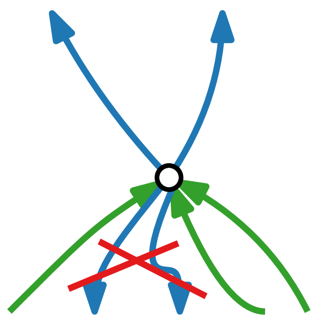


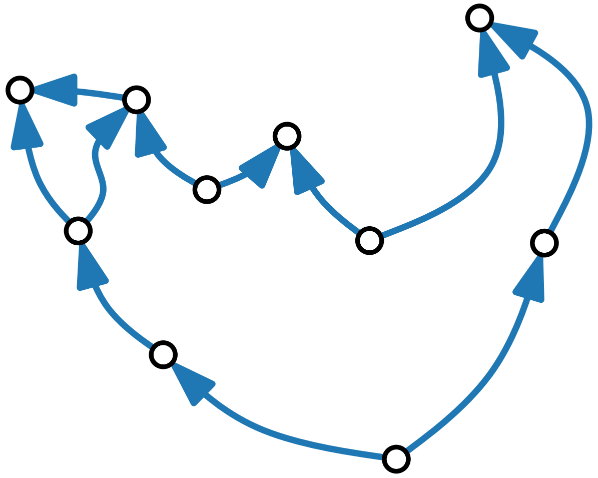


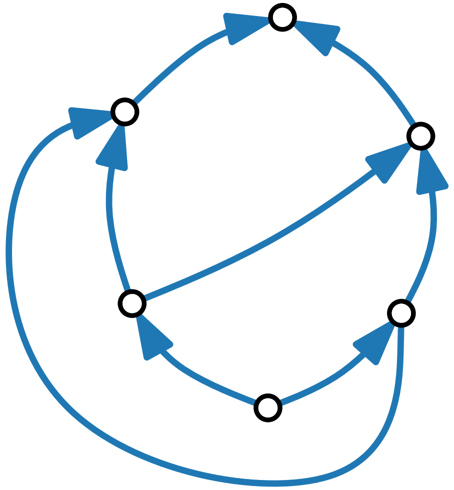


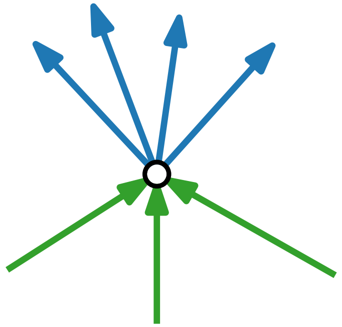
Upward Planarity – Necessary Conditions

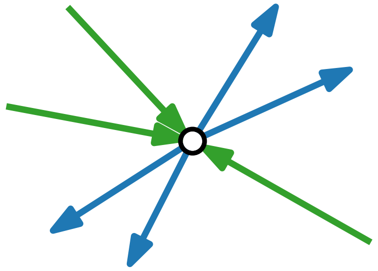
- For a digraph G to be upward planar, it has to be:
  - planar
  - acyclic





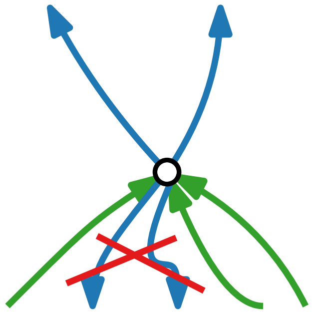


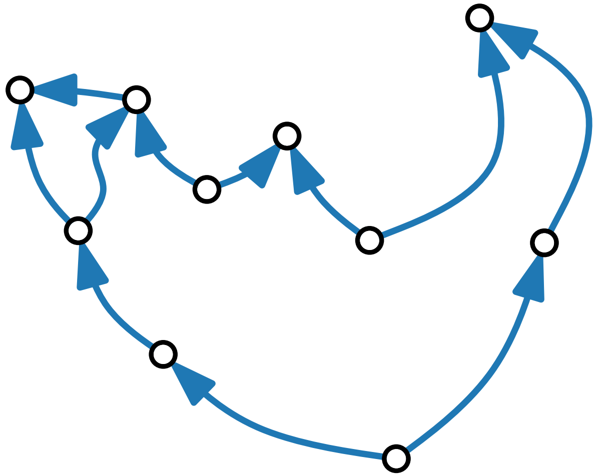


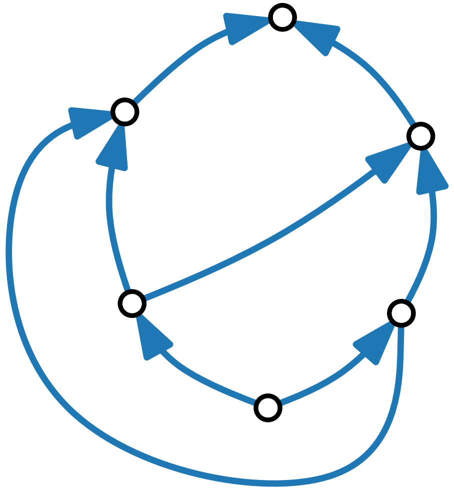


Upward Planarity – Necessary Conditions

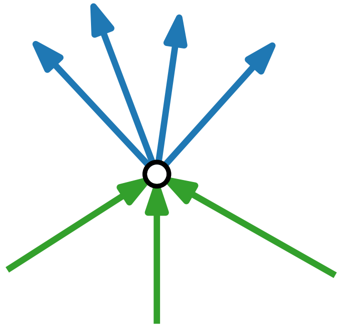
- For a digraph G to be upward planar, it has to be:
  - planar
  - acyclic



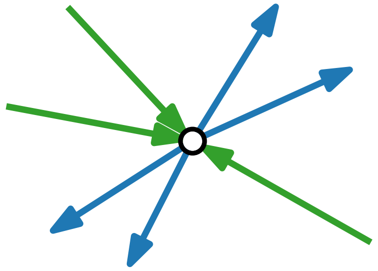




bimodal vertex

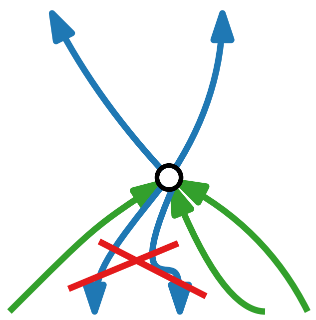


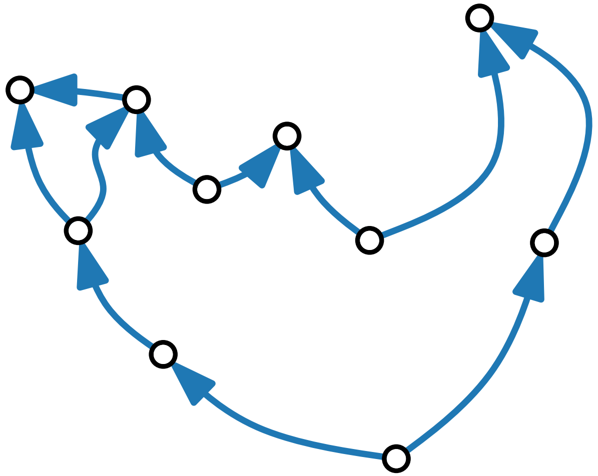
not bimodal

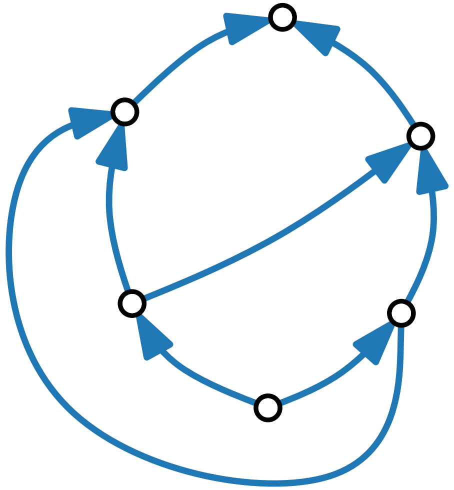


Upward Planarity – Necessary Conditions

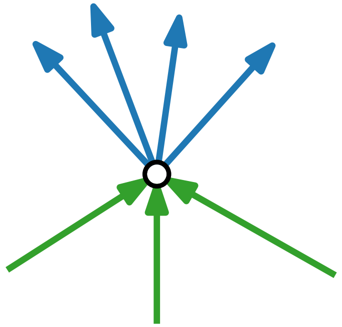
- For a digraph G to be upward planar, it has to be:
  - planar
  - acyclic
  - bimodal



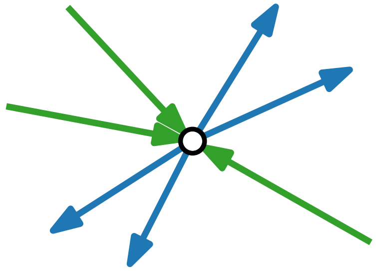




bimodal vertex

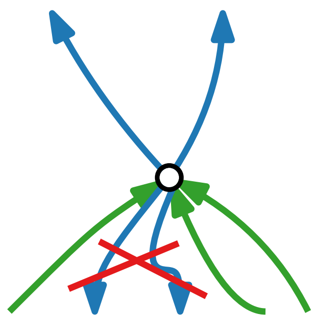


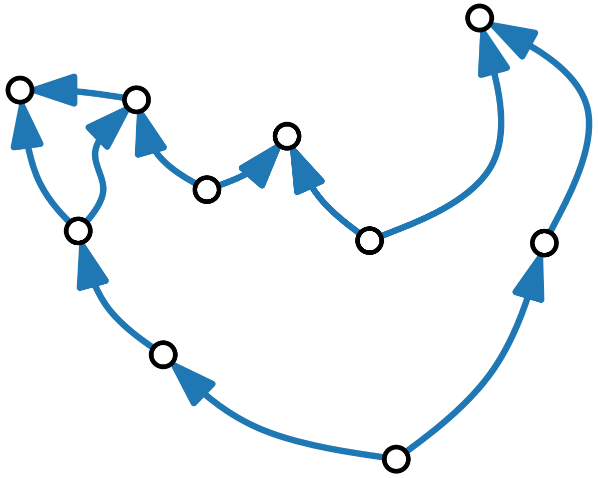
not bimodal

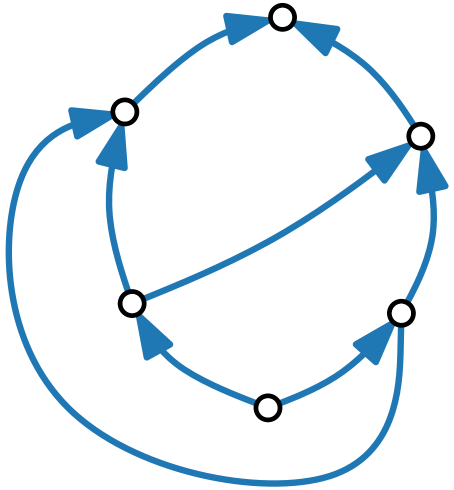


Upward Planarity – Necessary Conditions

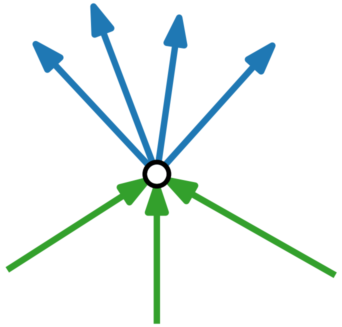
- For a digraph G to be upward planar, it has to be:
  - planar
  - acyclic
  - bimodal
- …but these conditions are *not sufficient*.



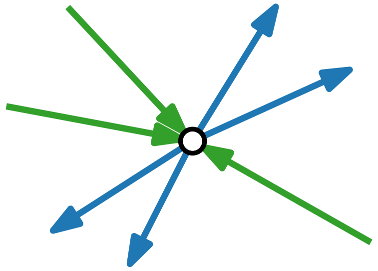




bimodal vertex



not bimodal



Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:

1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar *st*-digraph.

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

$\underbrace{\text{planar}}_{\text{no crossings}}$

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

no crossings

acyclic digraph with
a single source s and single sink t

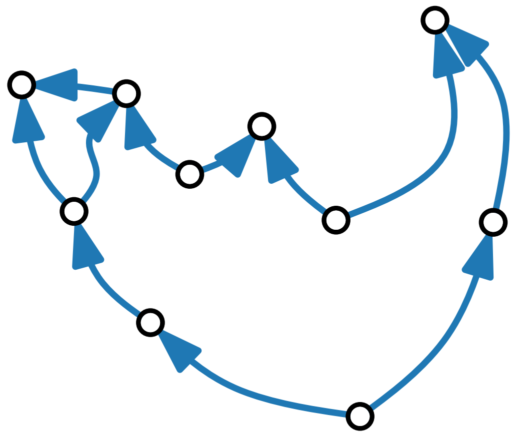
Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:

1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

no crossings

acyclic digraph with
a single source s and single sink t

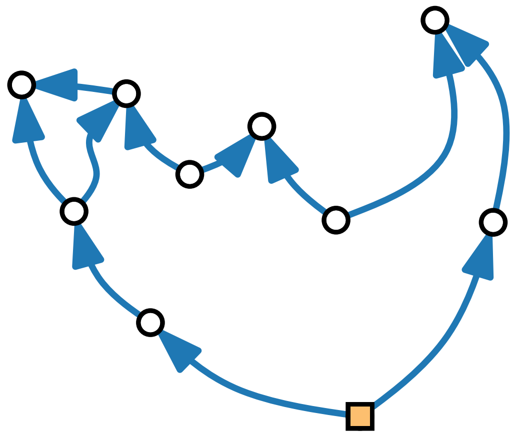


Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

no crossings

acyclic digraph with
a single source s and single sink t

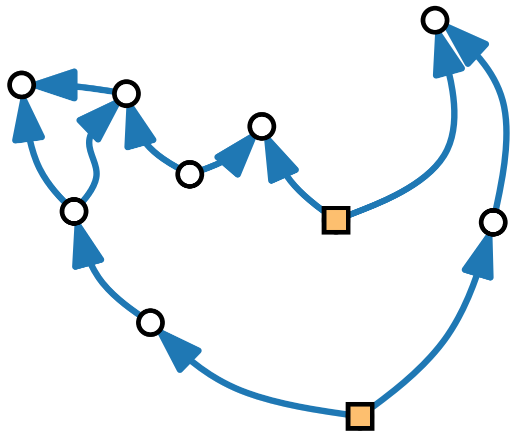


Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

no crossings

acyclic digraph with
a single source s and single sink t

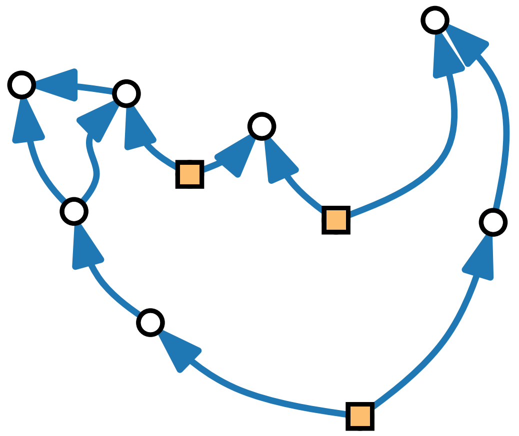


Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

no crossings

acyclic digraph with
a single source s and single sink t



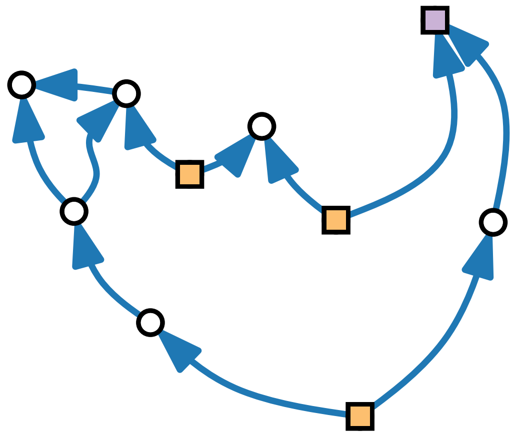
Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:

1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

no crossings

acyclic digraph with
a single source s and single sink t

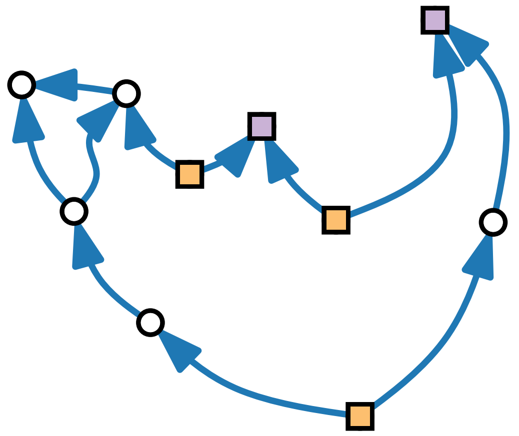


Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

no crossings

acyclic digraph with
a single source s and single sink t

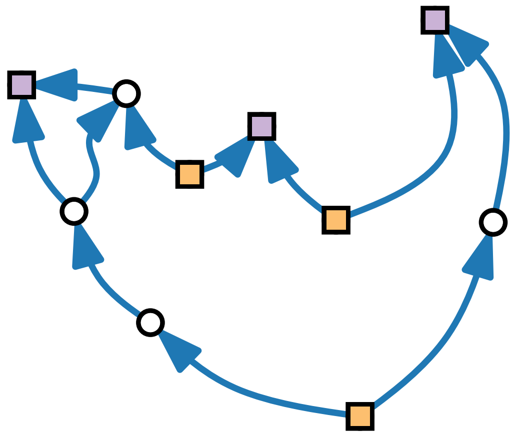


Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

no crossings

acyclic digraph with
a single source s and single sink t

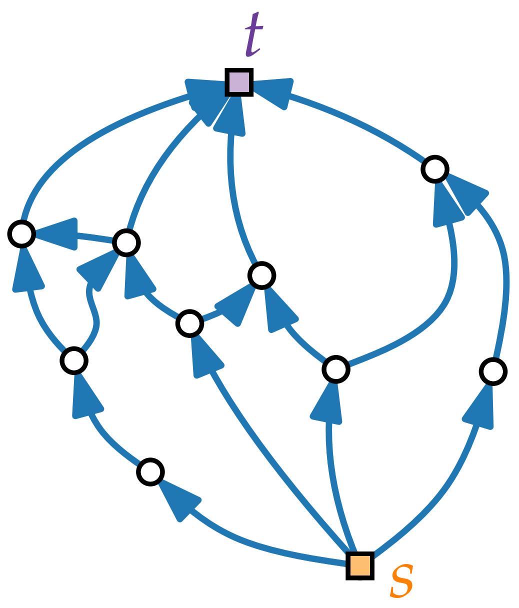


Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

no crossings

acyclic digraph with
a single source s and single sink t



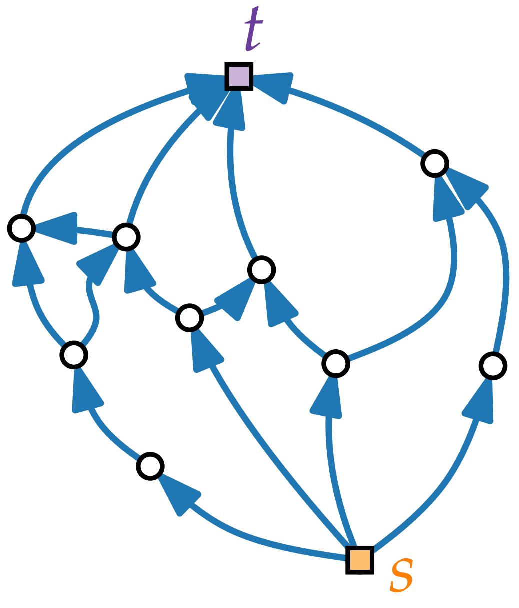
Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

Additionally:
Embedded such that s and t are on the outerface f_0.

no crossings

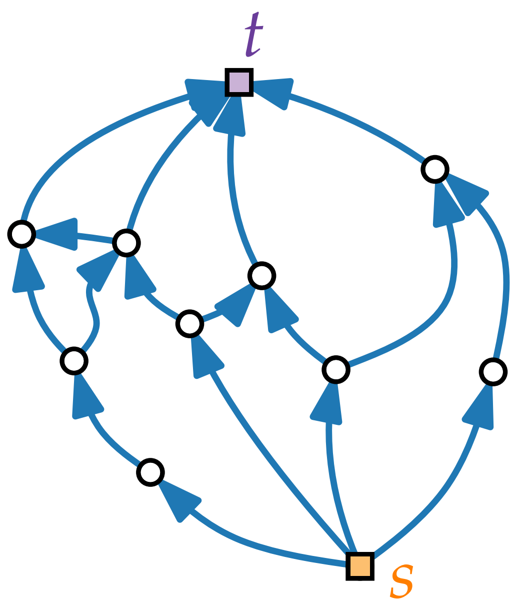
acyclic digraph with
a single source s and single sink t



Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:

1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.



no crossings

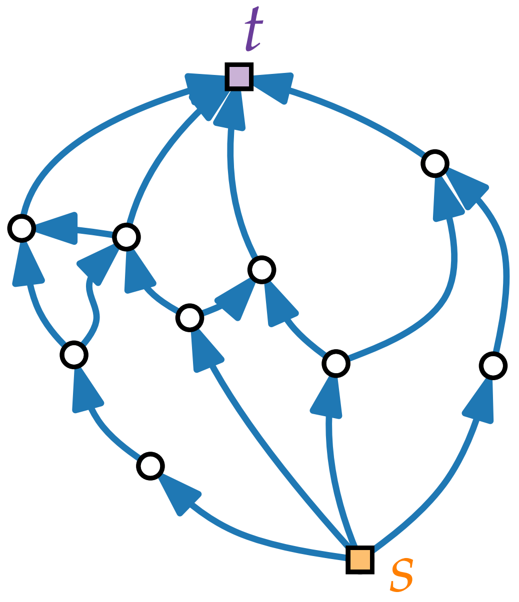
acyclic digraph with
a single source s and single sink t

Additionally:
Embedded such that s and t are on the outerface f_0.

or:
Edge (s, t) exists.

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

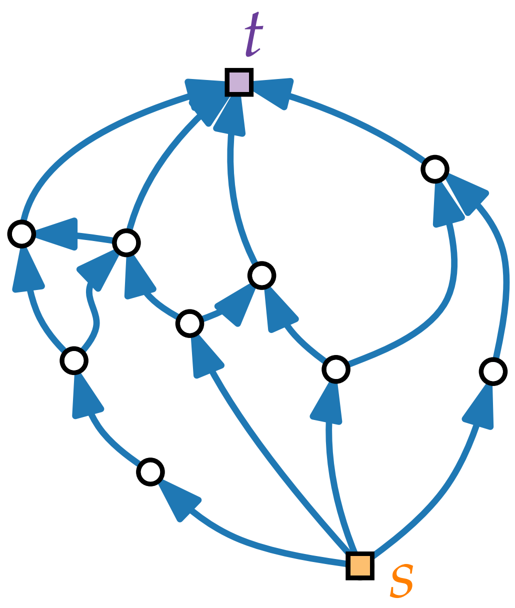


Proof.

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:

1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

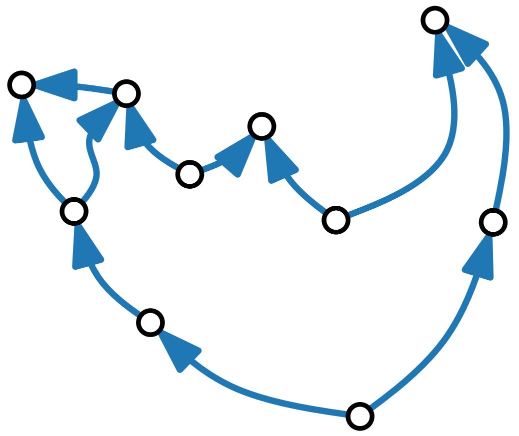


Proof.

(2) $\Rightarrow$ **(1)** By definition.

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar *st*-digraph.



Proof.

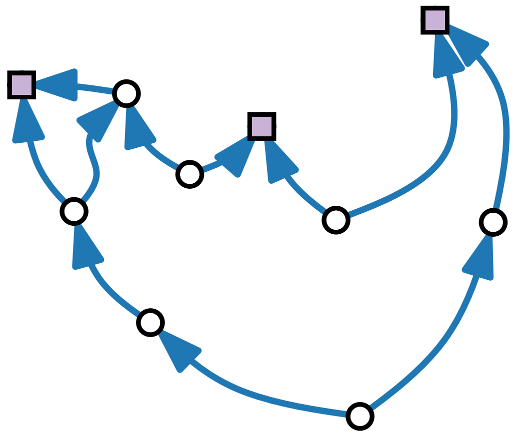
(2) $\Rightarrow$ **(1)** By definition. **(1)** $\Leftrightarrow$ **(3)** Example:

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

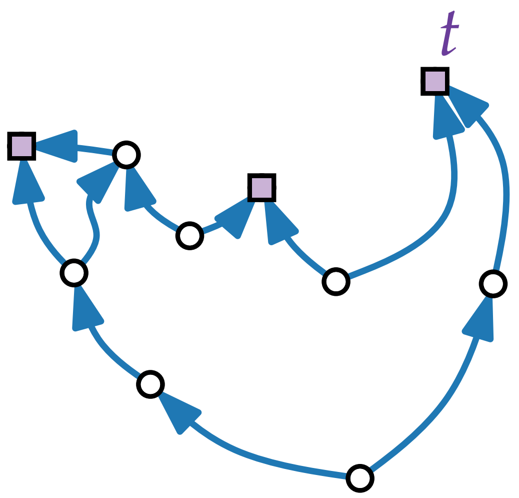
Proof.

(2) $\Rightarrow$ **(1)** By definition. **(1)** $\Leftrightarrow$ **(3)** Example:



Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar *st*-digraph.

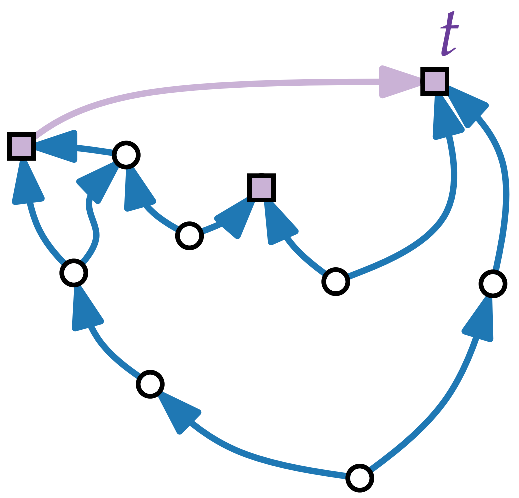


Proof.

(2) $\Rightarrow$ **(1)** By definition. **(1)** $\Leftrightarrow$ **(3)** Example:

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.



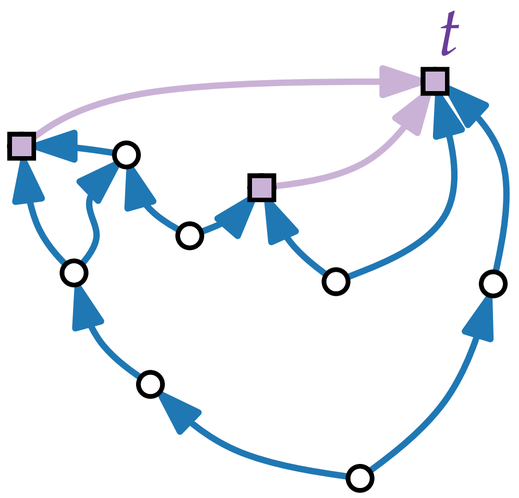
Proof.

(2) $\Rightarrow$ **(1)** By definition. **(1)** $\Leftrightarrow$ **(3)** Example:

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:

1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

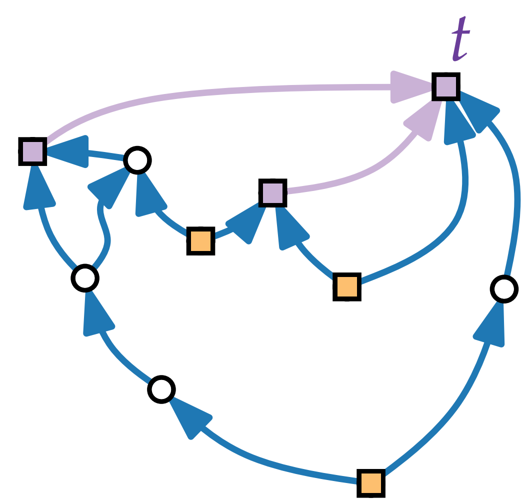


Proof.

(2) $\Rightarrow$ **(1)** By definition. **(1)** $\Leftrightarrow$ **(3)** Example:

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

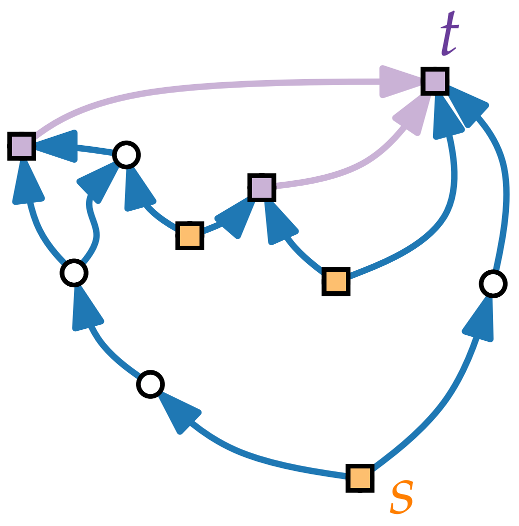


Proof.

(2) $\Rightarrow$ **(1)** By definition. **(1)** $\Leftrightarrow$ **(3)** Example:

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar *st*-digraph.



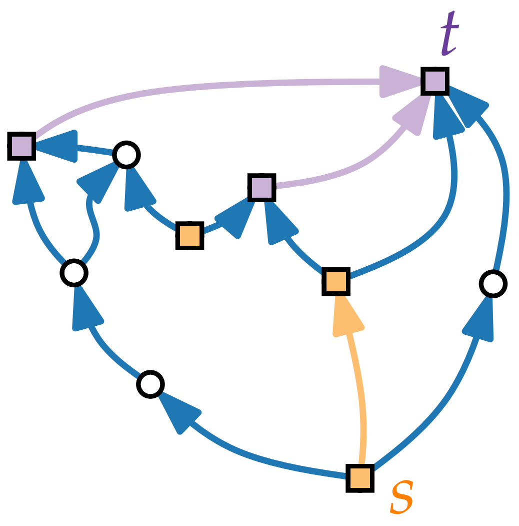
Proof.

(2) $\Rightarrow$ **(1)** By definition. **(1)** $\Leftrightarrow$ **(3)** Example:

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:

1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar *st*-digraph.

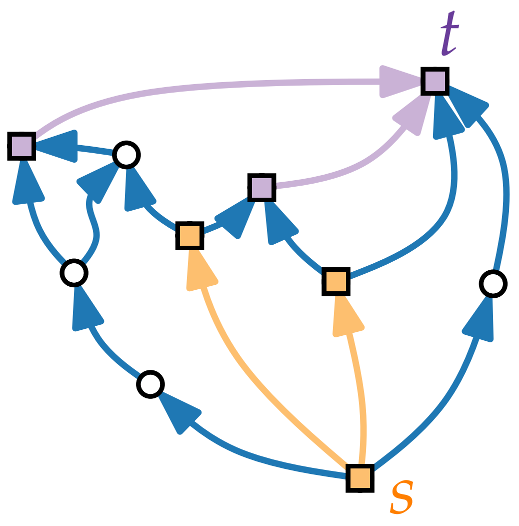


Proof.

(2) $\Rightarrow$ **(1)** By definition. **(1)** $\Leftrightarrow$ **(3)** Example:

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

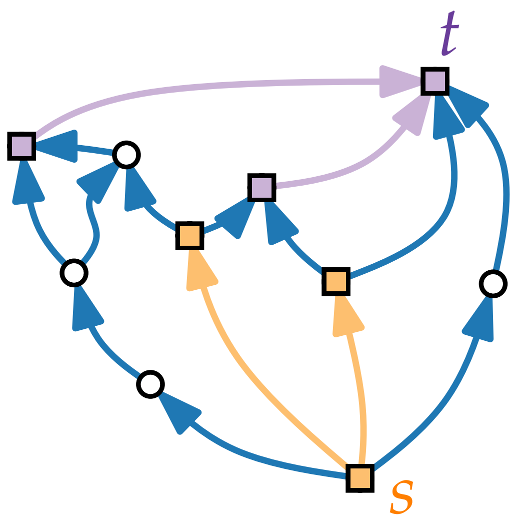


Proof.

(2) $\Rightarrow$ **(1)** By definition. **(1)** $\Leftrightarrow$ **(3)** Example:

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

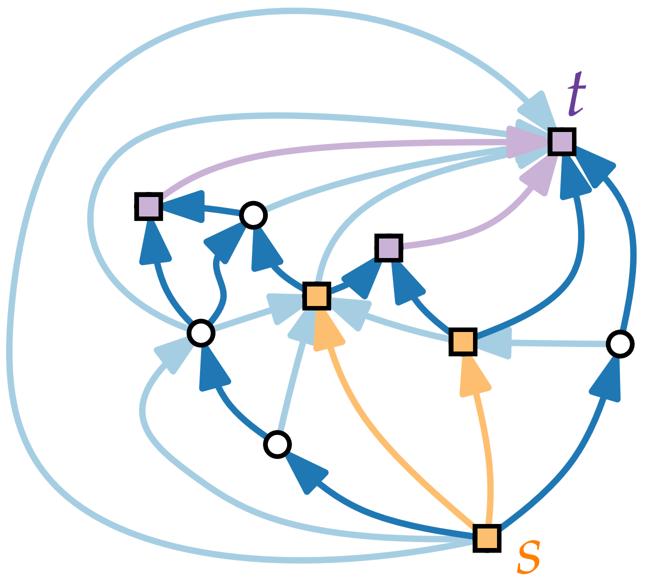


Proof.

(2) $\Rightarrow$ **(1)** By definition. **(1)** $\Leftrightarrow$ **(3)** Example:
(3) $\Rightarrow$ **(2)** Triangulate & construct drawing:

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

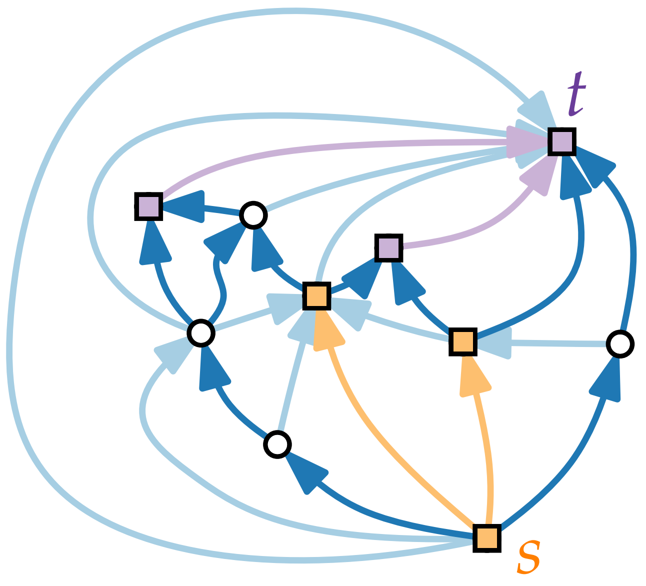


Proof.

(2) $\Rightarrow$ **(1)** By definition. **(1)** $\Leftrightarrow$ **(3)** Example:
(3) $\Rightarrow$ **(2)** Triangulate & construct drawing:

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.



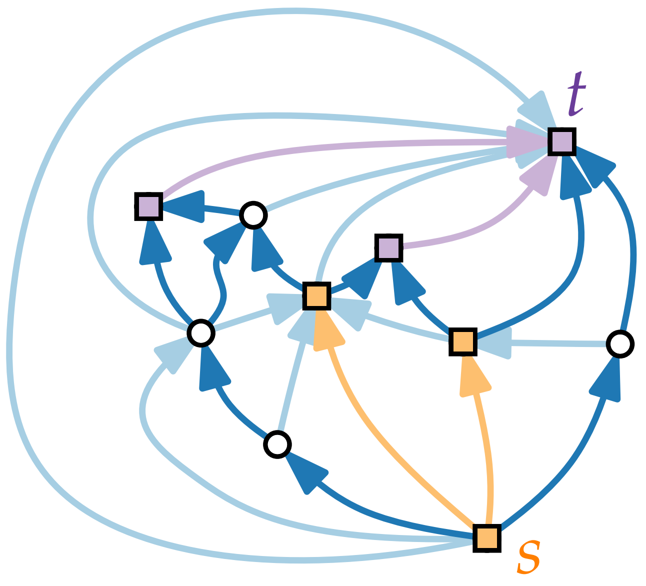
Proof.

(2) ⇒ (1) By definition. **(1) ⇔ (3)** Example:
(3) ⇒ (2) Triangulate & construct drawing:

Claim.
Can draw in prespecified triangle.

Upward Planarity – Characterization

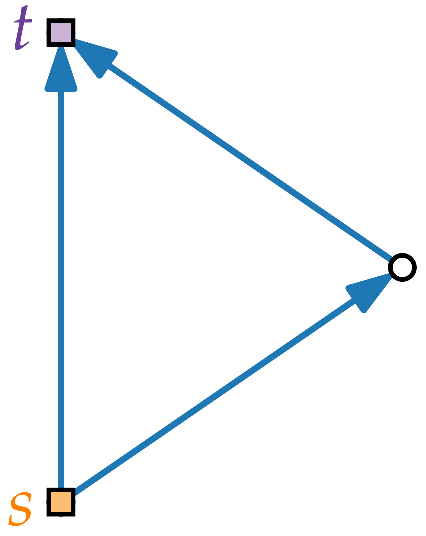
Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.



Proof.

(2) ⇒ **(1)** By definition. **(1)** ⇔ **(3)** Example:
(3) ⇒ **(2)** Triangulate & construct drawing:

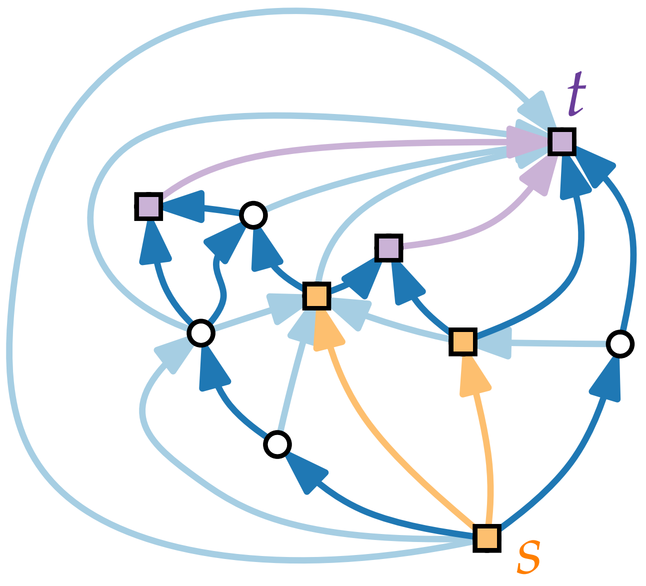
Claim.
Can draw in prespecified triangle.



Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:

1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

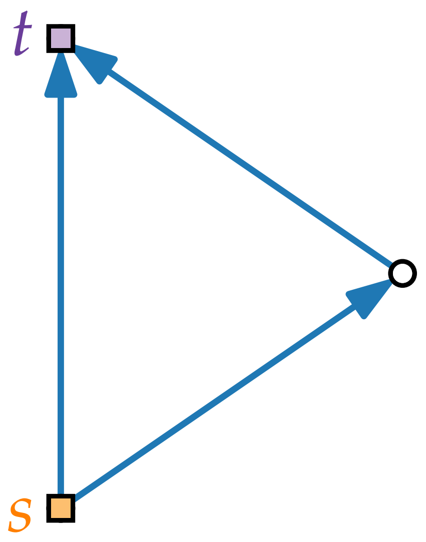


Proof.

(2) $\Rightarrow$ **(1)** By definition. **(1)** $\Leftrightarrow$ **(3)** Example:
(3) $\Rightarrow$ **(2)** Triangulate & construct drawing:

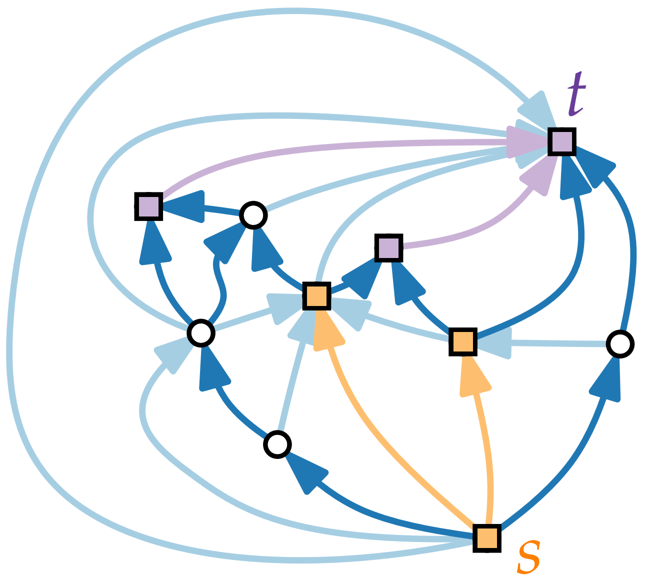
Claim.
Can draw in prespecified triangle.

Induction on n.



Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

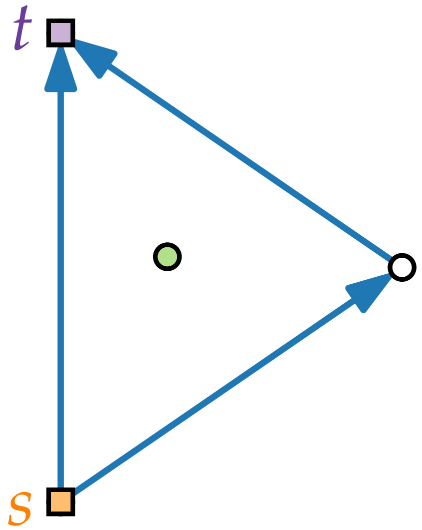


Proof.

(2) $\Rightarrow$ **(1)** By definition. **(1)** $\Leftrightarrow$ **(3)** Example:
(3) $\Rightarrow$ **(2)** Triangulate & construct drawing:

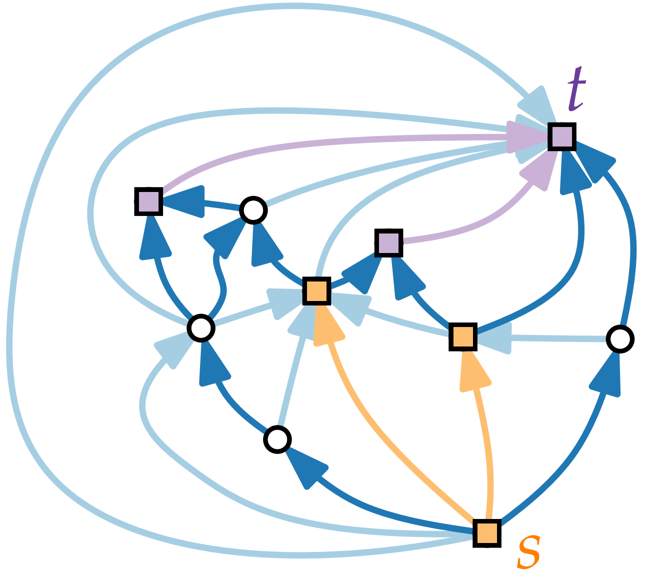
Claim.
Can draw in prespecified triangle.

Induction on n.



Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

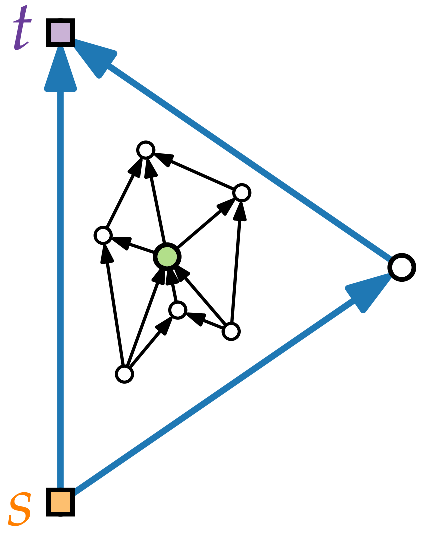


Proof.

(2) $\Rightarrow$ **(1)** By definition. **(1)** $\Leftrightarrow$ **(3)** Example:
(3) $\Rightarrow$ **(2)** Triangulate & construct drawing:

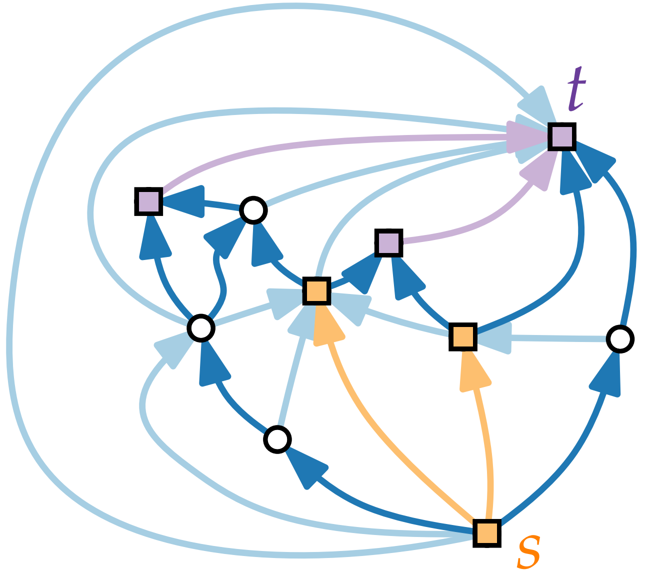
Claim.
Can draw in prespecified triangle.

Induction on n.



Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.



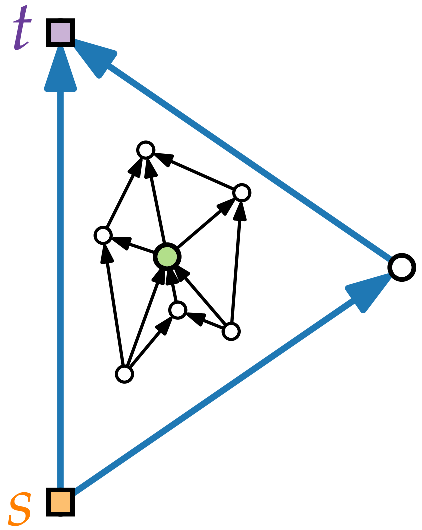
Proof.

(2) ⇒ (1) By definition. **(1) ⇔ (3)** Example:
(3) ⇒ (2) Triangulate & construct drawing:

Claim.
Can draw in prespecified triangle.

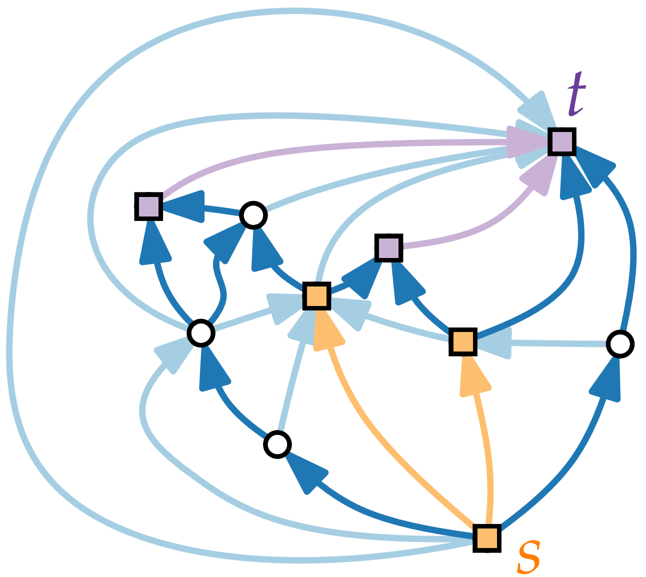
Induction on n.

Case 1: chord



Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.



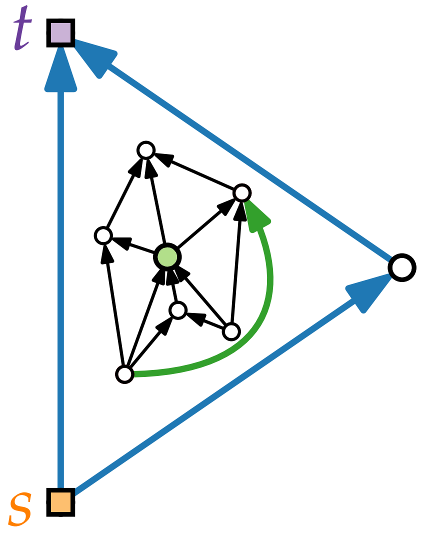
Proof.

(2) $\Rightarrow$ **(1)** By definition. **(1)** $\Leftrightarrow$ **(3)** Example:
(3) $\Rightarrow$ **(2)** Triangulate & construct drawing:

Claim.
Can draw in prespecified triangle.

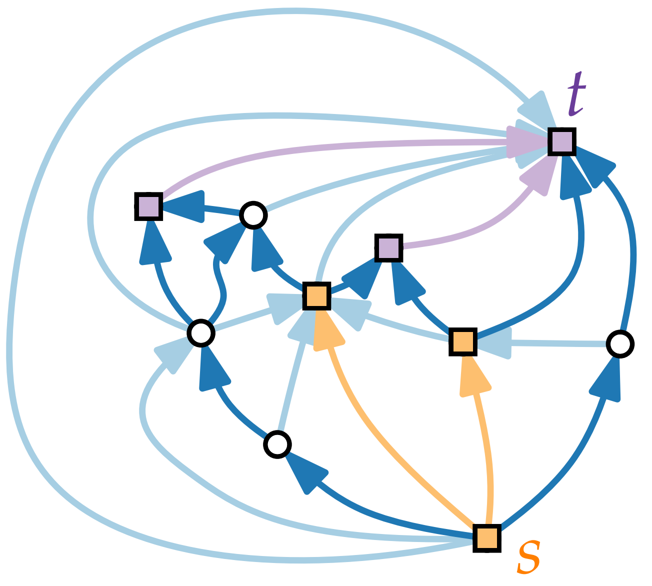
Induction on n.

Case 1: chord



Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.



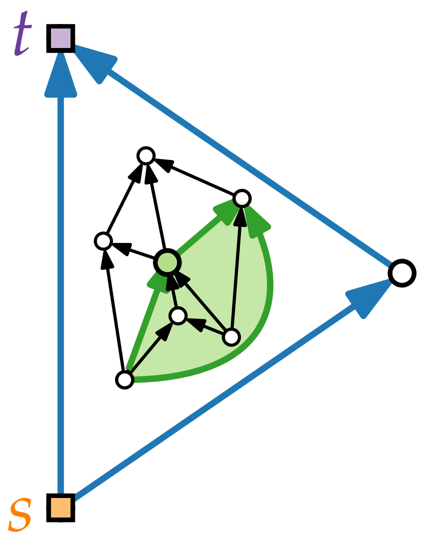
Proof.

(2) ⇒ (1) By definition. **(1) ⇔ (3)** Example:
(3) ⇒ (2) Triangulate & construct drawing:

Claim.
Can draw in prespecified triangle.

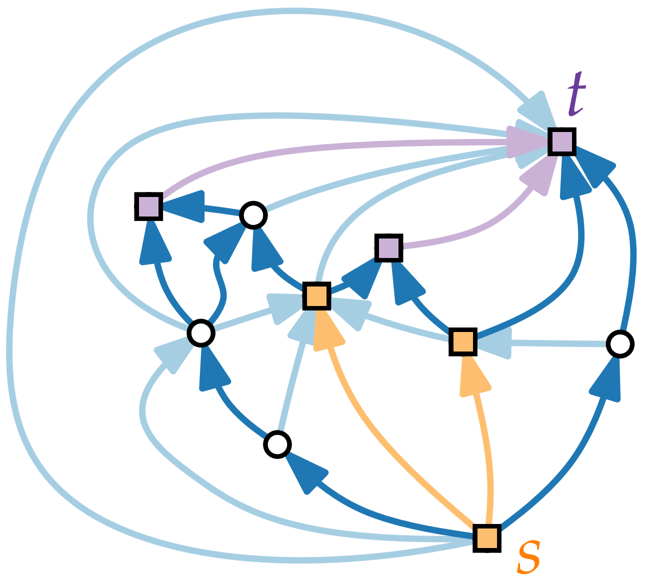
Induction on n.

Case 1: chord



Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.



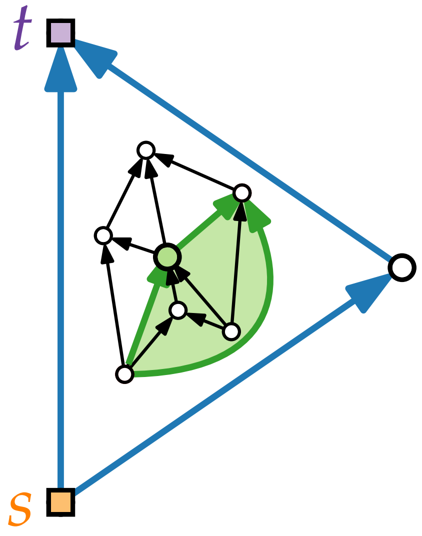
Proof.

(2) ⇒ (1) By definition. **(1) ⇔ (3)** Example:
(3) ⇒ (2) Triangulate & construct drawing:

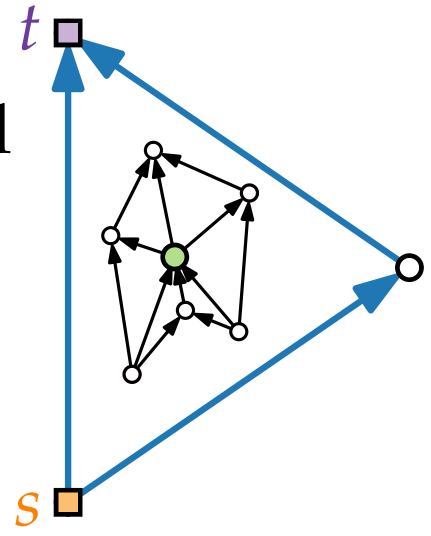
Claim.
Can draw in prespecified triangle.

Induction on n.

Case 1:
chord

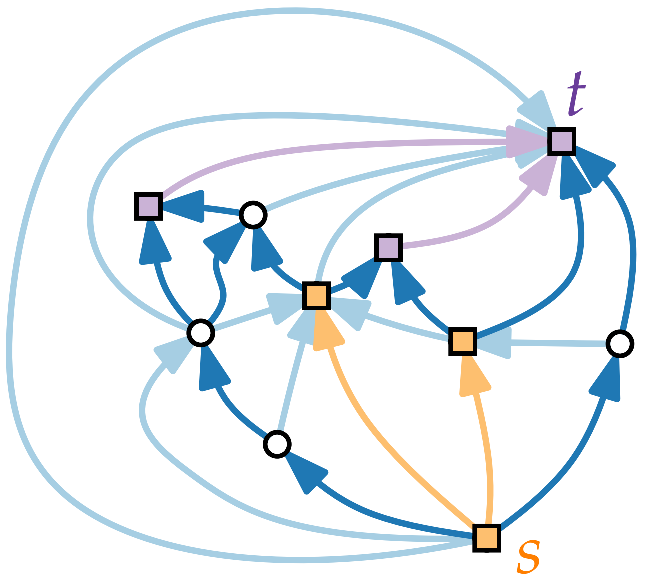


Case 2:
no chord



Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.



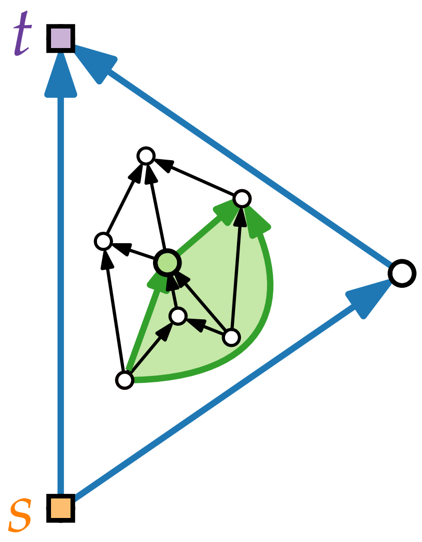
Proof.

(2) ⇒ **(1)** By definition. **(1)** ⇔ **(3)** Example:
(3) ⇒ **(2)** Triangulate & construct drawing:

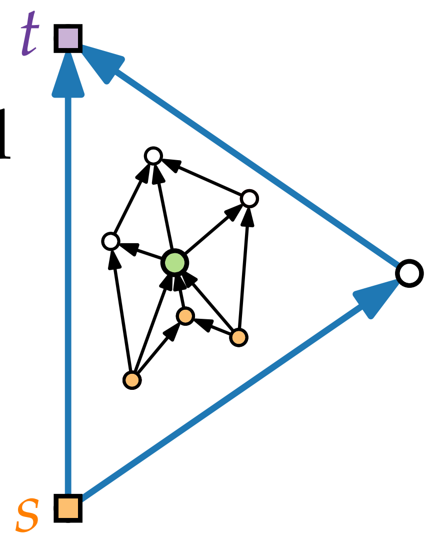
Claim.
Can draw in prespecified triangle.

Induction on n.

Case 1: chord



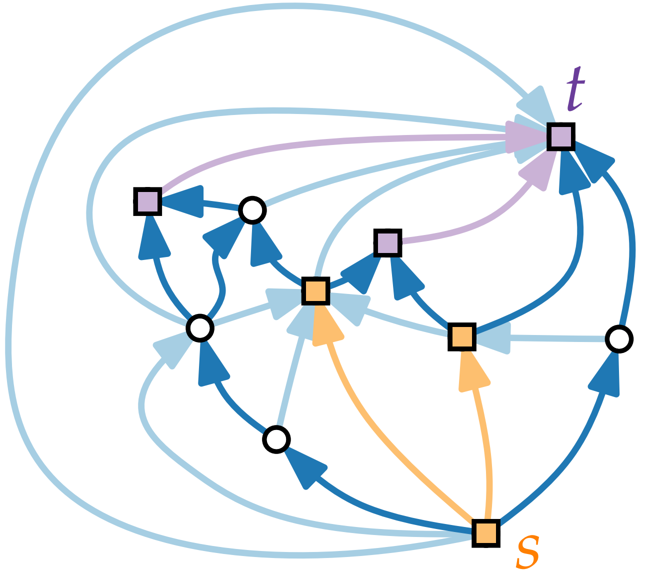
Case 2: no chord



Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:

1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.



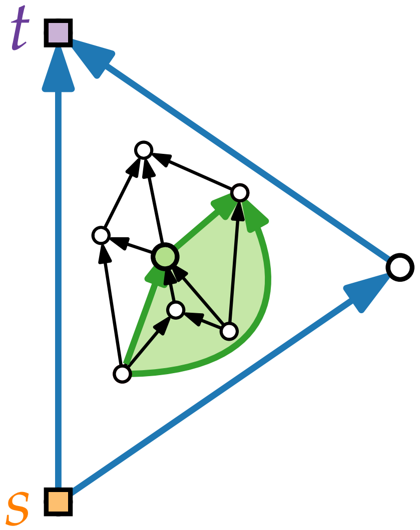
Proof.

(2) $\Rightarrow$ **(1)** By definition. **(1)** $\Leftrightarrow$ **(3)** Example:
(3) $\Rightarrow$ **(2)** Triangulate & construct drawing:

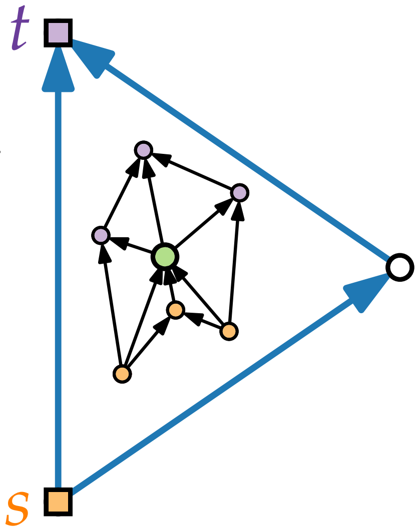
Claim.
Can draw in prespecified triangle.

Induction on n.

Case 1: chord

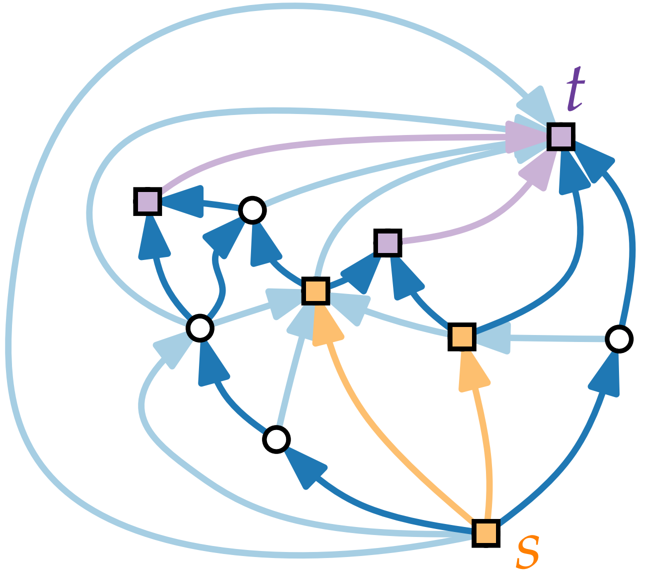


Case 2: no chord



Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.



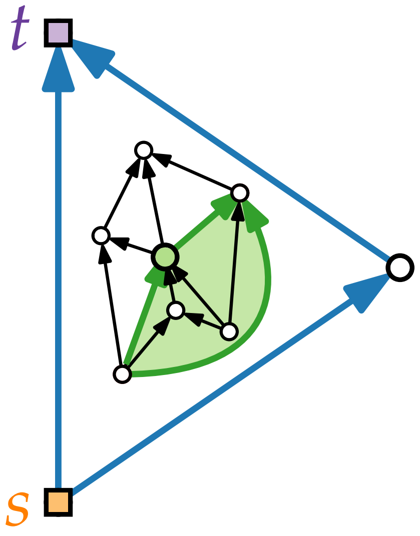
Proof.

(2) ⇒ **(1)** By definition. **(1)** ⇔ **(3)** Example:
(3) ⇒ **(2)** Triangulate & construct drawing:

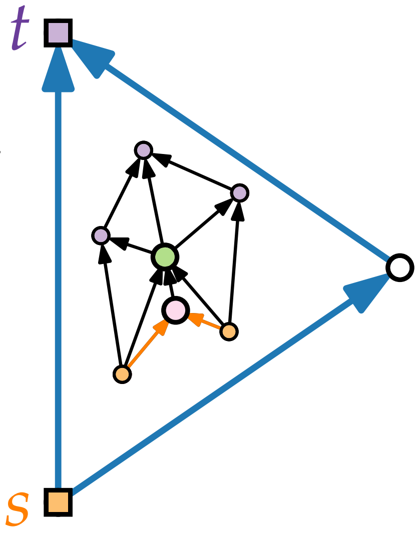
Claim.
Can draw in prespecified triangle.

Induction on n.

Case 1: chord

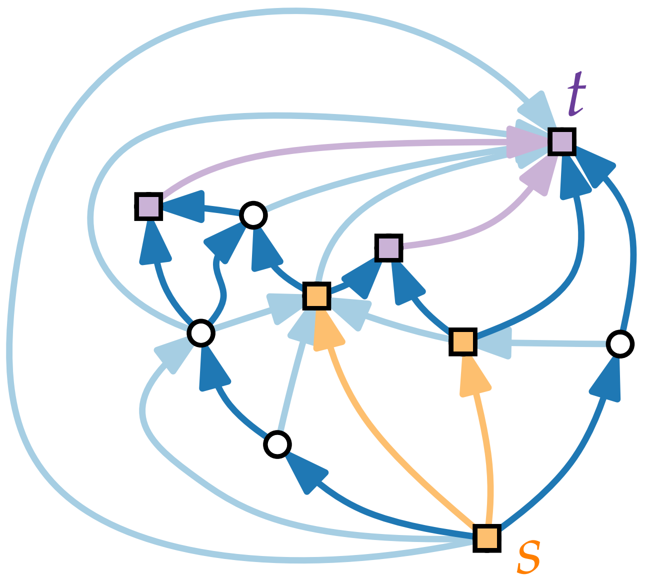


Case 2: no chord



Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.



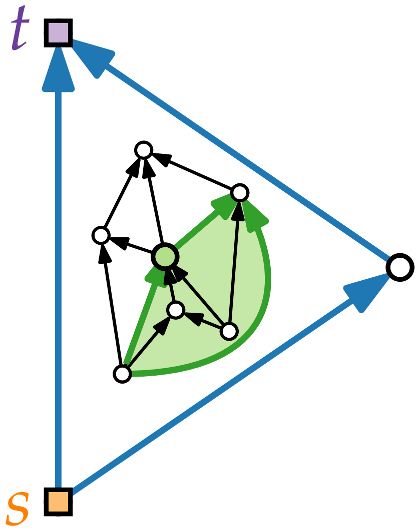
Proof.

(2) $\Rightarrow$ **(1)** By definition. **(1)** $\Leftrightarrow$ **(3)** Example:
(3) $\Rightarrow$ **(2)** Triangulate & construct drawing:

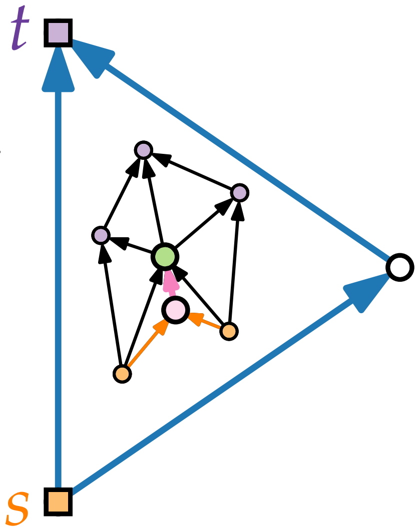
Claim.
Can draw in prespecified triangle.

Induction on n.

Case 1: chord

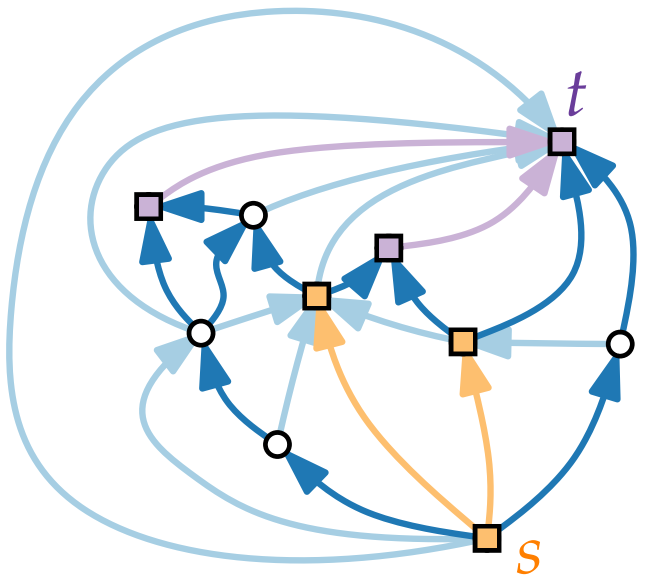


Case 2: no chord



Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.



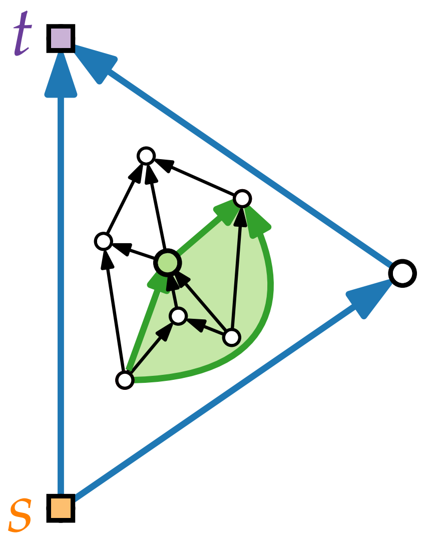
Proof.

(2) $\Rightarrow$ **(1)** By definition. **(1)** $\Leftrightarrow$ **(3)** Example:
(3) $\Rightarrow$ **(2)** Triangulate & construct drawing:

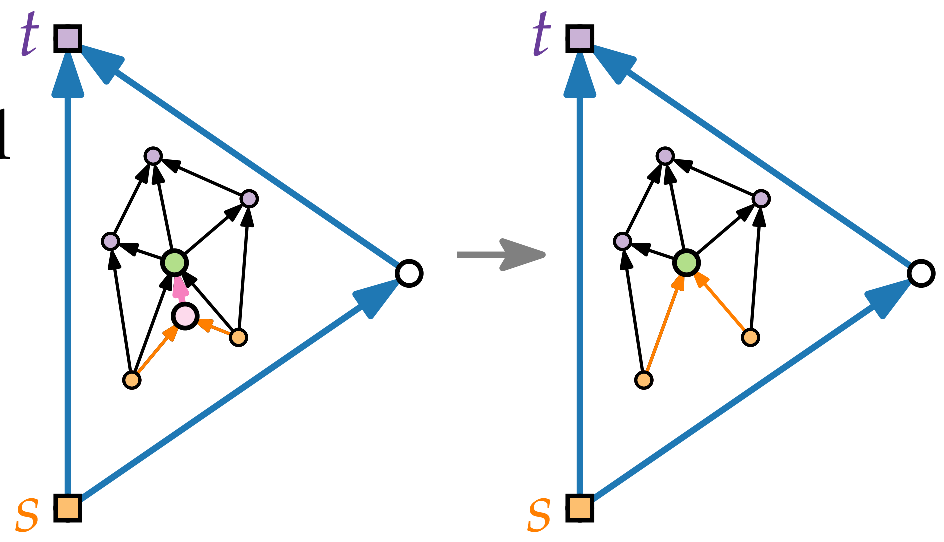
Claim.
Can draw in prespecified triangle.

Induction on n.

Case 1: chord

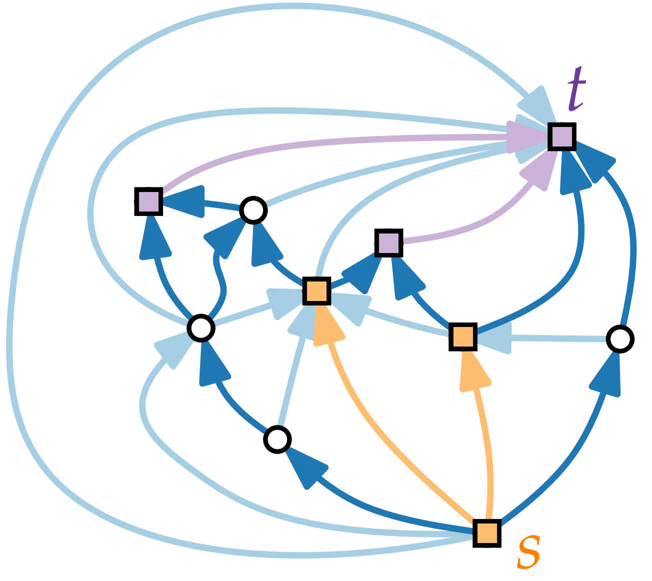


Case 2: no chord



Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.



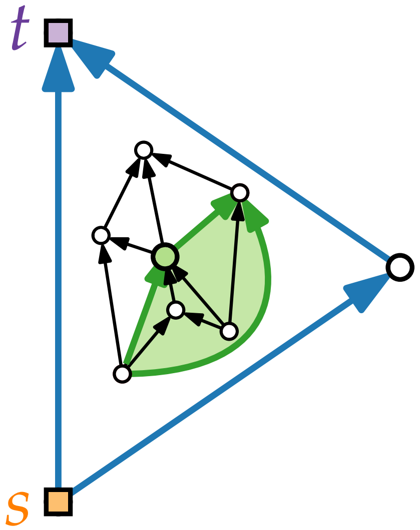
Proof.

(2) ⇒ **(1)** By definition. **(1)** ⇔ **(3)** Example:
(3) ⇒ **(2)** Triangulate & construct drawing:

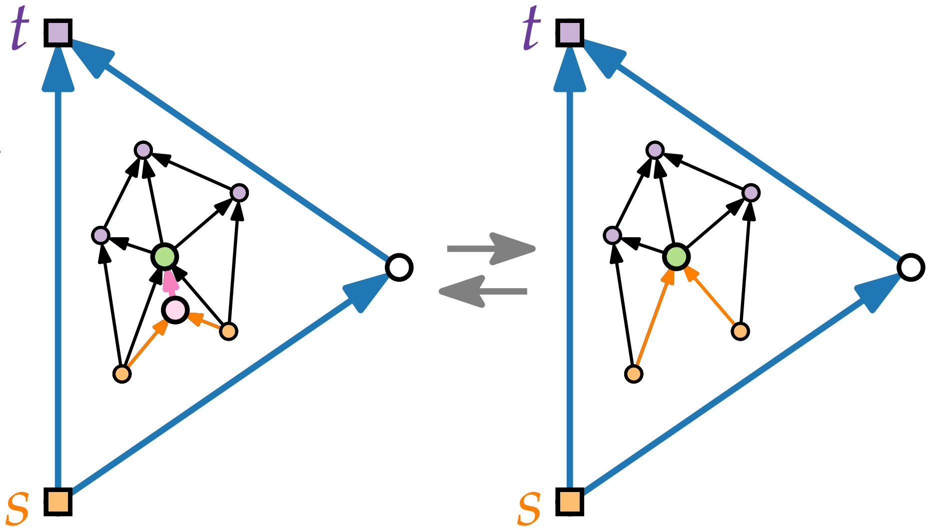
Claim.
Can draw in prespecified triangle.

Induction on n.

Case 1: chord

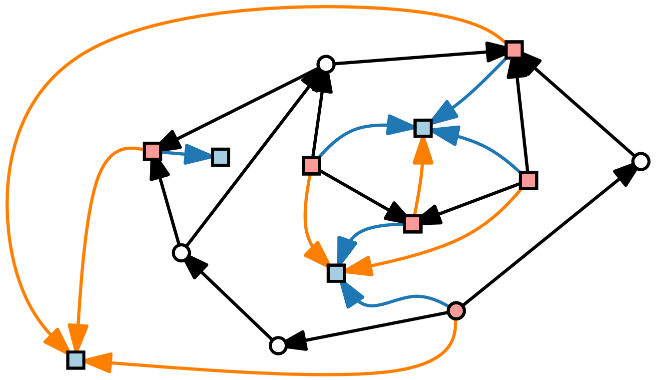


Case 2: no chord

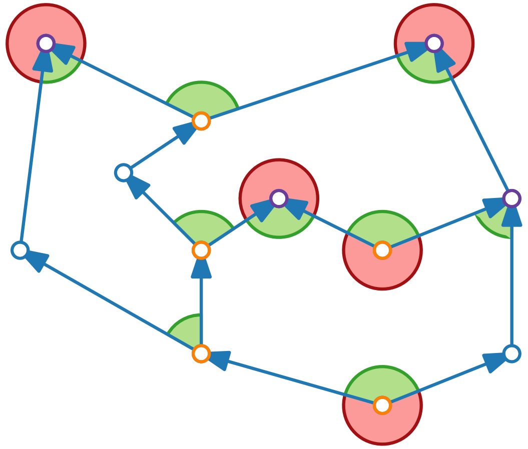


Universität Trier

Visualization of Graphs



Lecture 7:
Upward Planar Drawings



Part II:
Complexity

Philipp Kindermann

Upward Planarity – Complexity

Theorem. [Garg & Tamassia, 1995]
For a *planar acyclic* digraph it is in general NP-hard to decide whether it is upward planar.

Upward Planarity – Complexity

Theorem. [Garg & Tamassia, 1995]
For a *planar acyclic* digraph it is in general NP-hard to decide whether it is upward planar.

Theorem. [Bertolazzi et al., 1994]
For a *combinatorially embedded* planar digraph it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Upward Planarity – Complexity

Theorem. [Garg & Tamassia, 1995]
For a *planar acyclic* digraph it is in general NP-hard to decide whether it is upward planar.

Theorem. [Bertolazzi et al., 1994]
For a *combinatorially embedded* planar digraph it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Corollary.
For a *triconnected* planar digraph it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Upward Planarity – Complexity

Theorem. [Garg & Tamassia, 1995]
For a *planar acyclic* digraph it is in general NP-hard to decide whether it is upward planar.

Theorem. [Bertolazzi et al., 1994]
For a *combinatorially embedded* planar digraph it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Corollary.
For a *triconnected* planar digraph it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Theorem. [Hutton & Lubiw, 1996]
For a *single-source* acyclic digraph it can be tested in $\mathcal{O}(n)$ time whether it is upward planar.

Upward Planarity – Complexity

Theorem. [Garg & Tamassia, 1995]
For a *planar acyclic* digraph it is in general NP-hard to decide whether it is upward planar.

Theorem. [Bertolazzi et al., 1994]
For a *combinatorially embedded* planar digraph it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Corollary.
For a *triconnected* planar digraph it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Theorem. [Hutton & Lubiw, 1996]
For a *single-source* acyclic digraph it can be tested in $\mathcal{O}(n)$ time whether it is upward planar.

The Problem

Fixed Embedding Upward Planarity Testing.
Let $G = (V, E)$ be a plane digraph with set of faces F and outer face f_0.
Test whether G is upward planar (wrt to F, f_0).

The Problem

Fixed Embedding Upward Planarity Testing.
Let $G = (V, E)$ be a plane digraph with set of faces F and outer face f_0.
Test whether G is upward planar (wrt to F, f_0).

Idea.

- Find property that any upward planar drawing of G satisfies.

The Problem

Fixed Embedding Upward Planarity Testing.
Let $G = (V, E)$ be a plane digraph with set of faces F and outer face f_0.
Test whether G is upward planar (wrt to F, f_0).

Idea.

- Find property that any upward planar drawing of G satisfies.
- Formalize property.

The Problem

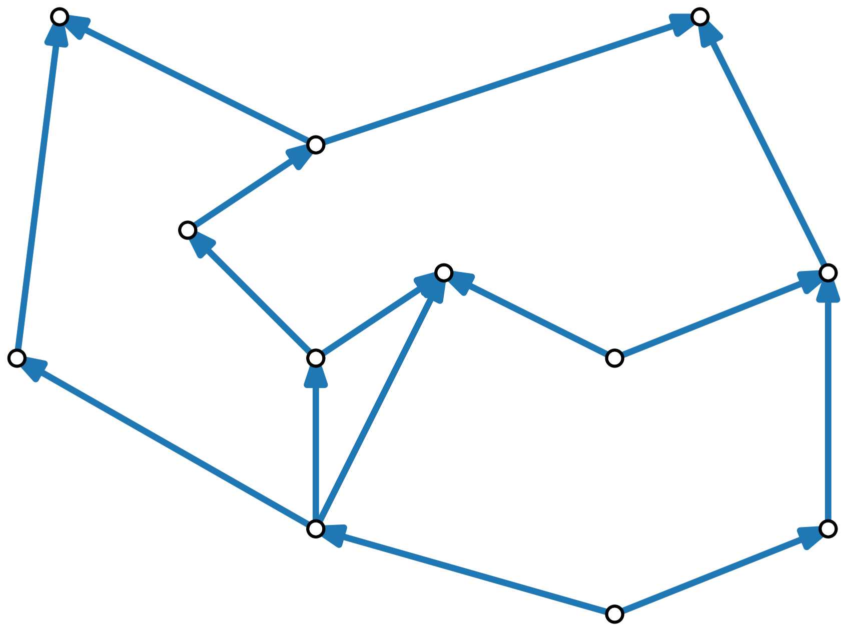
Fixed Embedding Upward Planarity Testing.
Let $G = (V, E)$ be a plane digraph with set of faces F and outer face f_0.
Test whether G is upward planar (wrt to F, f_0).

Idea.

- Find property that any upward planar drawing of G satisfies.
- Formalize property.
- Find algorithm to test property.

Angles, Local Sources & Sinks

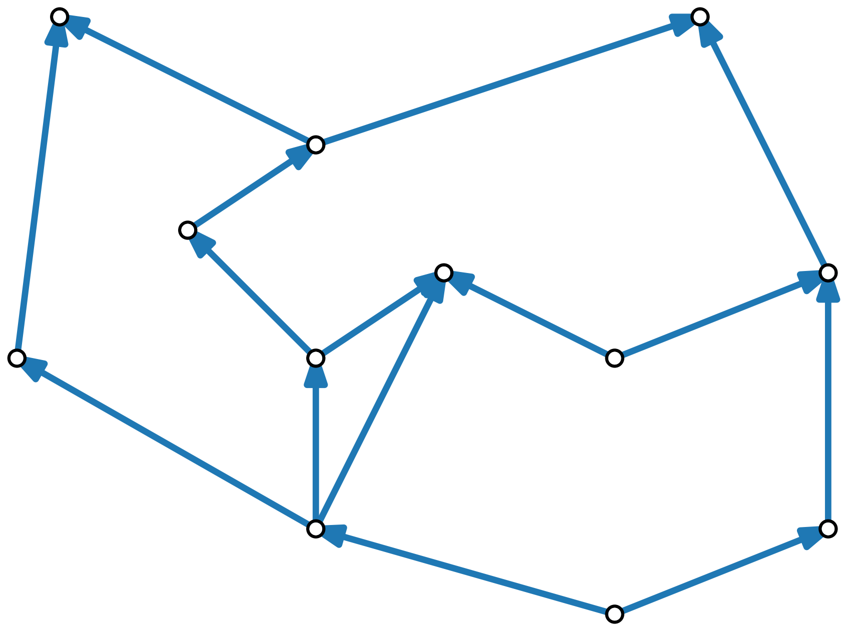
Definitions.



Angles, Local Sources & Sinks

Definitions.

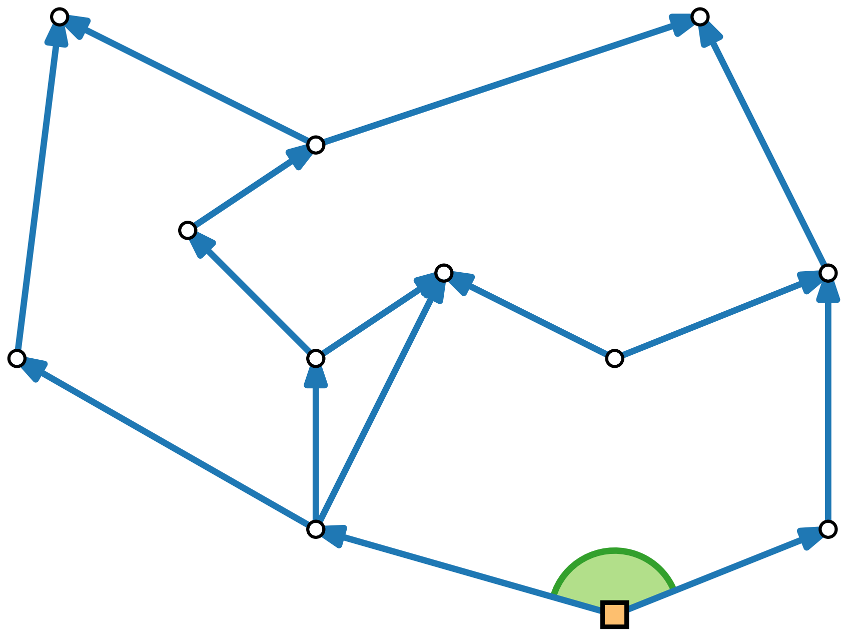
- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.



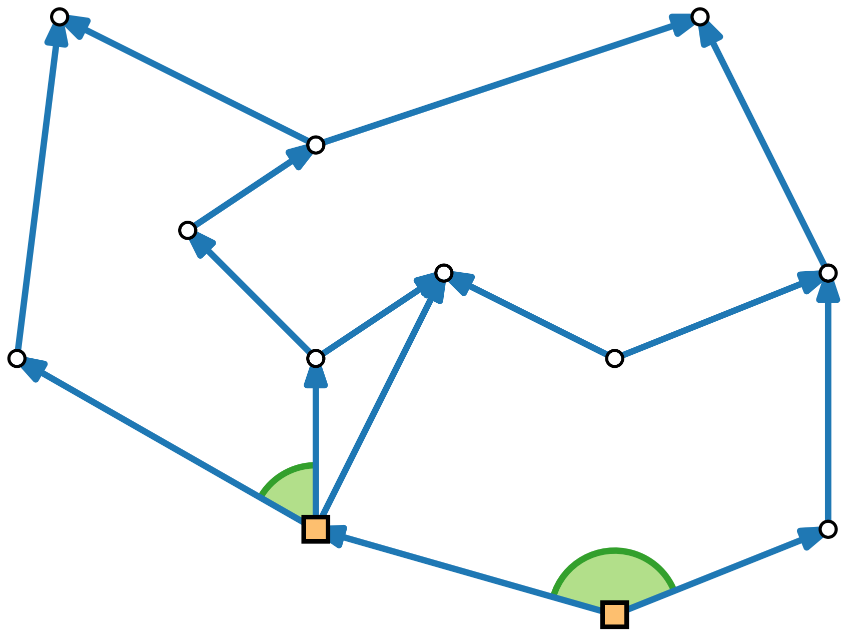
Angles, Local Sources & Sinks

Definitions.

- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.



Angles, Local Sources & Sinks



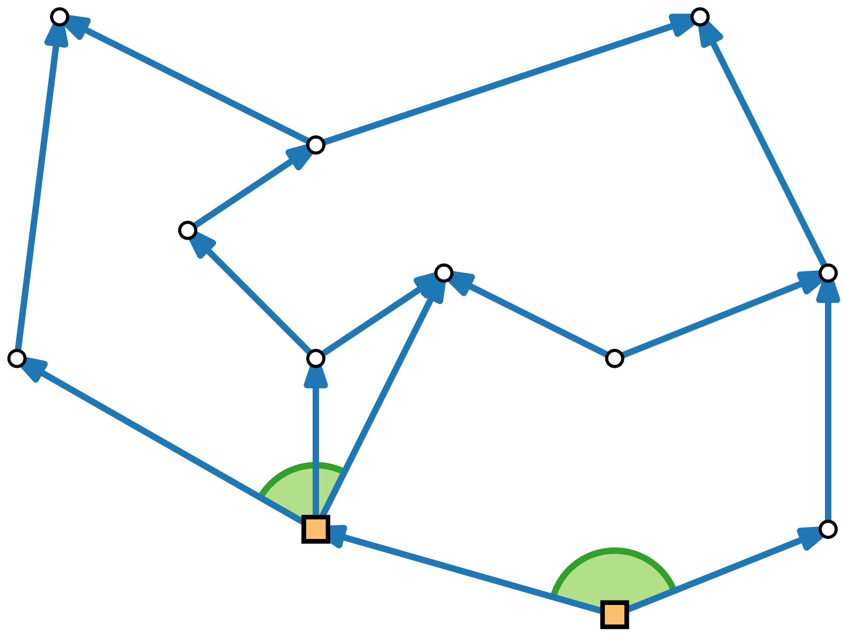
Definitions.

- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.

Angles, Local Sources & Sinks

Definitions.

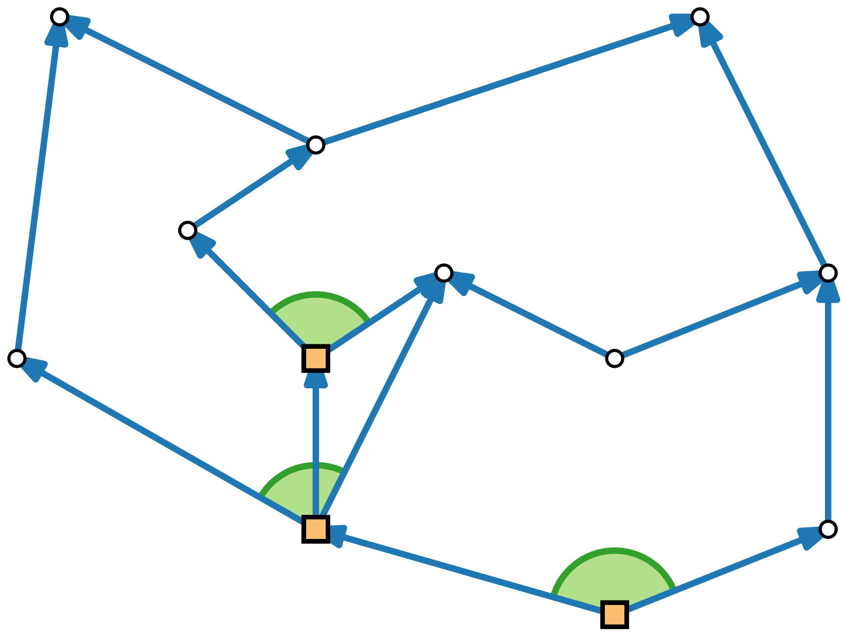
- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.



Angles, Local Sources & Sinks

Definitions.

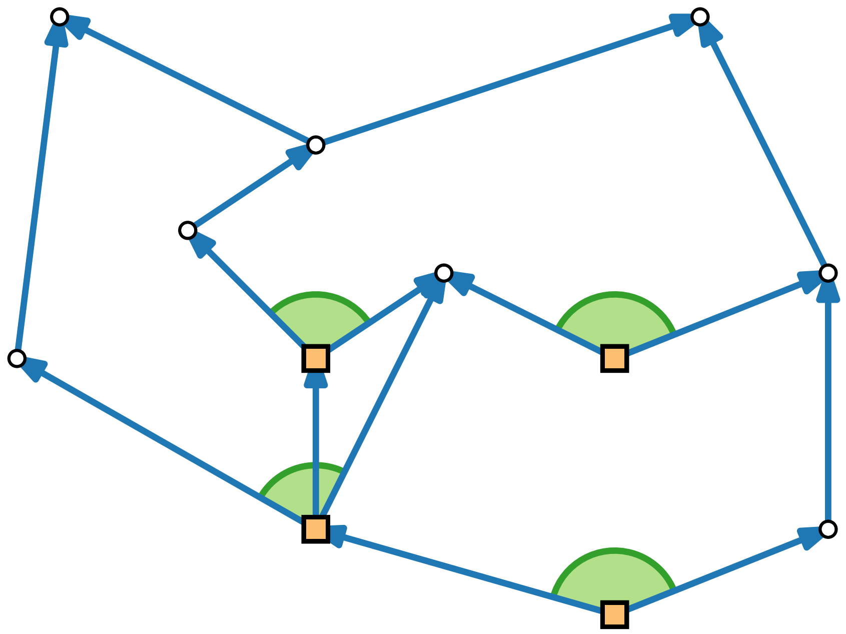
- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.



Angles, Local Sources & Sinks

Definitions.

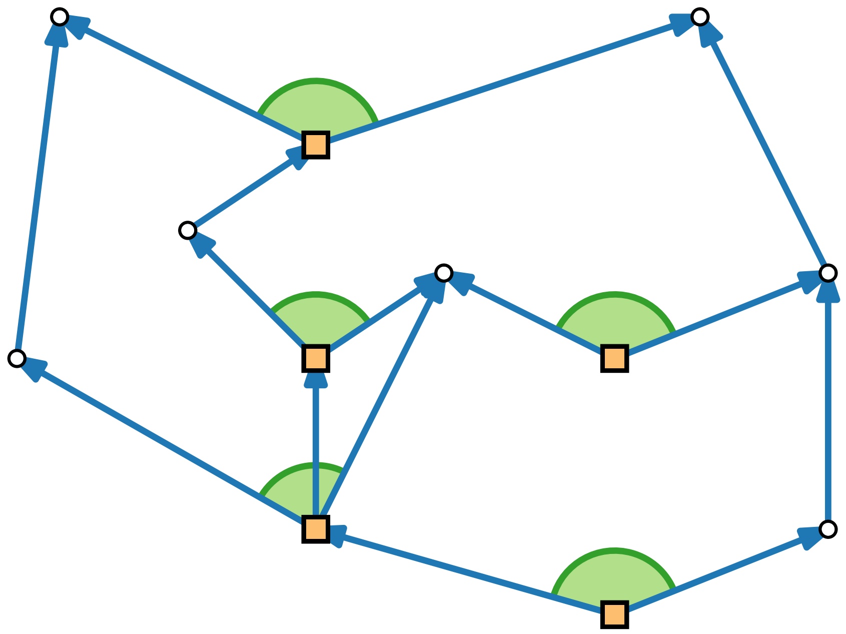
- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.



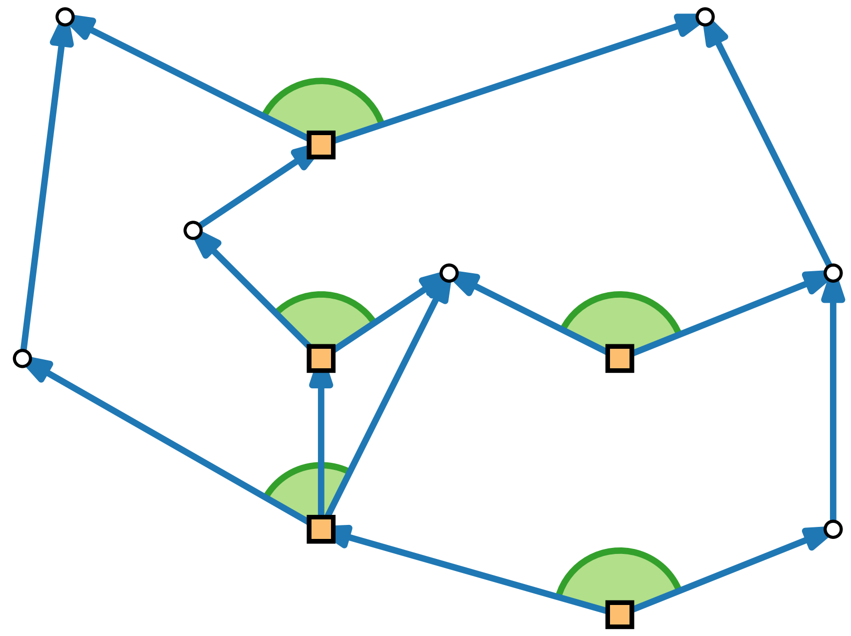
Angles, Local Sources & Sinks

Definitions.

- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.



Angles, Local Sources & Sinks



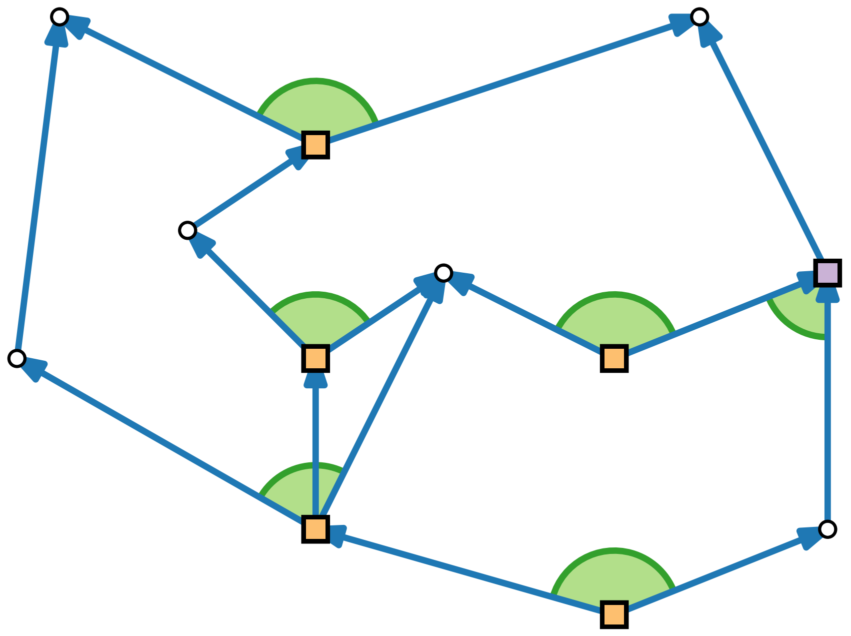
Definitions.

- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt a face f if v has two incoming edges on ∂f.

Angles, Local Sources & Sinks

Definitions.

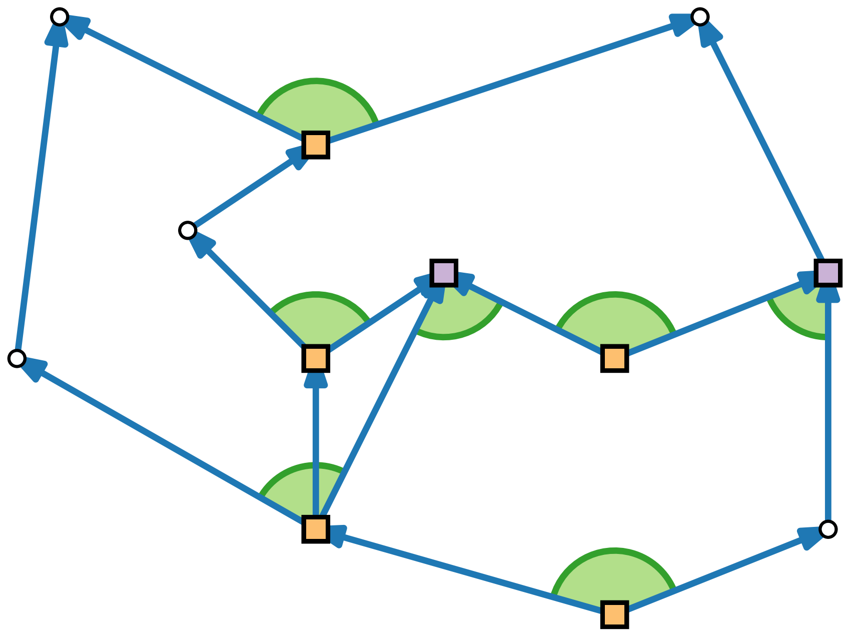
- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt a face f if v has two incoming edges on ∂f.



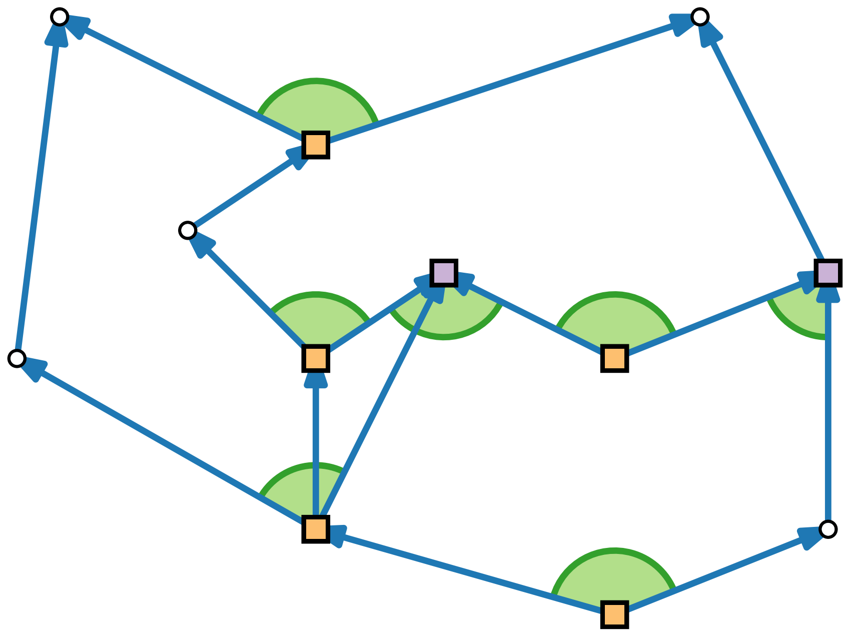
Angles, Local Sources & Sinks

Definitions.

- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt a face f if v has two incoming edges on ∂f.



Angles, Local Sources & Sinks



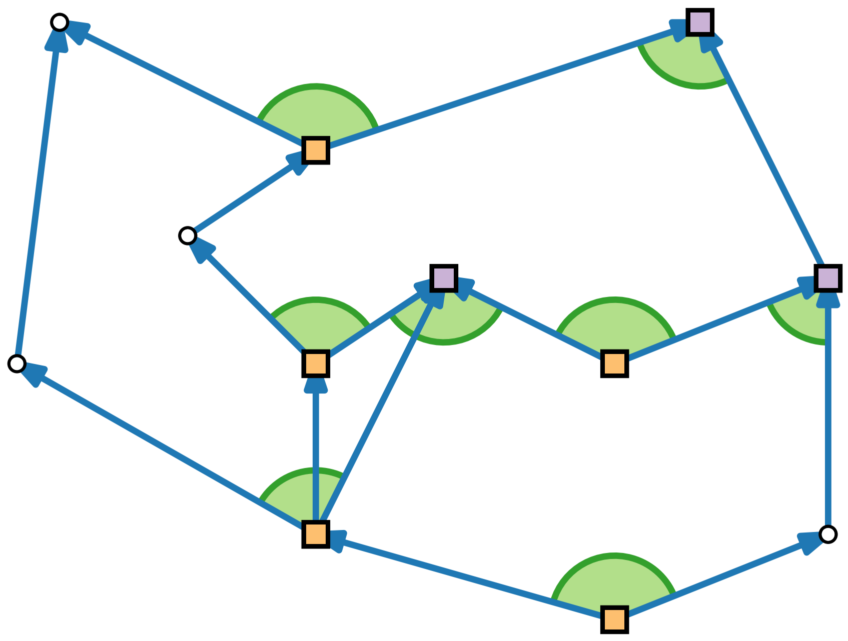
Definitions.

- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt a face f if v has two incoming edges on ∂f.

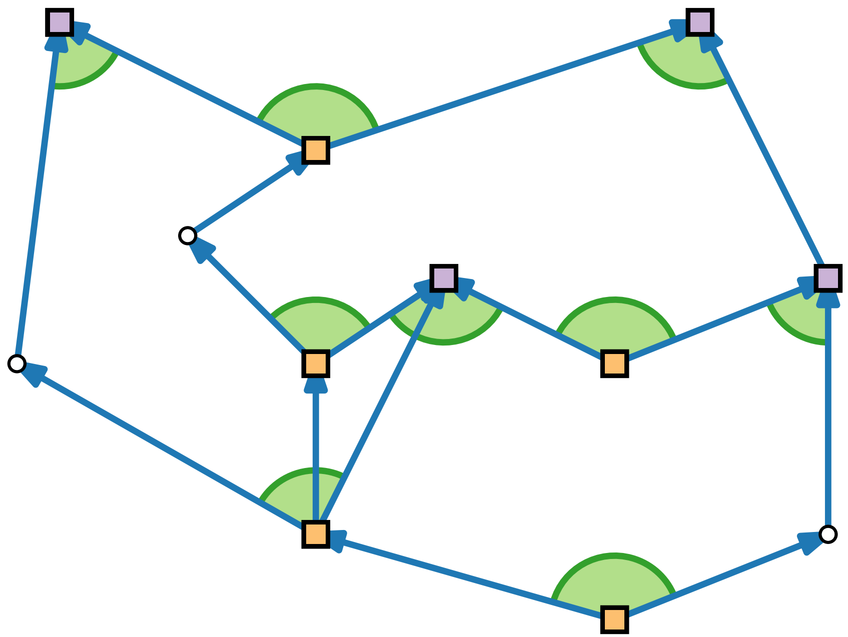
Angles, Local Sources & Sinks

Definitions.

- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt a face f if v has two incoming edges on ∂f.



Angles, Local Sources & Sinks



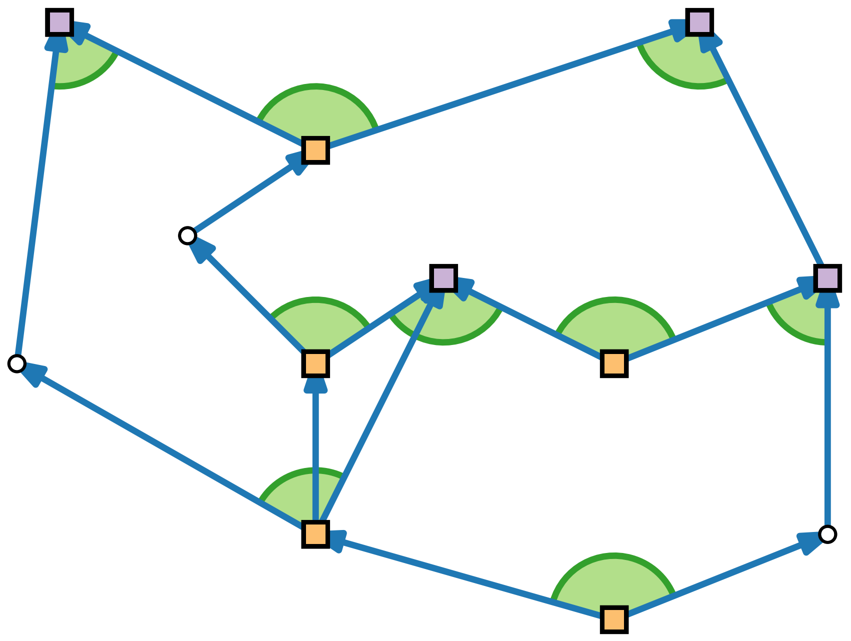
Definitions.

- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt a face f if v has two incoming edges on ∂f.

Angles, Local Sources & Sinks

Definitions.

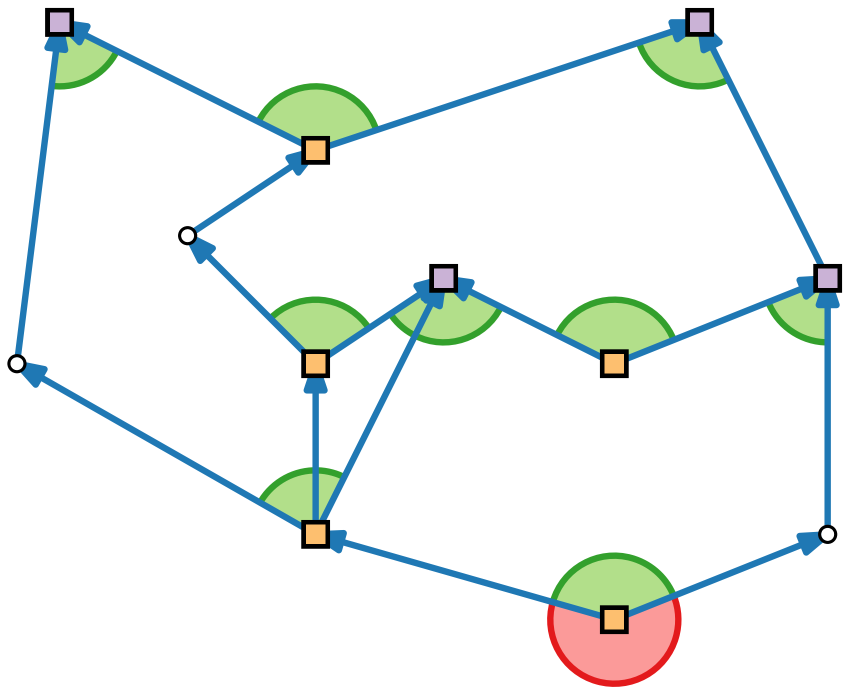
- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt a face f if v has two incoming edges on ∂f.
- An angle α at a local source / sink is **large** when $\alpha > \pi$ and **small** otherwise.



Angles, Local Sources & Sinks

Definitions.

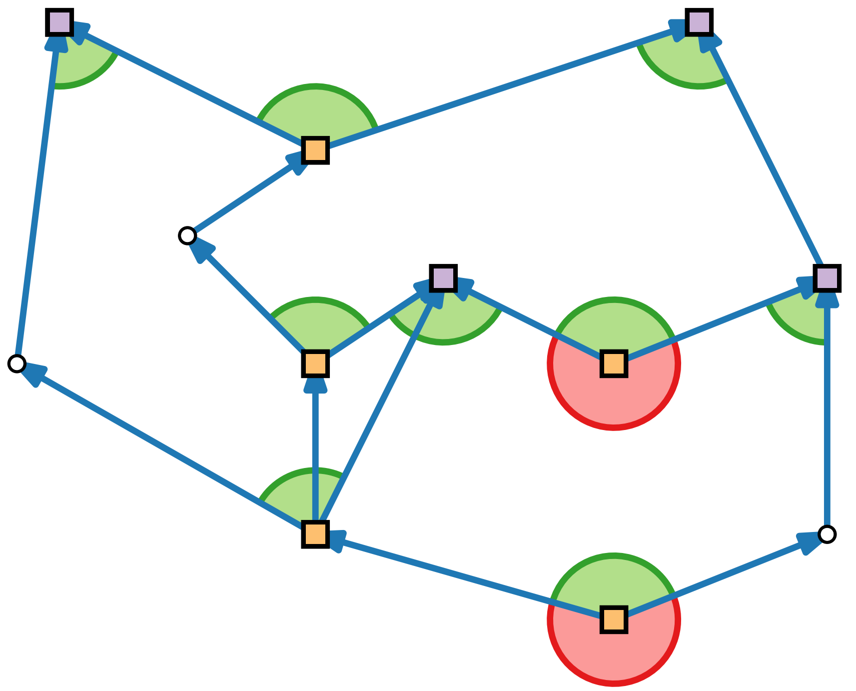
- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt a face f if v has two incoming edges on ∂f.
- An angle α at a local source / sink is **large** when $\alpha > \pi$ and **small** otherwise.



Angles, Local Sources & Sinks

Definitions.

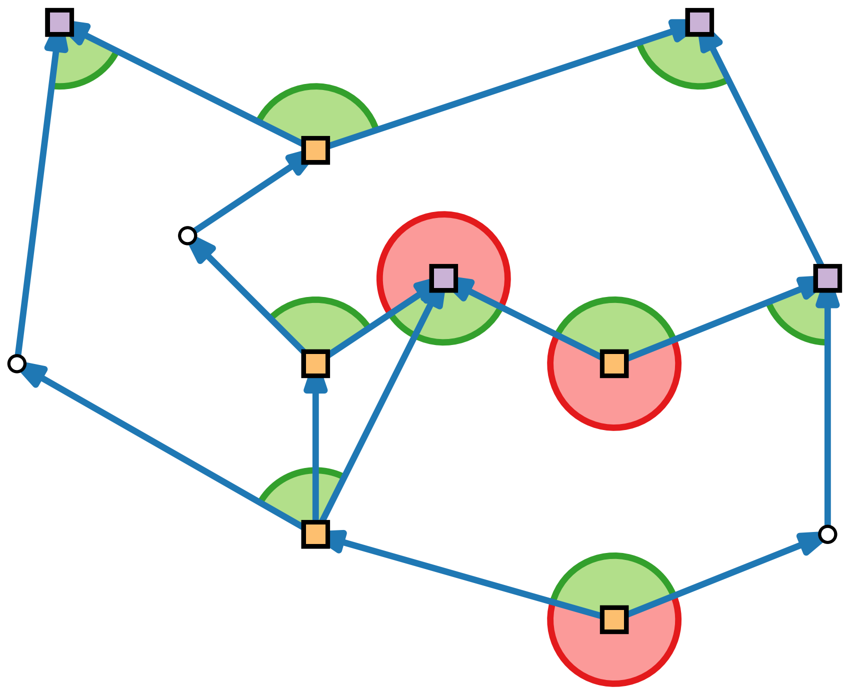
- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt a face f if v has two incoming edges on ∂f.
- An angle α at a local source / sink is **large** when $\alpha > \pi$ and **small** otherwise.



Angles, Local Sources & Sinks

Definitions.

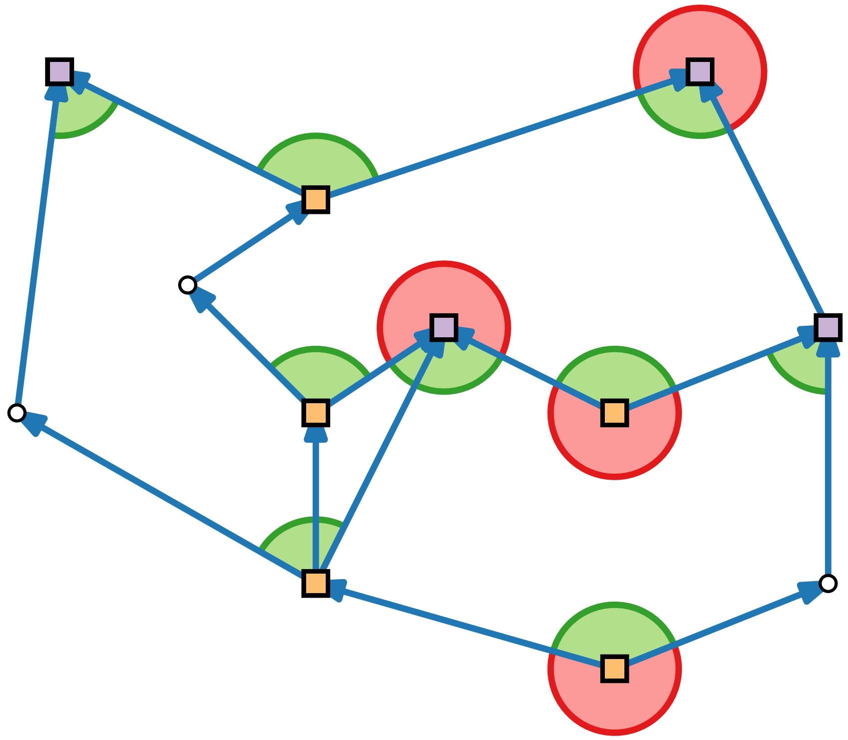
- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt a face f if v has two incoming edges on ∂f.
- An angle α at a local source / sink is **large** when $\alpha > \pi$ and **small** otherwise.



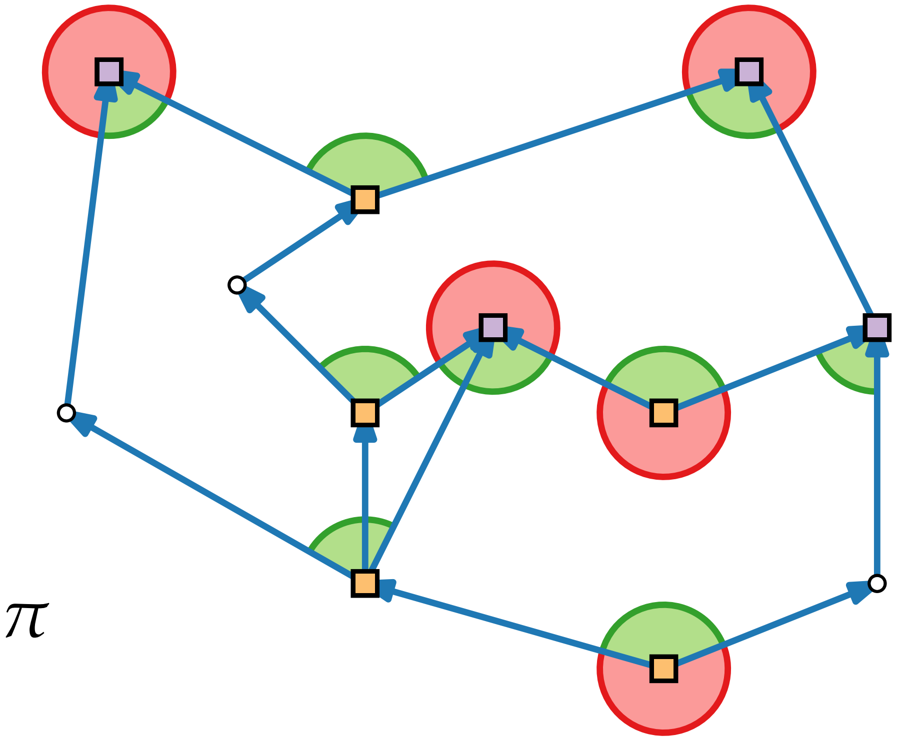
Angles, Local Sources & Sinks

Definitions.

- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt a face f if v has two incoming edges on ∂f.
- An angle α at a local source / sink is **large** when $\alpha > \pi$ and **small** otherwise.



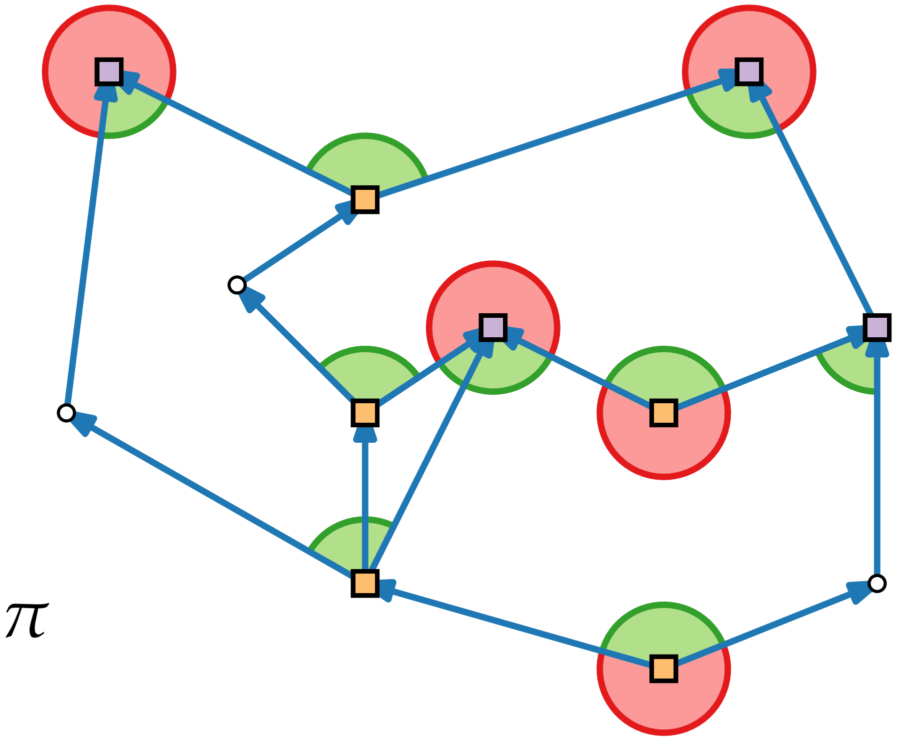
Angles, Local Sources & Sinks



Definitions.

- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt a face f if v has two incoming edges on ∂f.
- An angle α at a local source / sink is **large** when $\alpha > \pi$ and **small** otherwise.

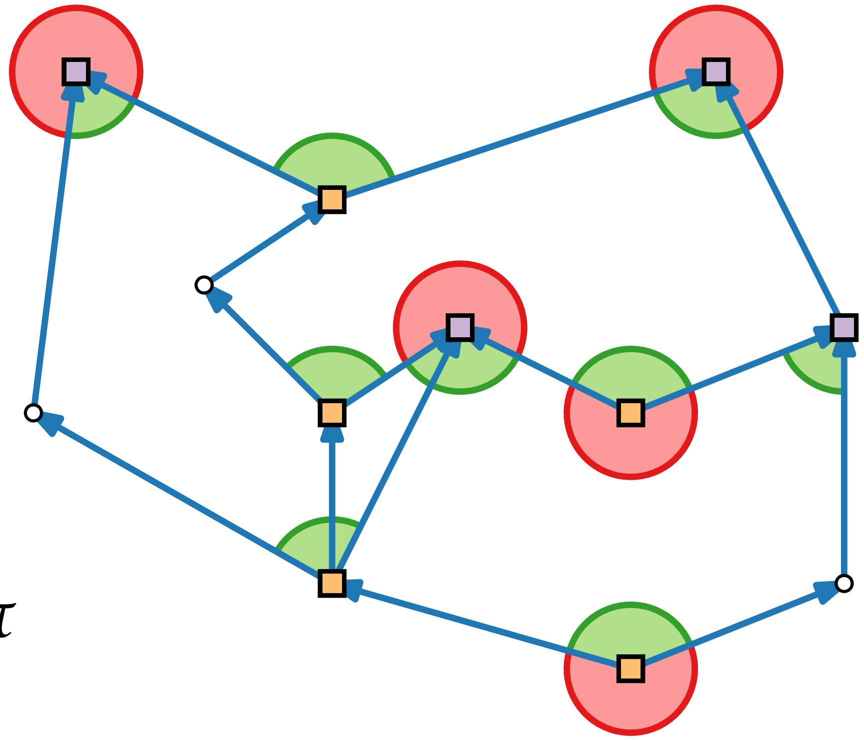
Angles, Local Sources & Sinks



Definitions.

- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt a face f if v has two incoming edges on ∂f.
- An angle α at a local source / sink is **large** when $\alpha > \pi$ and **small** otherwise.
- $L(v)$ = # large angles at v

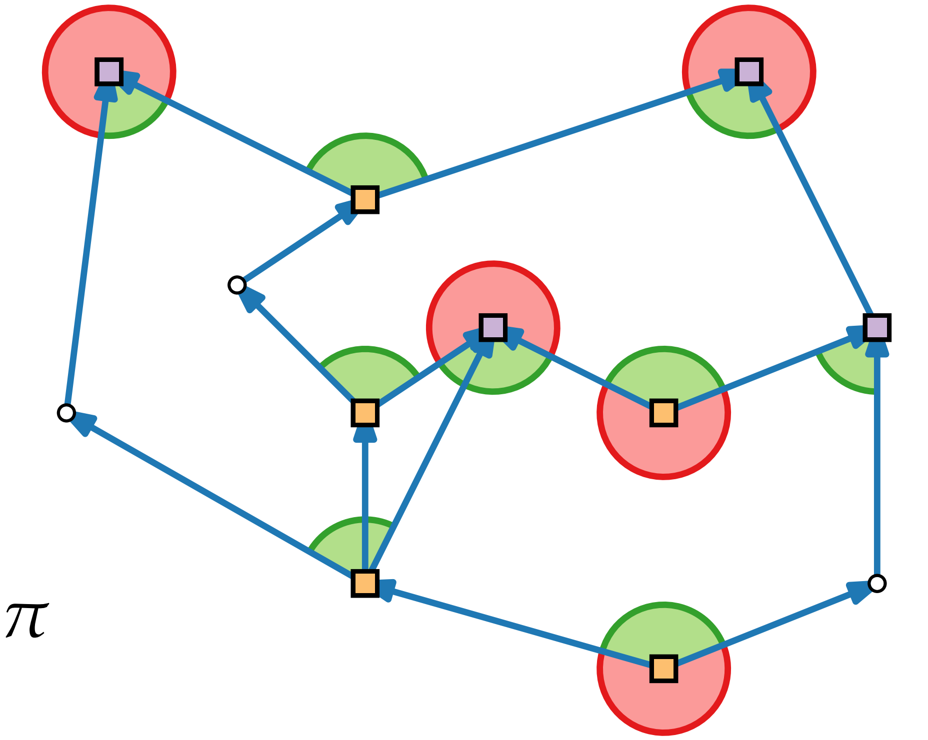
Angles, Local Sources & Sinks



Definitions.

- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt a face f if v has two incoming edges on ∂f.
- An angle α at a local source / sink is **large** when $\alpha > \pi$ and **small** otherwise.
- $L(v)$ = # large angles at v
- $L(f)$ = # large angles in f

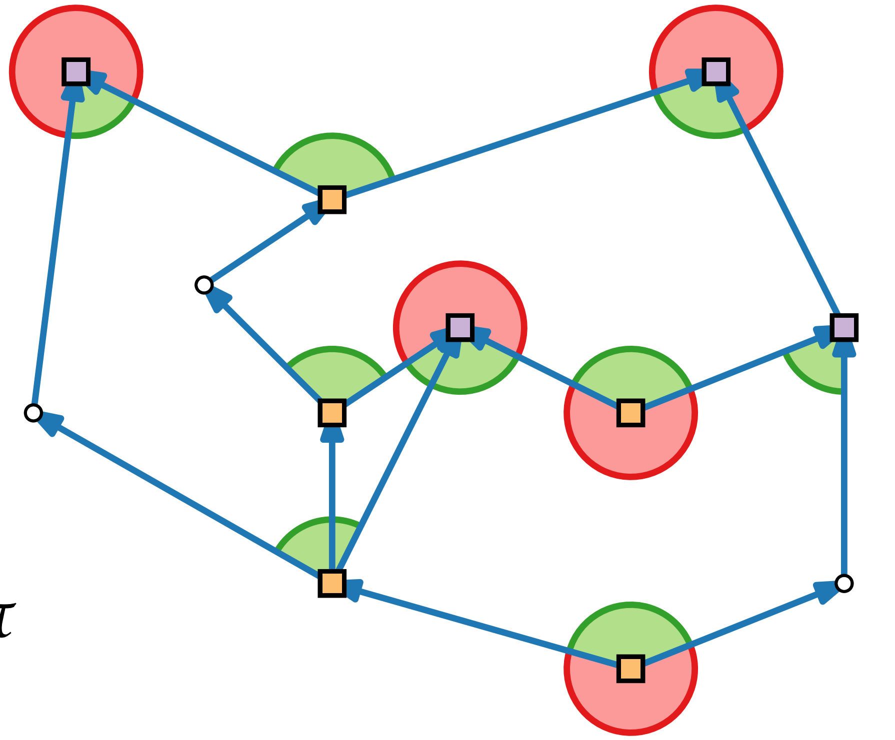
Angles, Local Sources & Sinks



Definitions.

- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt a face f if v has two incoming edges on ∂f.
- An angle α at a local source / sink is **large** when $\alpha > \pi$ and **small** otherwise.
- $L(v)$ = # large angles at v
- $L(f)$ = # large angles in f
- $S(v)$ & $S(f)$ for # small angles

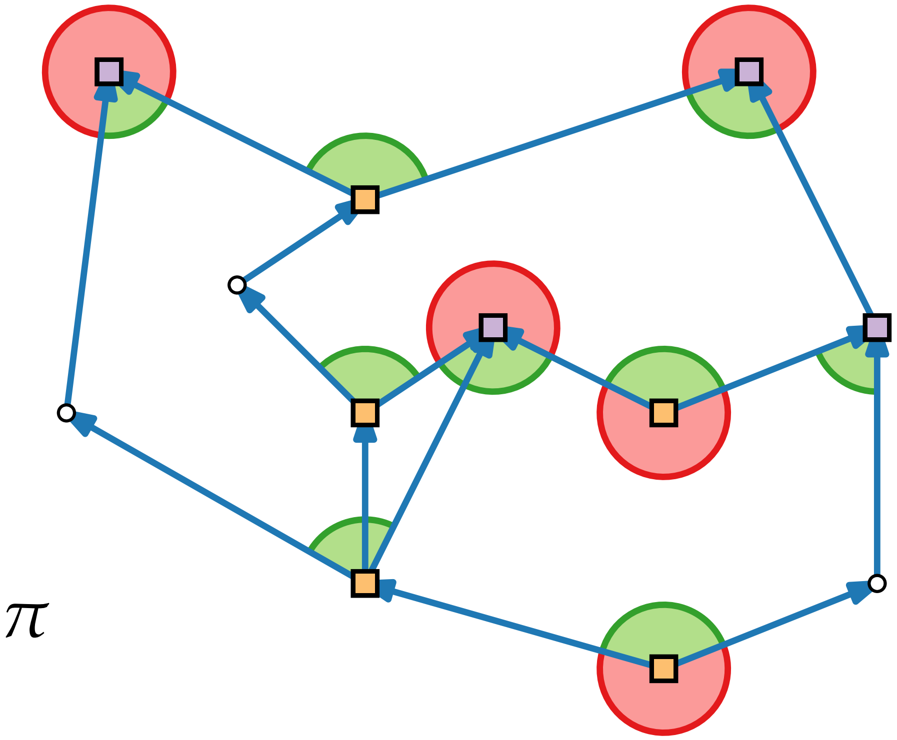
Angles, Local Sources & Sinks



Definitions.

- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt a face f if v has two incoming edges on ∂f.
- An angle α at a local source / sink is **large** when $\alpha > \pi$ and **small** otherwise.
- $L(v)$ = # large angles at v
- $L(f)$ = # large angles in f
- $S(v)$ & $S(f)$ for # small angles
- $A(f)$ = # local sources wrt f

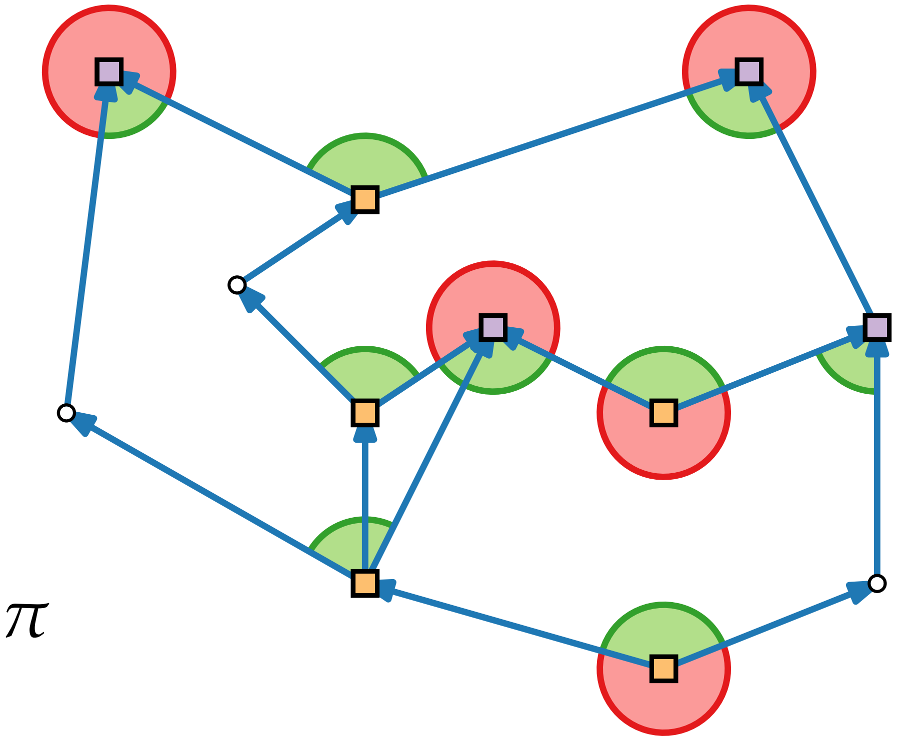
Angles, Local Sources & Sinks



Definitions.

- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt a face f if v has two incoming edges on ∂f.
- An angle α at a local source / sink is **large** when $\alpha > \pi$ and **small** otherwise.
- $L(v)$ = # large angles at v
- $L(f)$ = # large angles in f
- $S(v)$ & $S(f)$ for # small angles
- $A(f)$ = # local sources wrt f
  = # local sinks wrt f

Angles, Local Sources & Sinks



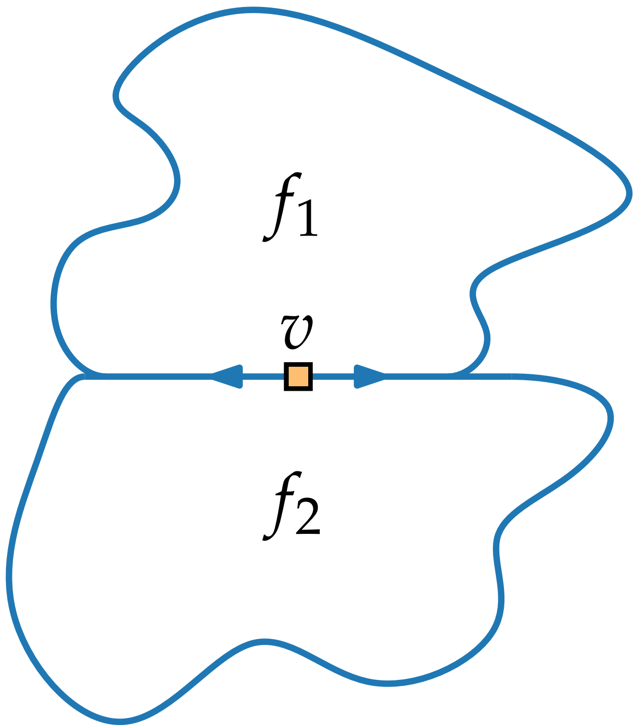
Definitions.

- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt a face f if v has two incoming edges on ∂f.
- An angle α at a local source / sink is **large** when $\alpha > \pi$ and **small** otherwise.
- $L(v)$ = # large angles at v
- $L(f)$ = # large angles in f
- $S(v)$ & $S(f)$ for # small angles
- $A(f)$ = # local sources wrt f
  = # local sinks wrt f

Lemma 1.
$L(f) + S(f) = 2A(f)$

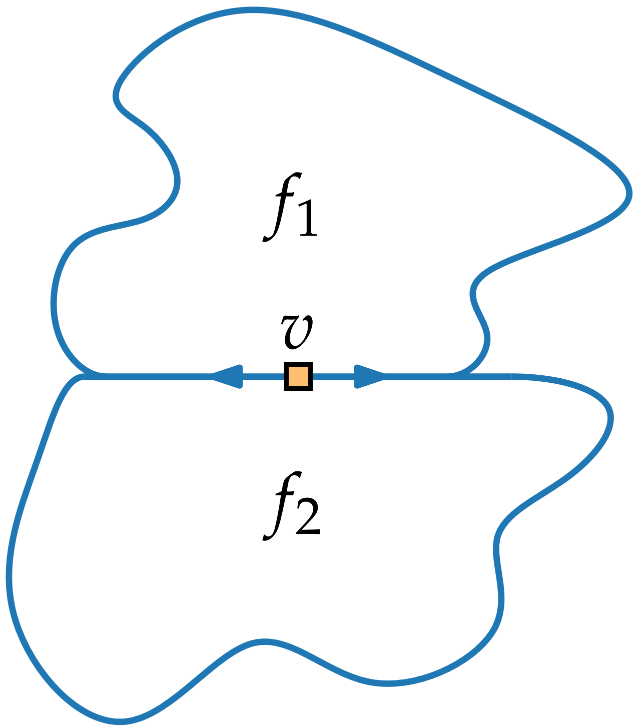
Assignment Problem

- Vertex v is a global source.



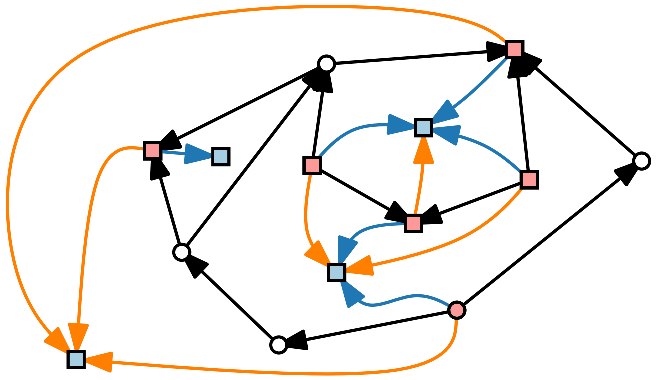
Assignment Problem

- Vertex v is a global source.
- At which face does v have a **large** angle?

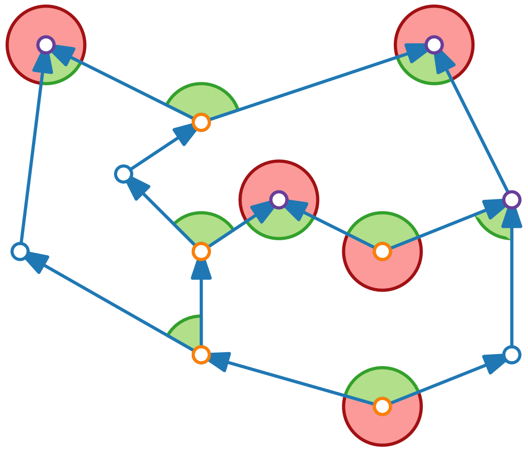


Universität Trier

Visualization of Graphs



Lecture 7:
Upward Planar Drawings



Part III:
Angle Relations

Philipp Kindermann

Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

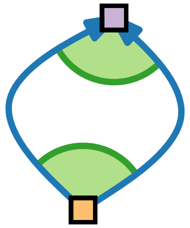
Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

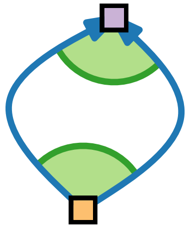


Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

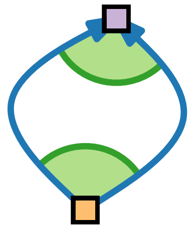
- $L(f) = 0$  $\Rightarrow S(f) = 2$

Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

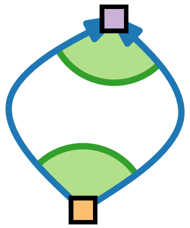
- $L(f) = 0$  $\Rightarrow S(f) = 2$
- $L(f) \geq 1$

Angle Relations

Lemma 2.
$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$



$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

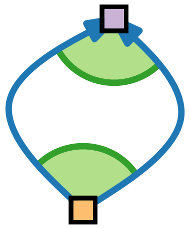
Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

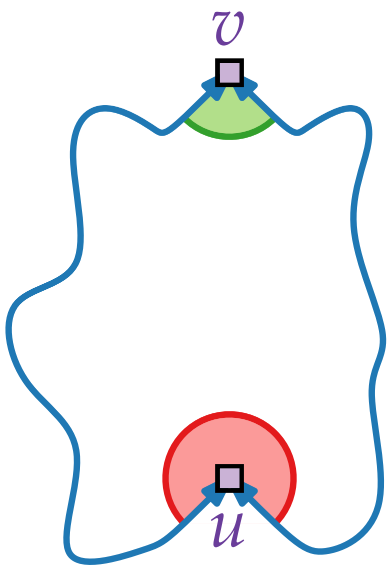


$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- sink v with small angle:

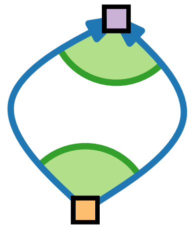


Angle Relations

Lemma 2.

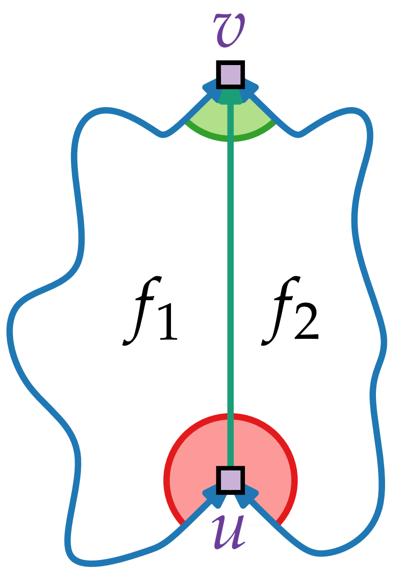
$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$  $\Rightarrow S(f) = 2$
- $L(f) \geq 1$

Split f with edge from a large angle at a “low” sink u to

- sink v with small angle:



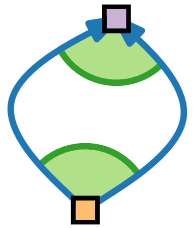
Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

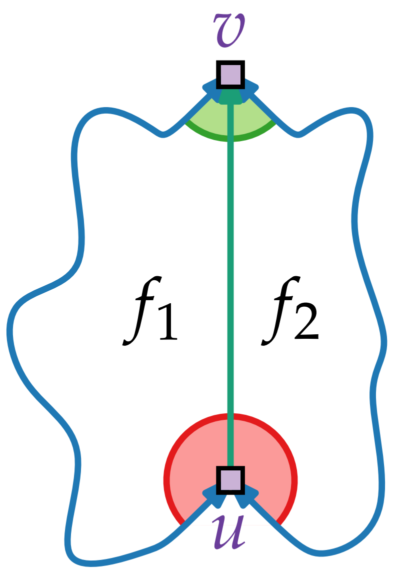


$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- sink v with small angle:



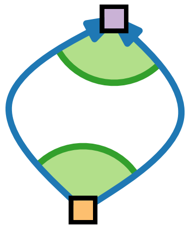
$L(f) - S(f)$

Angle Relations

Lemma 2.
$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

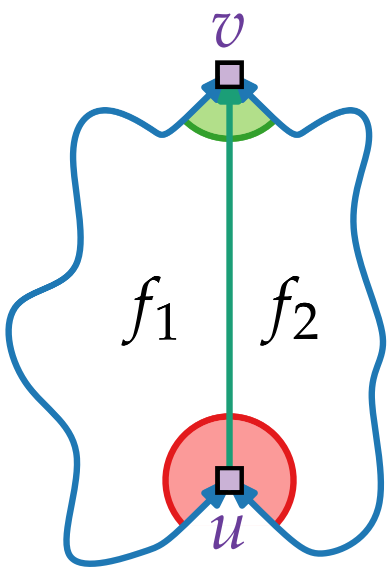


$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- sink v with small angle:



$$L(f) - S(f) = L(f_1) + L(f_2) + 1$$

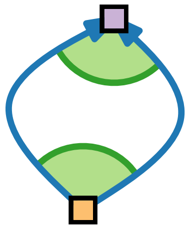
Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

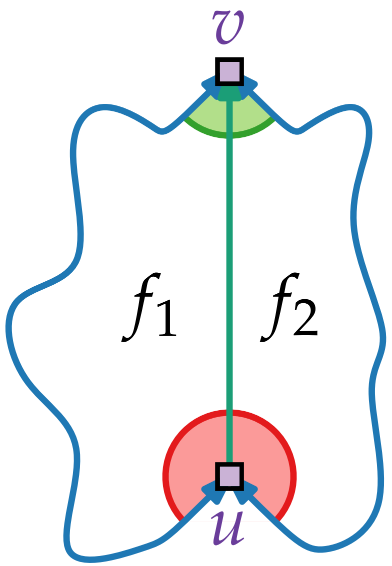


$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- sink v with small angle:



$$L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1)$$

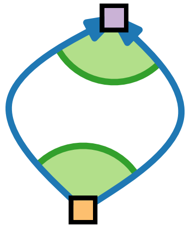
Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

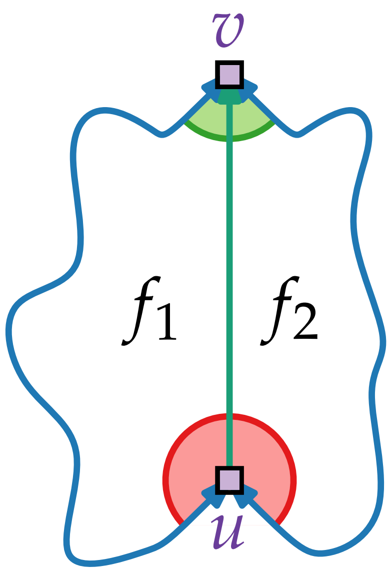


$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- sink v with small angle:



-2 -2

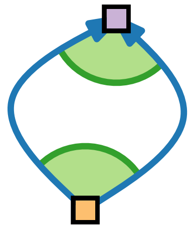
$$L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1)$$

Angle Relations

Lemma 2.
$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

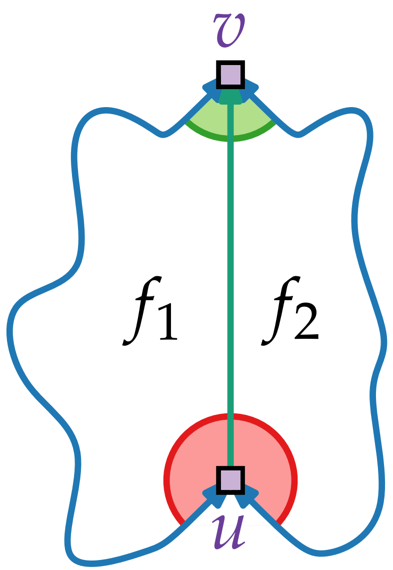


$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- sink v with small angle:



$$\begin{aligned} L(f) - S(f) &= \underbrace{L(f_1)}_{-2} + \underbrace{L(f_2)}_{-2} + 1 \\ &\quad - (S(f_1) + S(f_2) - 1) \\ &= -2 \end{aligned}$$

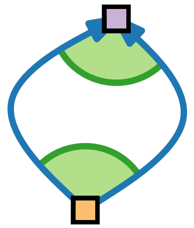
Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

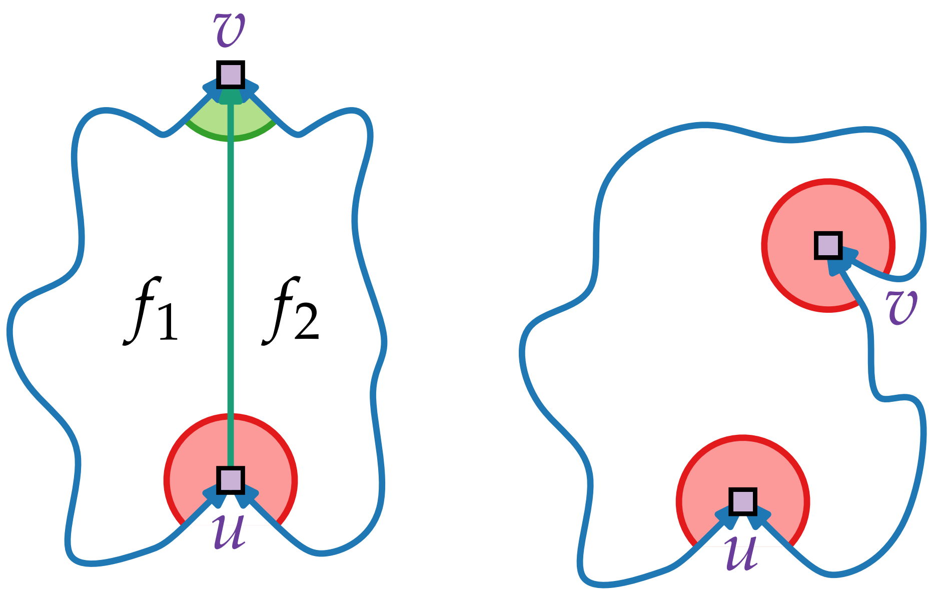


$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- sink v with small/large angle:



$$L(f) - S(f) = \underbrace{L(f_1)}_{-2} + \underbrace{L(f_2)}_{-2} + 1 - (S(f_1) + S(f_2) - 1) = -2$$

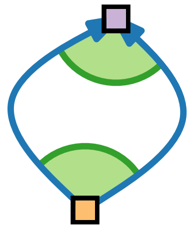
Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

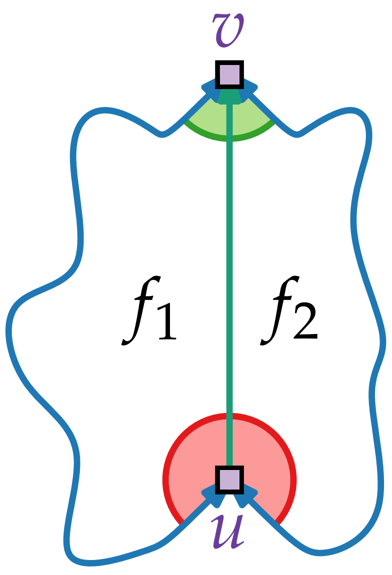


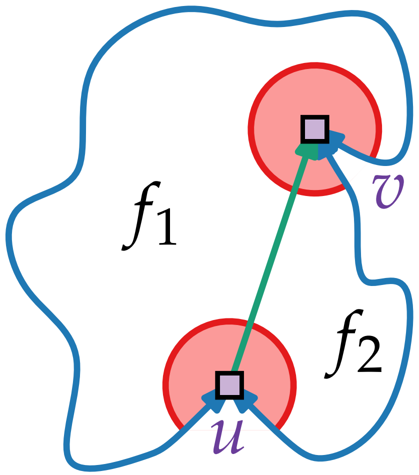
$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- sink v with small/large angle:





-2 -2

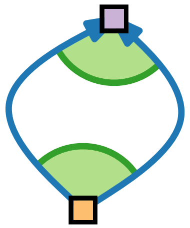
$$\begin{aligned} L(f) - S(f) &= L(f_1) + L(f_2) + 1 \\ &\quad - (S(f_1) + S(f_2) - 1) \\ &= -2 \end{aligned}$$

Angle Relations

Lemma 2.

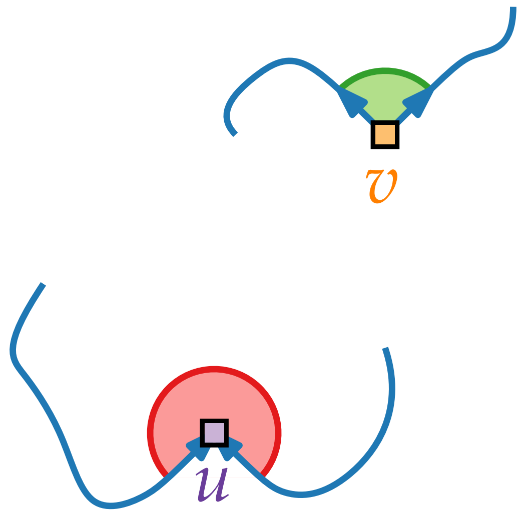
$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$  $\Rightarrow S(f) = 2$
- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

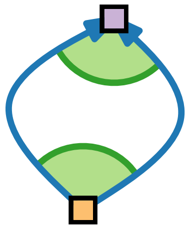
- source v with small angle:



Angle Relations

Lemma 2.
$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

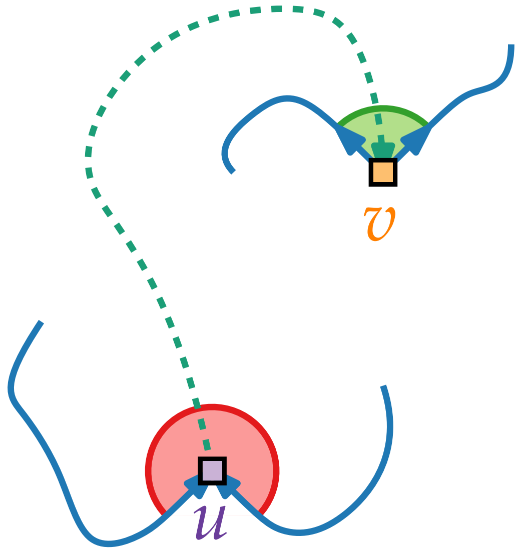
Proof by induction.

- $L(f) = 0$  $\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- source v with small angle:

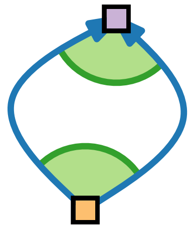


Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

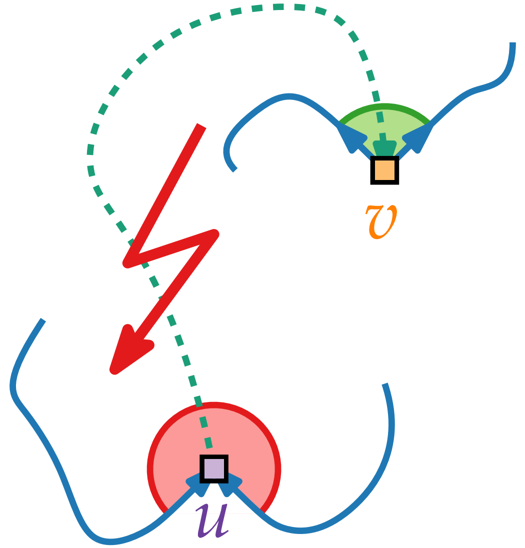
Proof by induction.

- $L(f) = 0$  $\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- source v with ~~small~~ angle:

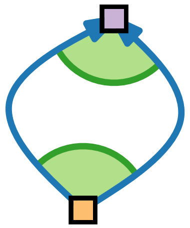


Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

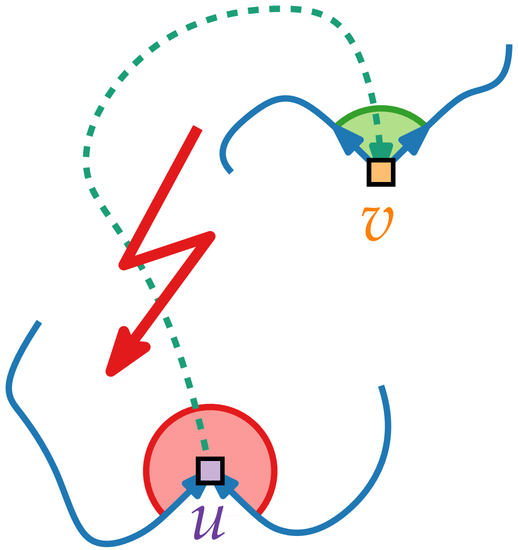
Proof by induction.

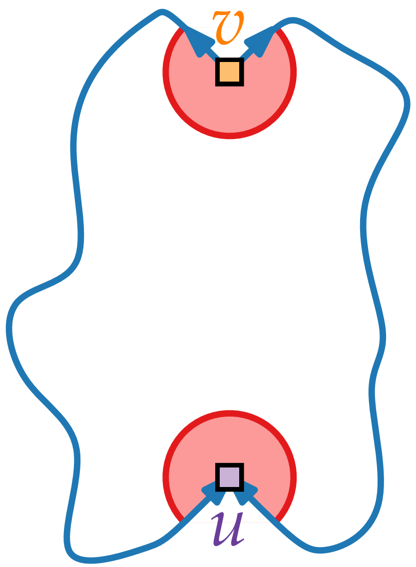
- $L(f) = 0$  $\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- source v with ~~small~~/large angle:





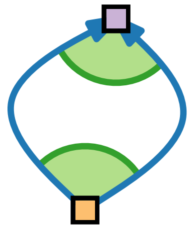
Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

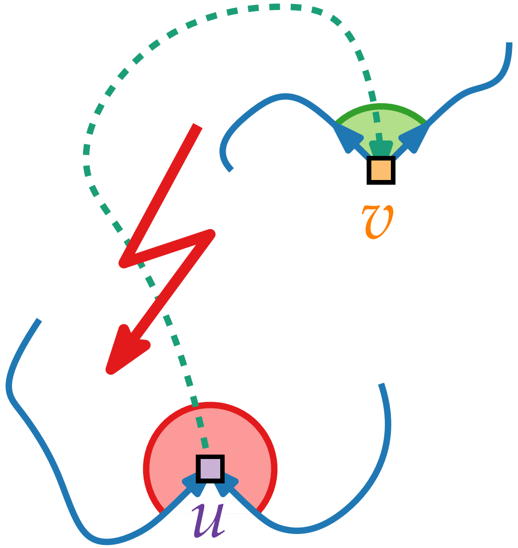


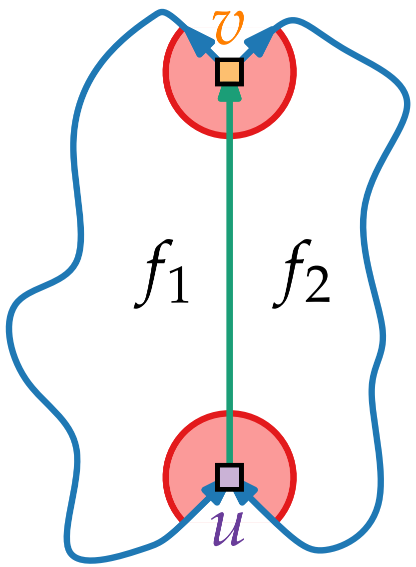
$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- source v with ~~small~~/large angle:



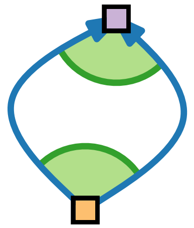


Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

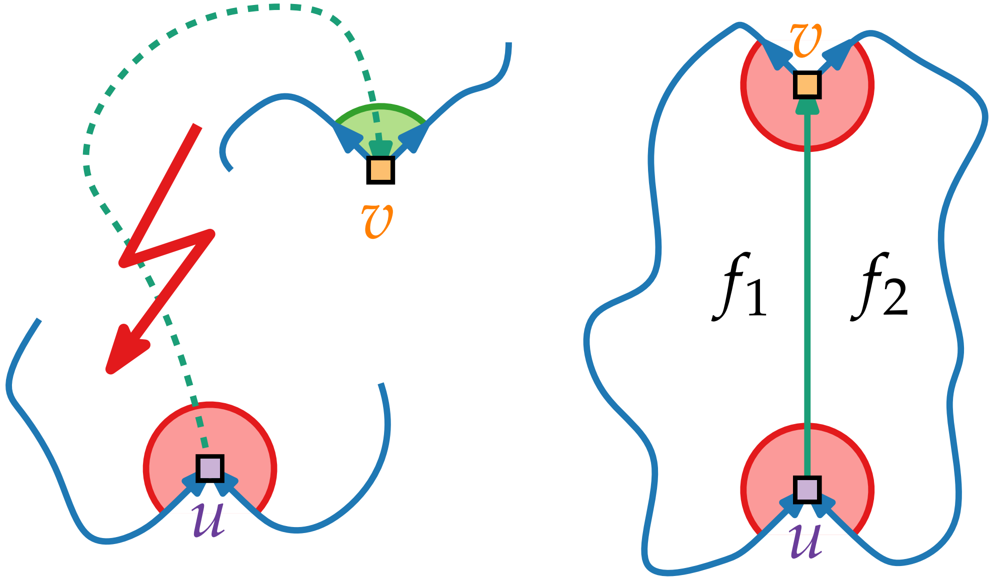
Proof by induction.

- $L(f) = 0$  $\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- source v with ~~small~~/large angle:



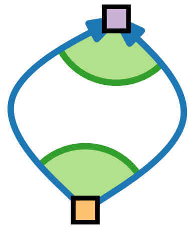
$L(f) - S(f)$

Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

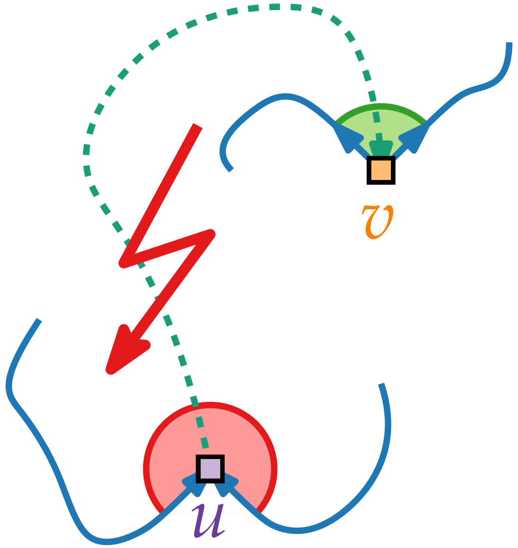
Proof by induction.

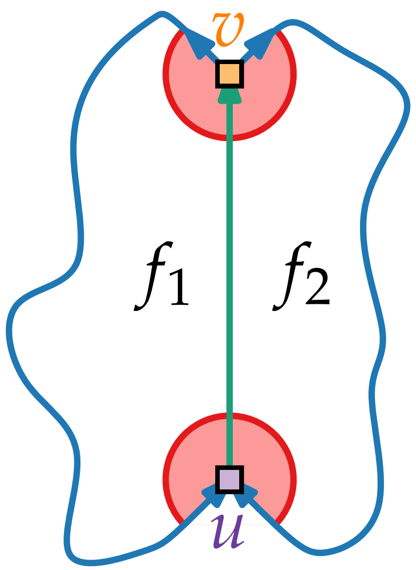
- $L(f) = 0$  $\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- source v with ~~small~~/large angle:





$$L(f) - S(f) = L(f_1) + L(f_2) + 2$$

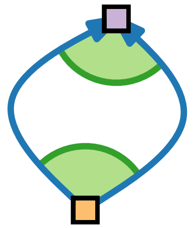
Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

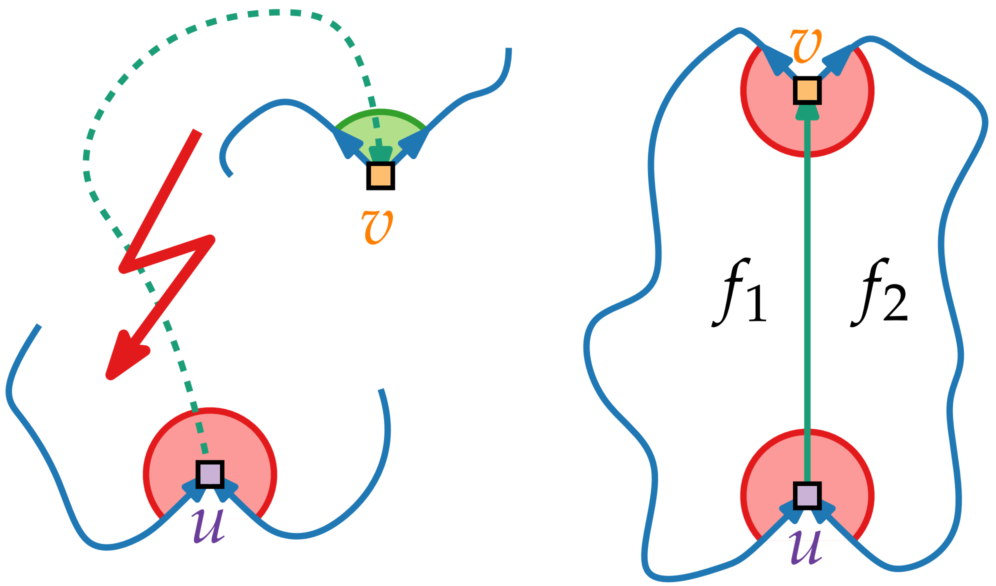


$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- source v with ~~small~~/large angle:



$$L(f) - S(f) = L(f_1) + L(f_2) + 2 - (S(f_1) + S(f_2))$$

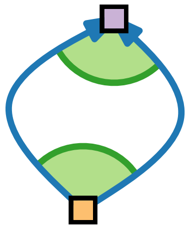
Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

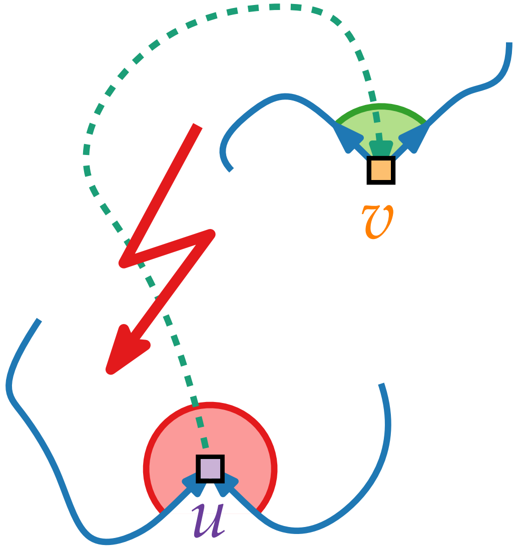


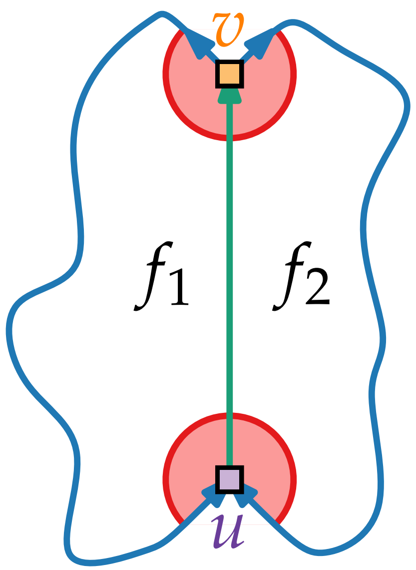
$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- source v with ~~small~~/large angle:





$$L(f) - S(f) = \underbrace{L(f_1)}_{} + L(f_2) + 2 - (S(f_1) + S(f_2))$$

-2 -2

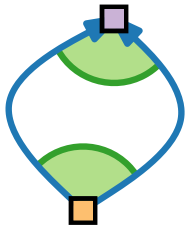
$$L(f) - S(f) = L(f_1) + L(f_2) + 2 - (S(f_1) + S(f_2))$$

Angle Relations

Lemma 2.

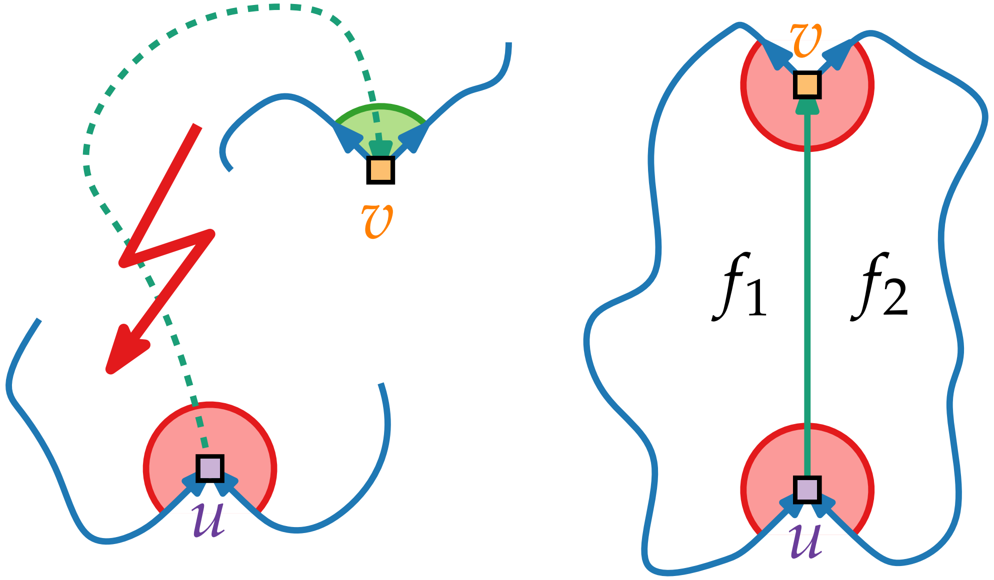
$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$  $\Rightarrow S(f) = 2$
- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- source v with ~~small~~/large angle:



$$L(f) - S(f) = L(f_1) + L(f_2) + 2 - (S(f_1) + S(f_2)) = -2$$

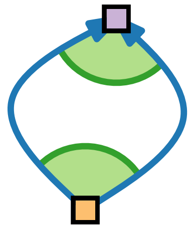
(with $L(f_1) - S(f_1)$ and $L(f_2) - S(f_2)$ each annotated -2)

Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

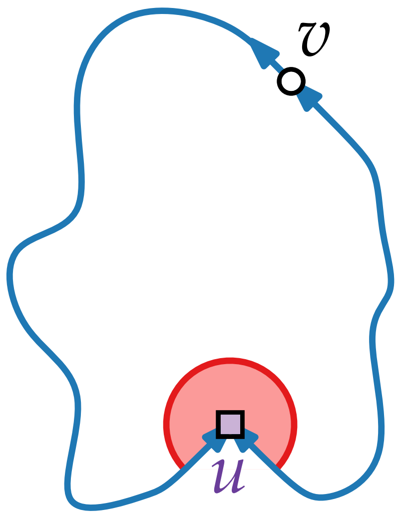
Proof by induction.

- $L(f) = 0$  $\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- vertex v that is neither source nor sink:



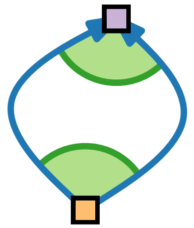
Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

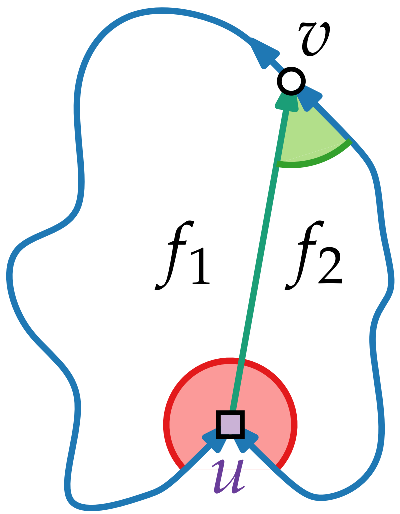
- $L(f) = 0$ $\Rightarrow S(f) = 2$



- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- vertex v that is neither source nor sink:



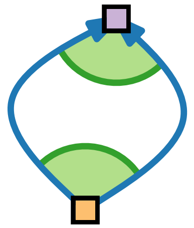
Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

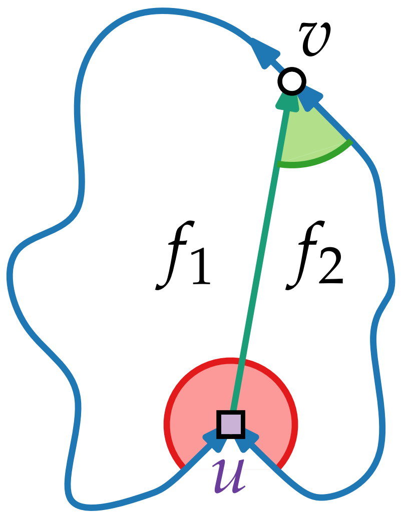


$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- vertex v that is neither source nor sink:



$L(f) - S(f)$

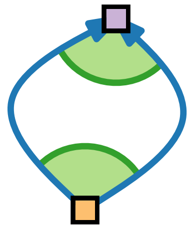
Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

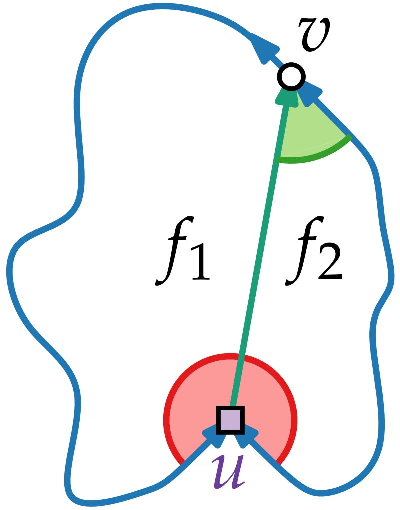


$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- vertex v that is neither source nor sink:



$$L(f) - S(f) = L(f_1) + L(f_2) + 1$$

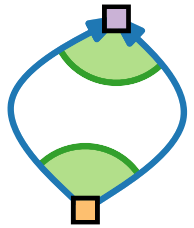
Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

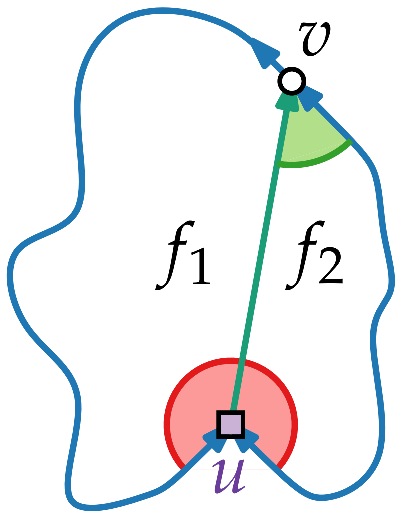


$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- vertex v that is neither source nor sink:



$$L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1)$$

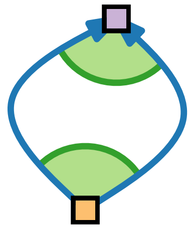
Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

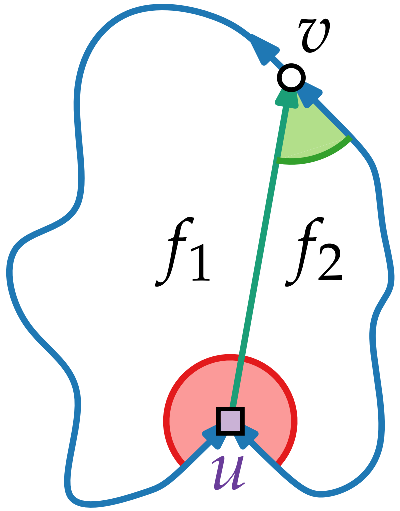


$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- vertex v that is neither source nor sink:



-2 -2

$$L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1)$$

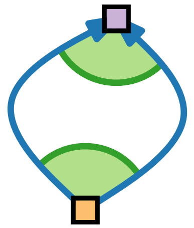
Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

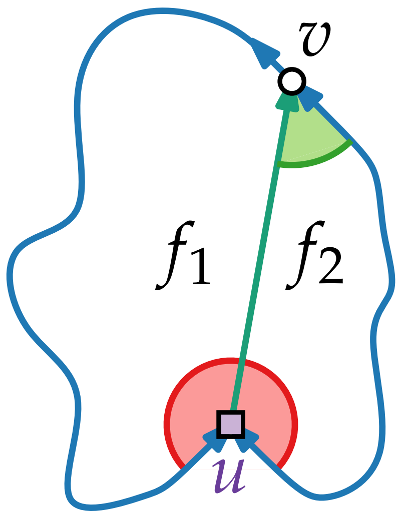
- $L(f) = 0$ $\Rightarrow S(f) = 2$



- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- vertex v that is neither source nor sink:



$$\begin{aligned} L(f) - S(f) &= L(f_1) + L(f_2) + 1 \\ &\quad - (S(f_1) + S(f_2) - 1) \\ &= -2 \end{aligned}$$

(-2 above $L(f_1) - S(f_1)$; -2 above $L(f_2) - S(f_2)$)

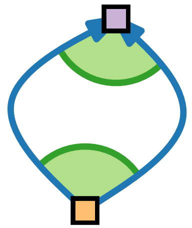
Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

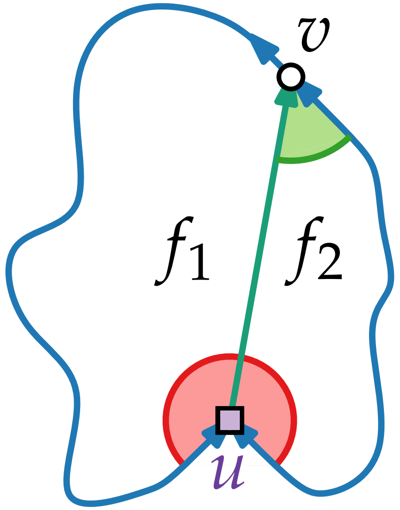


$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- vertex v that is neither source nor sink:



$$L(f) - S(f) = \underbrace{L(f_1)}_{-2} + \underbrace{L(f_2)}_{-2} + 1 - (S(f_1) + S(f_2) - 1) = -2$$

- Otherwise "high" source u exists.

Number of Large Angles

Lemma 3.
In every upward planar drawing of G holds that

Number of Large Angles

Lemma 3.
In every upward planar drawing of G holds that

- for each vertex $v \in V$: $L(v) =$

Number of Large Angles

Lemma 3.
In every upward planar drawing of G holds that

- for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex,} \\ \end{cases}$

Number of Large Angles

Lemma 3.
In every upward planar drawing of G holds that

- for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex,} \\ 1 & v \text{ source / sink;} \end{cases}$

Number of Large Angles

Lemma 3.
In every upward planar drawing of G holds that

- for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex,} \\ 1 & v \text{ source / sink;} \end{cases}$

- for each face f: $L(f) =$

Number of Large Angles

Lemma 3.
In every upward planar drawing of G holds that

- for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex,} \\ 1 & v \text{ source / sink;} \end{cases}$
- for each face f: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ & \end{cases}$

Number of Large Angles

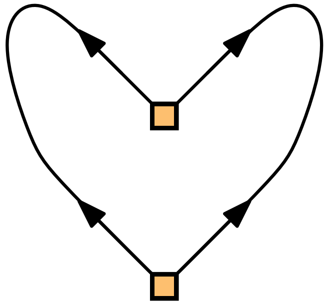
Lemma 3.
In every upward planar drawing of G holds that

- for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex,} \\ 1 & v \text{ source / sink;} \end{cases}$
- for each face f: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$

Number of Large Angles

Lemma 3.
In every upward planar drawing of G holds that

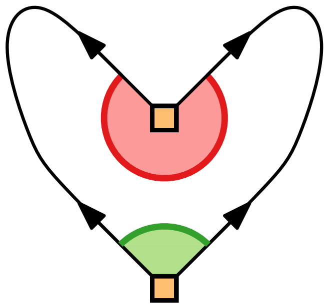
- for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex,} \\ 1 & v \text{ source / sink;} \end{cases}$
- for each face f: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$



Number of Large Angles

Lemma 3.
In every upward planar drawing of G holds that

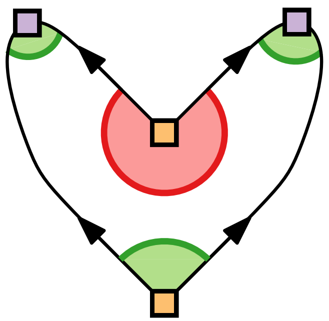
- for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex,} \\ 1 & v \text{ source / sink;} \end{cases}$
- for each face f: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$



Number of Large Angles

Lemma 3.
In every upward planar drawing of G holds that

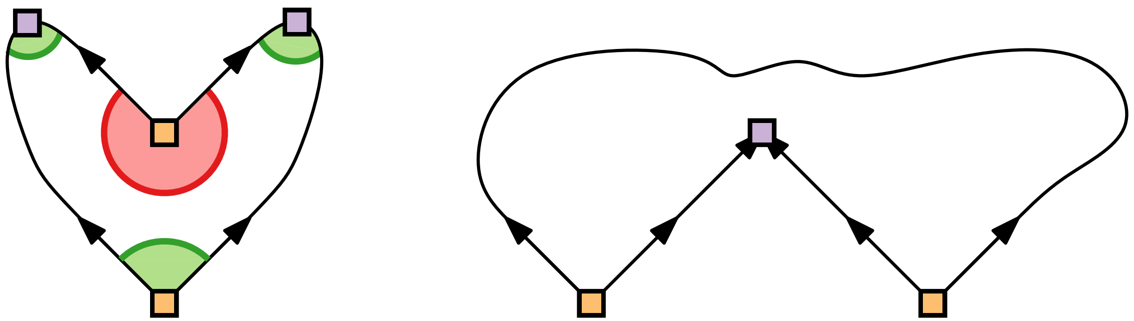
- for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex,} \\ 1 & v \text{ source / sink;} \end{cases}$
- for each face f: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$



Number of Large Angles

Lemma 3.
In every upward planar drawing of G holds that

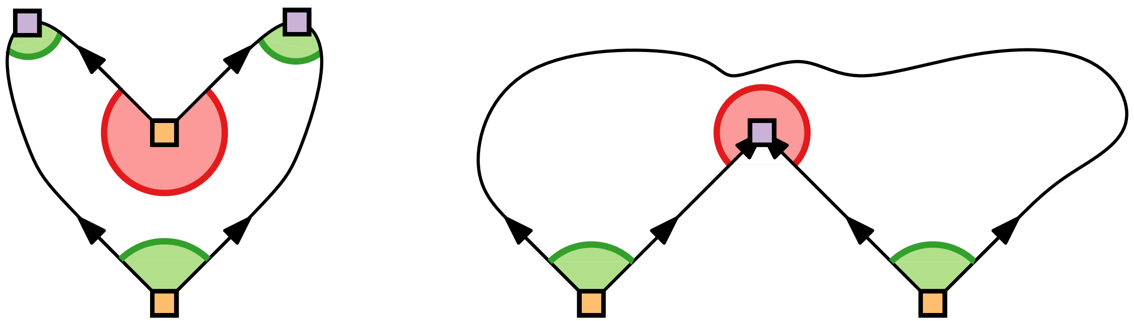
- for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex,} \\ 1 & v \text{ source / sink;} \end{cases}$
- for each face f: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$



Number of Large Angles

Lemma 3.
In every upward planar drawing of G holds that

- for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex,} \\ 1 & v \text{ source / sink;} \end{cases}$
- for each face f: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$

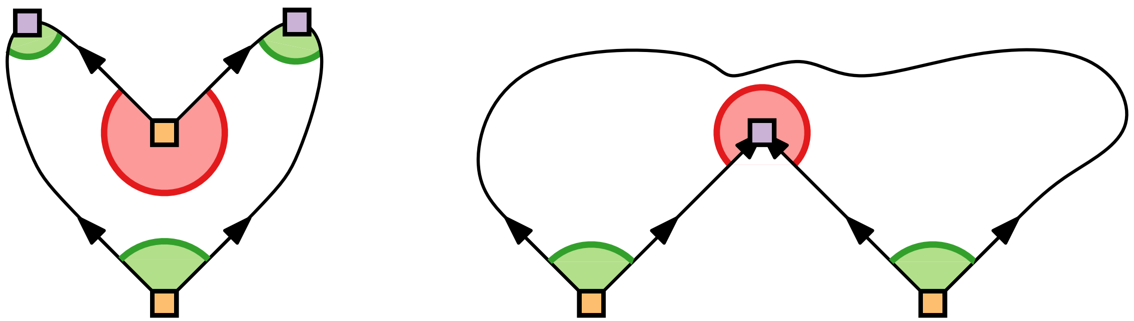


Number of Large Angles

Lemma 3.
In every upward planar drawing of G holds that

- for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex,} \\ 1 & v \text{ source / sink;} \end{cases}$
- for each face f: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$

Proof.

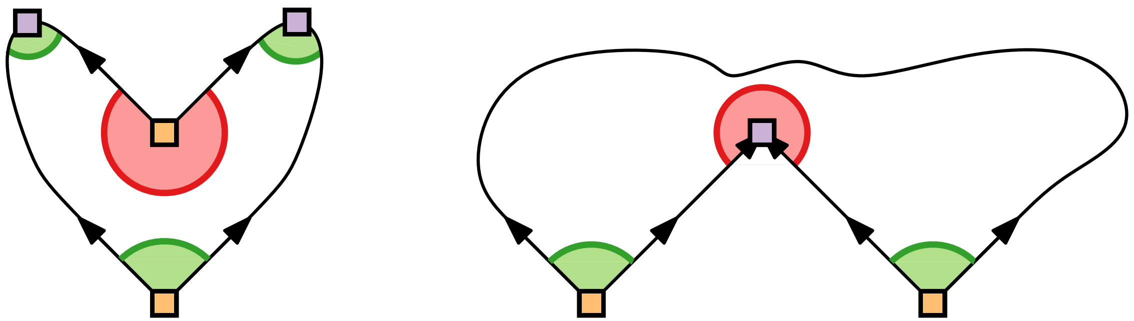


Number of Large Angles

Lemma 3.
In every upward planar drawing of G holds that

- for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex}, \\ 1 & v \text{ source / sink}; \end{cases}$
- for each face f: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$

Proof. Lemma 1: $L(f) + S(f) = 2A(f)$

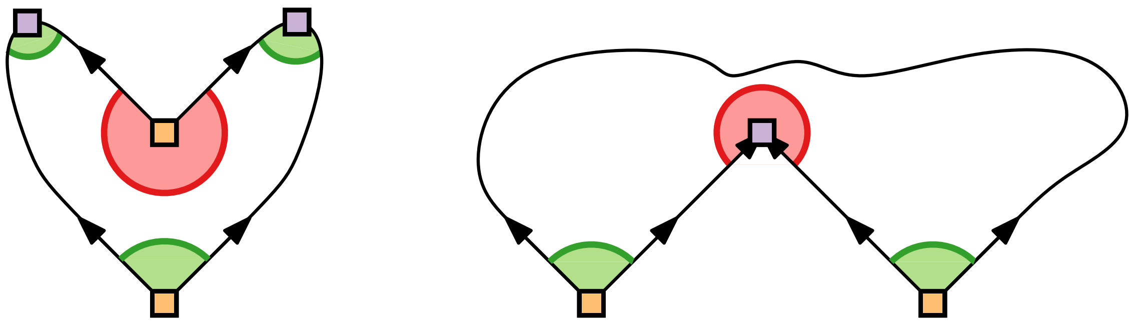


Number of Large Angles

Lemma 3.
In every upward planar drawing of G holds that

- for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex,} \\ 1 & v \text{ source / sink;} \end{cases}$
- for each face f: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$

Proof. Lemma 1: $L(f) + S(f) = 2A(f)$
Lemma 2: $L(f) - S(f) = \pm 2$.

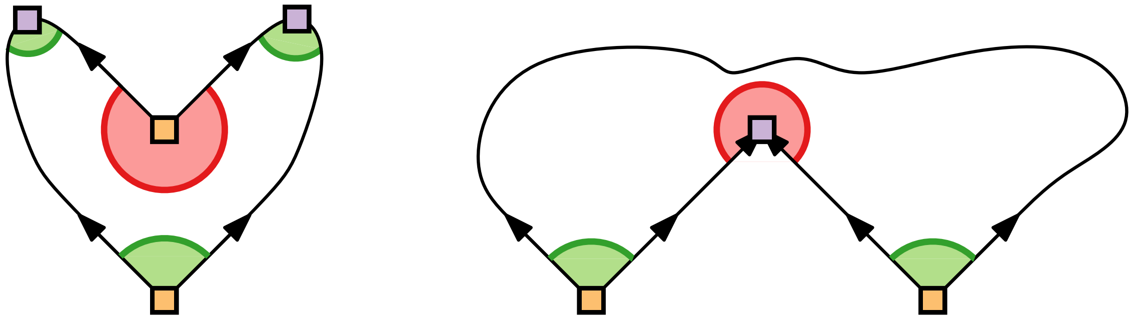


Number of Large Angles

Lemma 3.
In every upward planar drawing of G holds that

- for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex,} \\ 1 & v \text{ source / sink;} \end{cases}$
- for each face f: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$

Proof. Lemma 1: $L(f) + S(f) = 2A(f)$
Lemma 2: $L(f) - S(f) = \pm 2.$
$\Rightarrow 2L(f) = 2A(f) \pm 2.$



Assignment of Large Angles to Faces

Let S and T be the sets of sources and sinks, respectively.

Assignment of Large Angles to Faces

Let S and T be the sets of sources and sinks, respectively.

Definition.
A **consistent assignment** $\Phi \colon S \cup T \to F$ is a mapping where

Assignment of Large Angles to Faces

Let S and T be the sets of sources and sinks, respectively.

Definition.
A **consistent assignment** $\Phi\colon S \cup T \to F$ is a mapping where

$$\Phi\colon v \mapsto \text{ incident face, where } v \text{ forms large angle}$$

such that

Assignment of Large Angles to Faces

Let S and T be the sets of sources and sinks, respectively.

Definition.
A **consistent assignment** $\Phi\colon S \cup T \to F$ is a mapping where

$$\Phi\colon v \mapsto \text{ incident face, where } v \text{ forms large angle}$$

such that

$$|\Phi^{-1}(f)| =$$

Assignment of Large Angles to Faces

Let S and T be the sets of sources and sinks, respectively.

Definition.
A **consistent assignment** $\Phi\colon S \cup T \to F$ is a mapping where

$$\Phi\colon v \mapsto \text{ incident face, where } v \text{ forms large angle}$$

such that

$$|\Phi^{-1}(f)| = L(f) =$$

Assignment of Large Angles to Faces

Let S and T be the sets of sources and sinks, respectively.

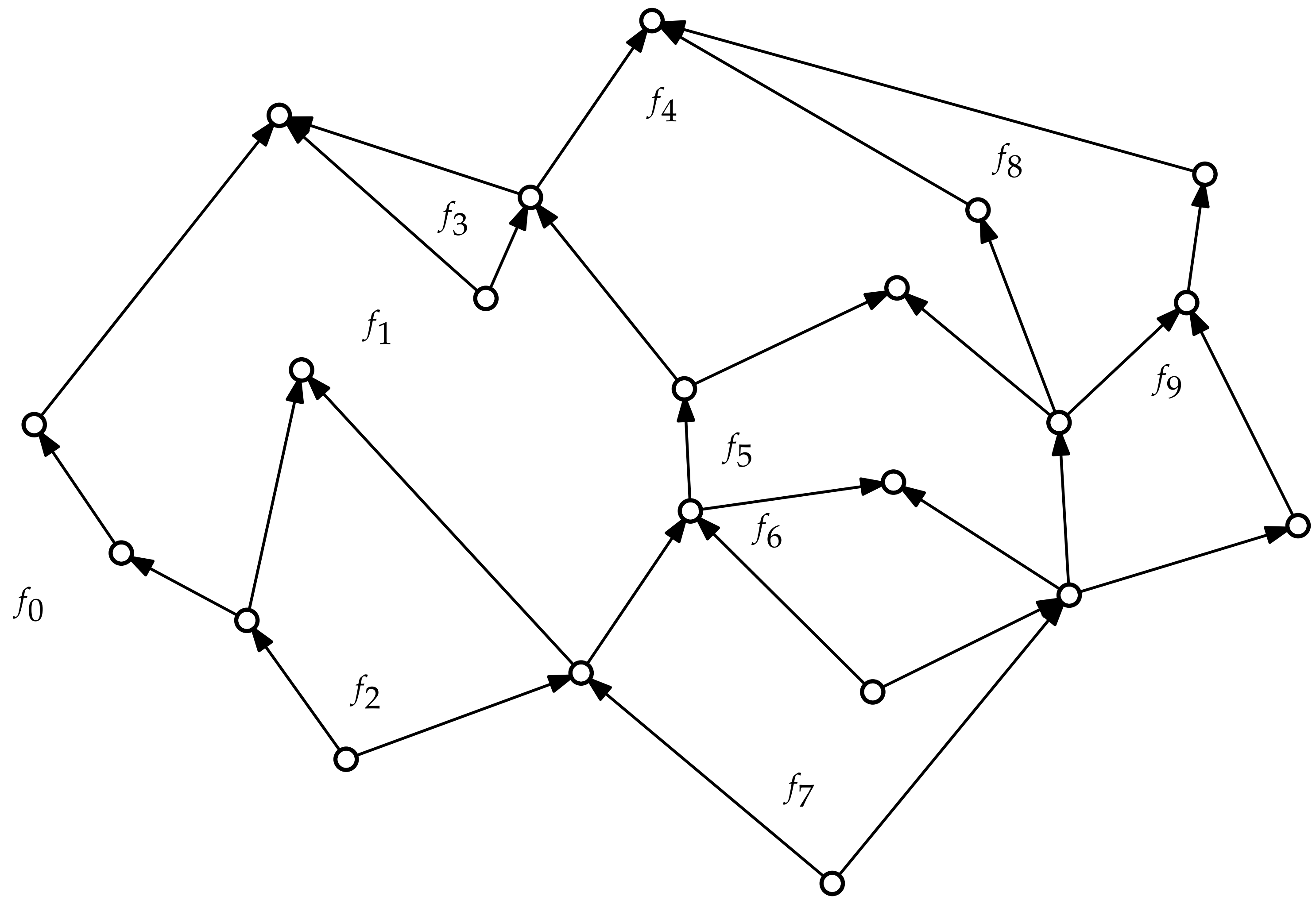
Definition.
A **consistent assignment** $\Phi\colon S \cup T \to F$ is a mapping where

$$\Phi\colon v \mapsto \text{ incident face, where } v \text{ forms large angle}$$

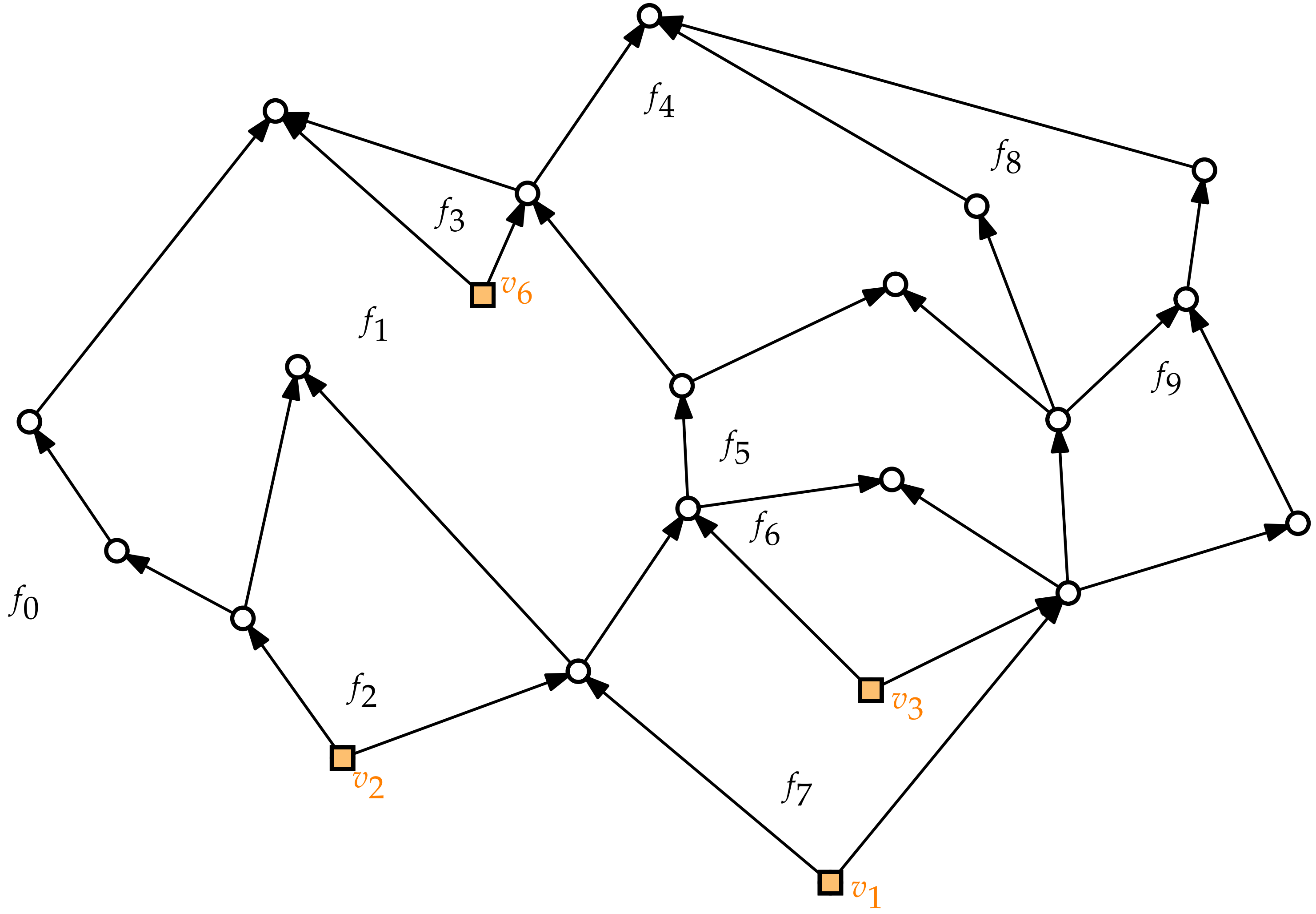
such that

$$|\Phi^{-1}(f)| = L(f) = \begin{cases} A(f) - 1 & \text{if } f \neq f_0, \\ A(f) + 1 & \text{if } f = f_0. \end{cases}$$

Example of Angle to Face Assignment

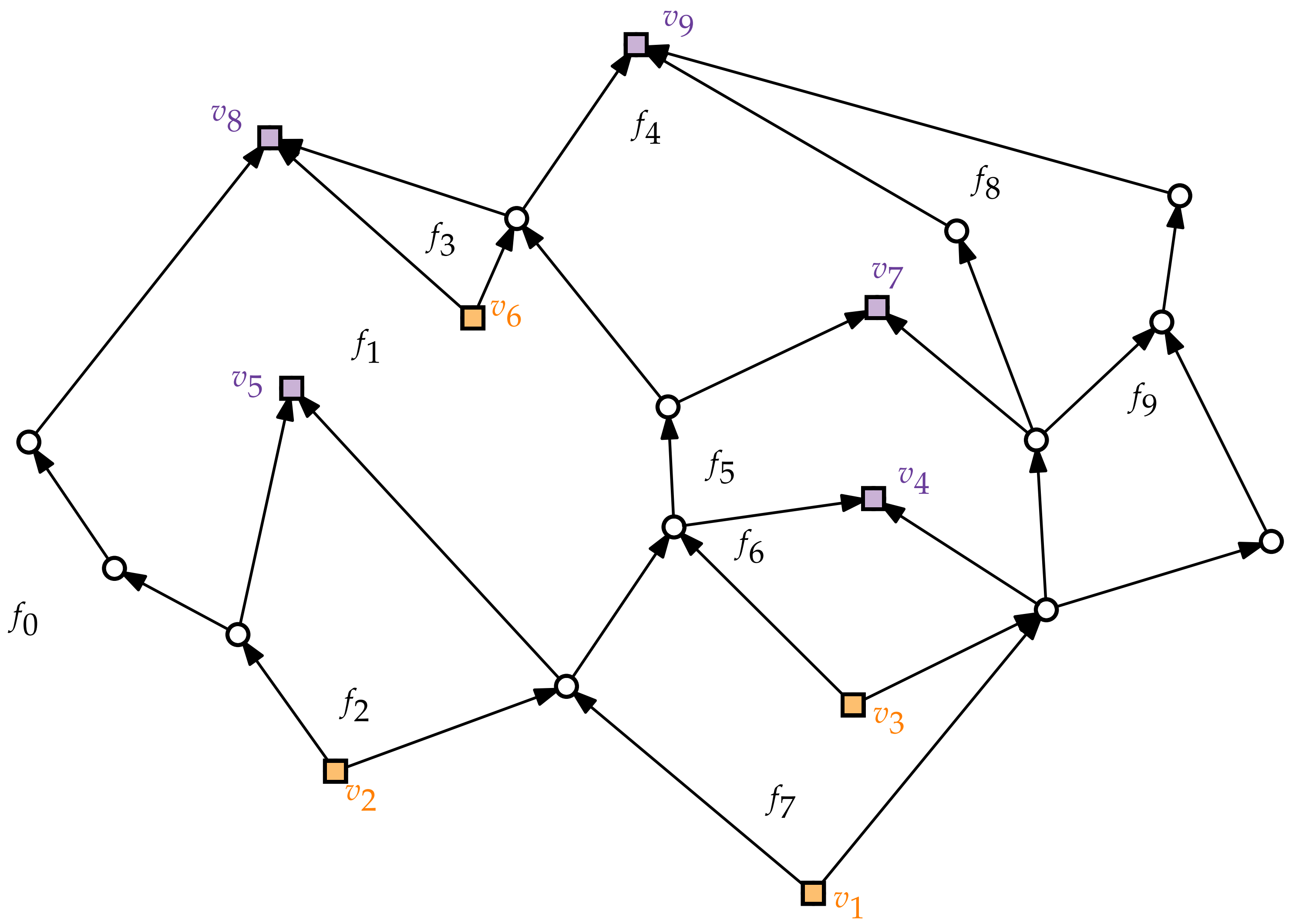


Example of Angle to Face Assignment



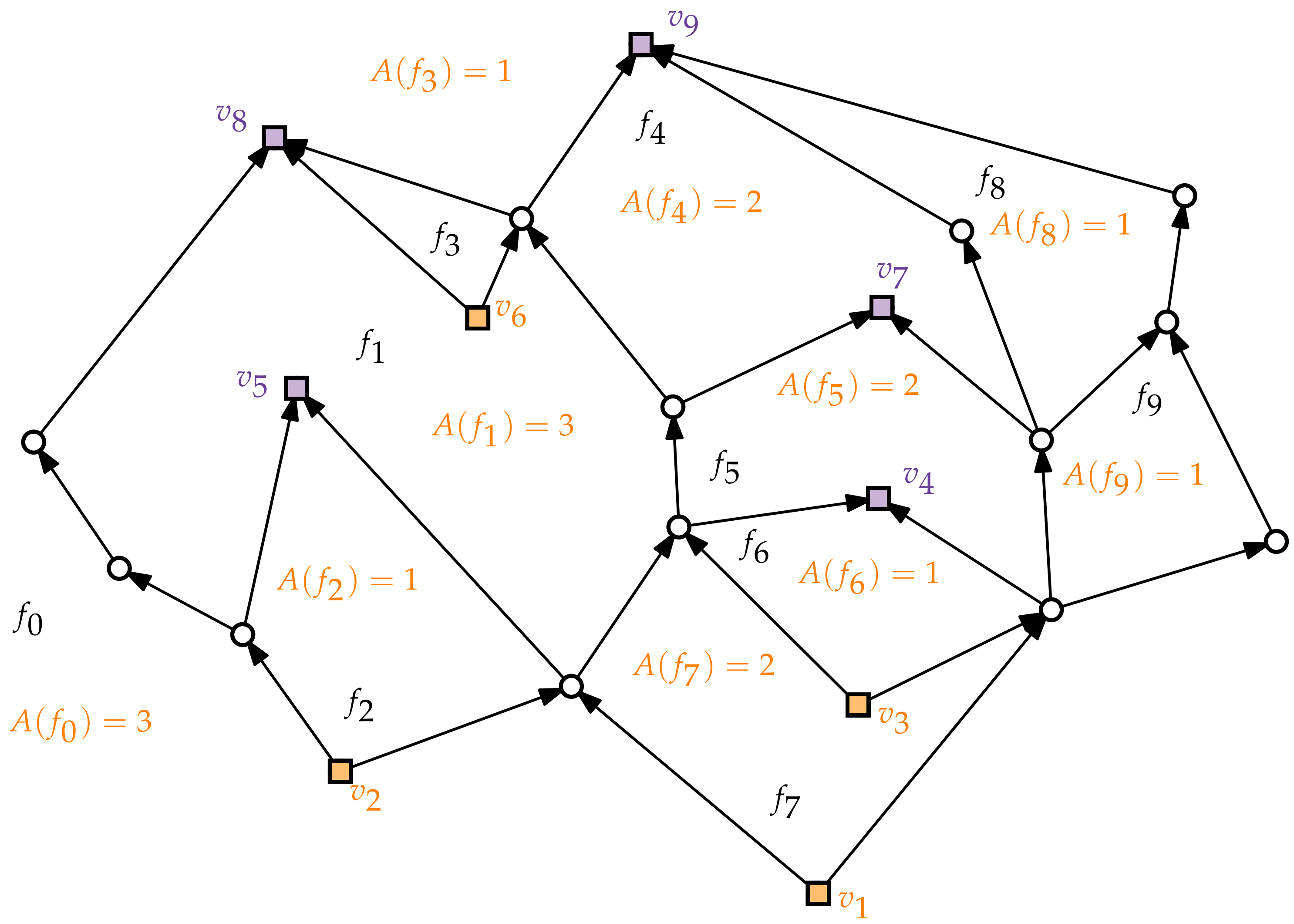
global sources

Example of Angle to Face Assignment



- global sources & sinks

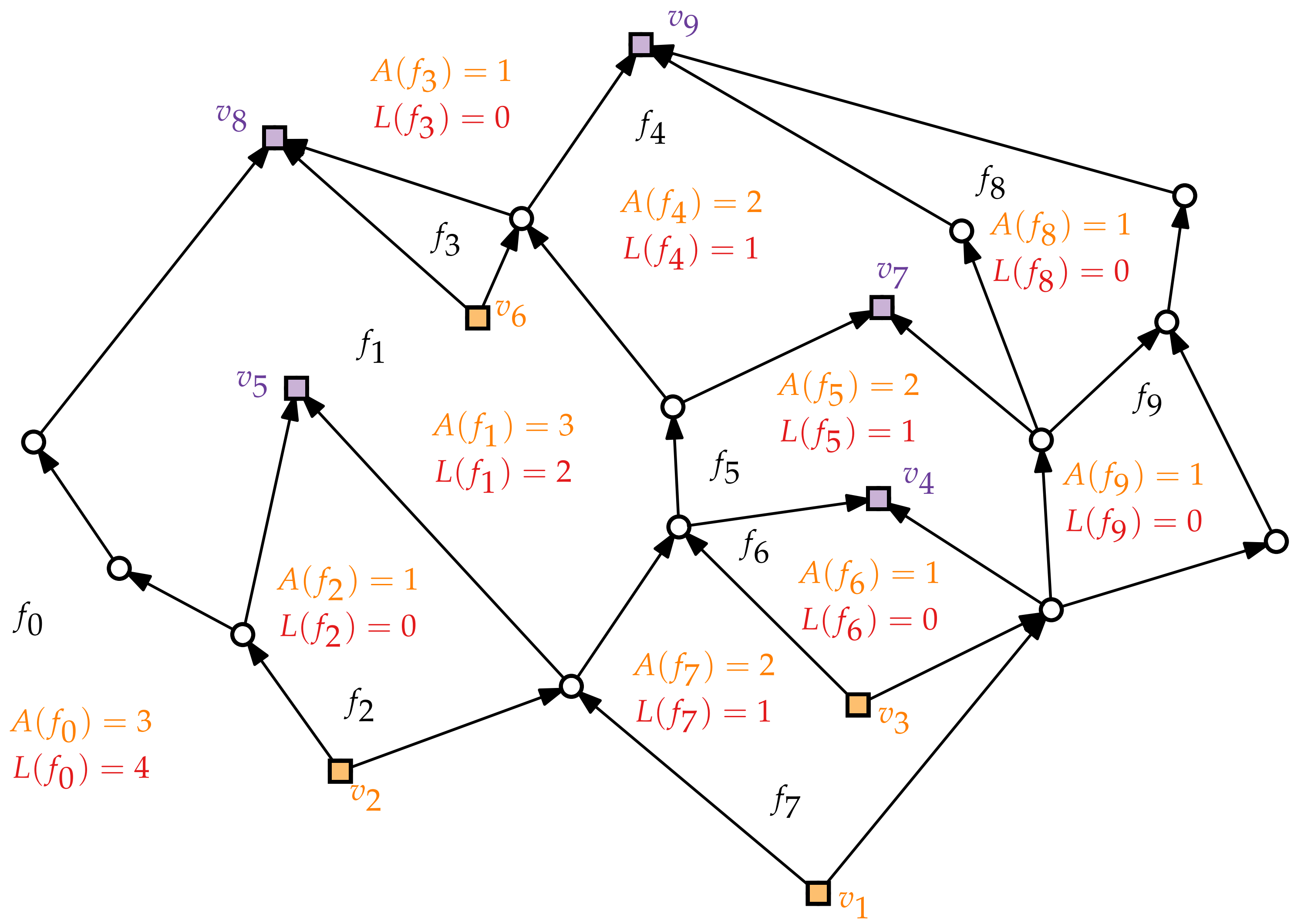
Example of Angle to Face Assignment



- global sources & sinks

$A(f)$ # sources / sinks of f

Example of Angle to Face Assignment

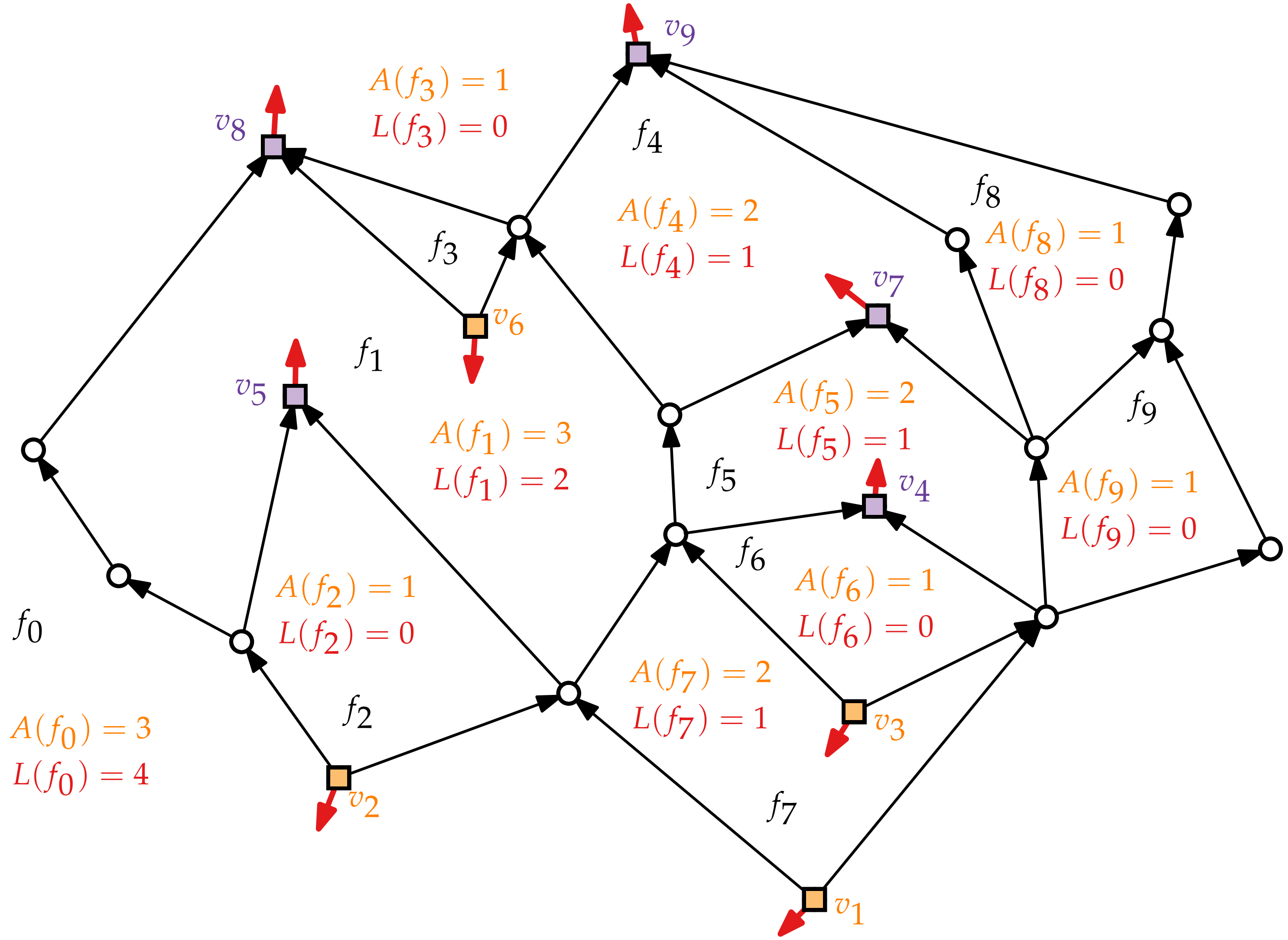


- global sources & sinks

$A(f)$ # sources / sinks of f

$L(f)$ # large angles of f

Example of Angle to Face Assignment



▫ global sources & sinks

$A(f)$ # sources / sinks of f

$L(f)$ # large angles of f

assignment
$\Phi : S \cup T \to F$

Finding a Consistent Assignment

Idea.
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

Finding a Consistent Assignment

Idea.
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

Flow network.
$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W =$
- $E' =$
- $\ell(e) =$
- $u(e) =$
- $b(w) =$

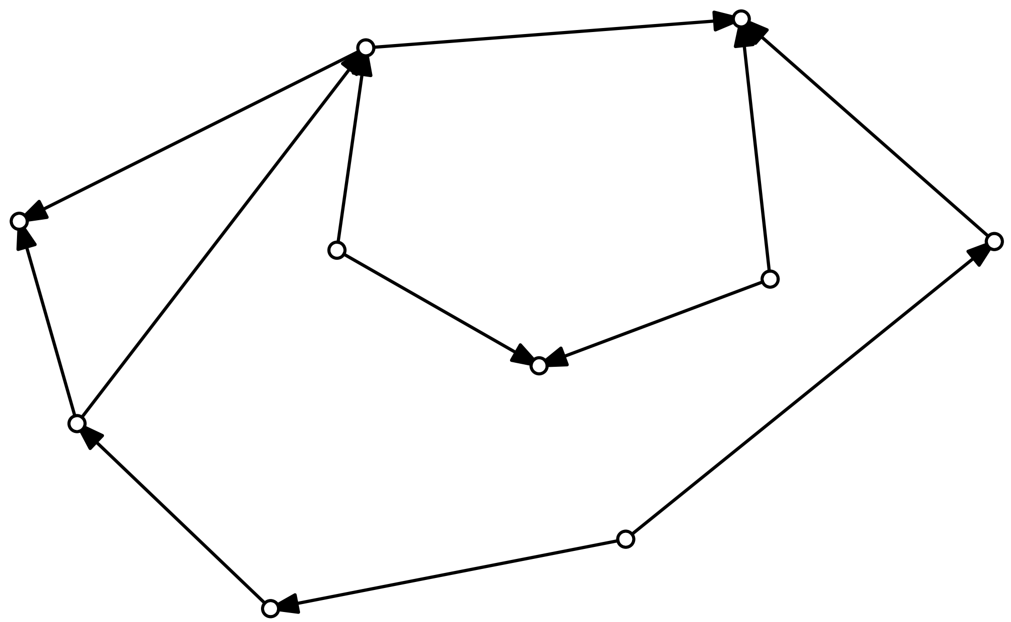
Finding a Consistent Assignment

Idea.
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

Flow network.
$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W =$
- $E' =$
- $\ell(e) =$
- $u(e) =$
- $b(w) =$

Example.



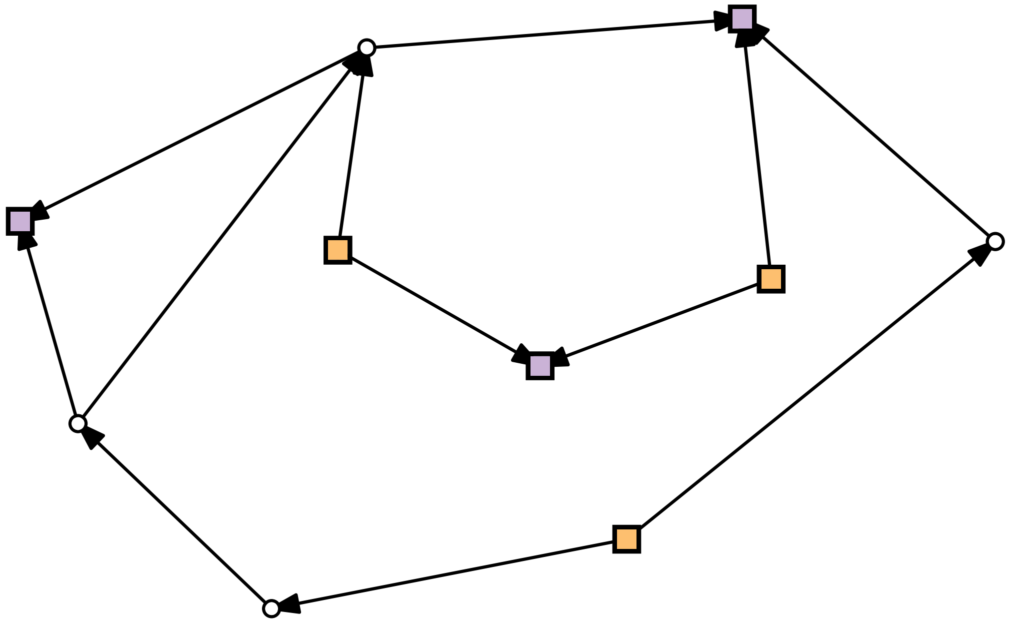
Finding a Consistent Assignment

Idea.
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

Flow network.
$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W = \{v \in V \mid v$ source or sink$\} \cup F$
- $E' =$
- $\ell(e) =$
- $u(e) =$
- $b(w) =$

Example.



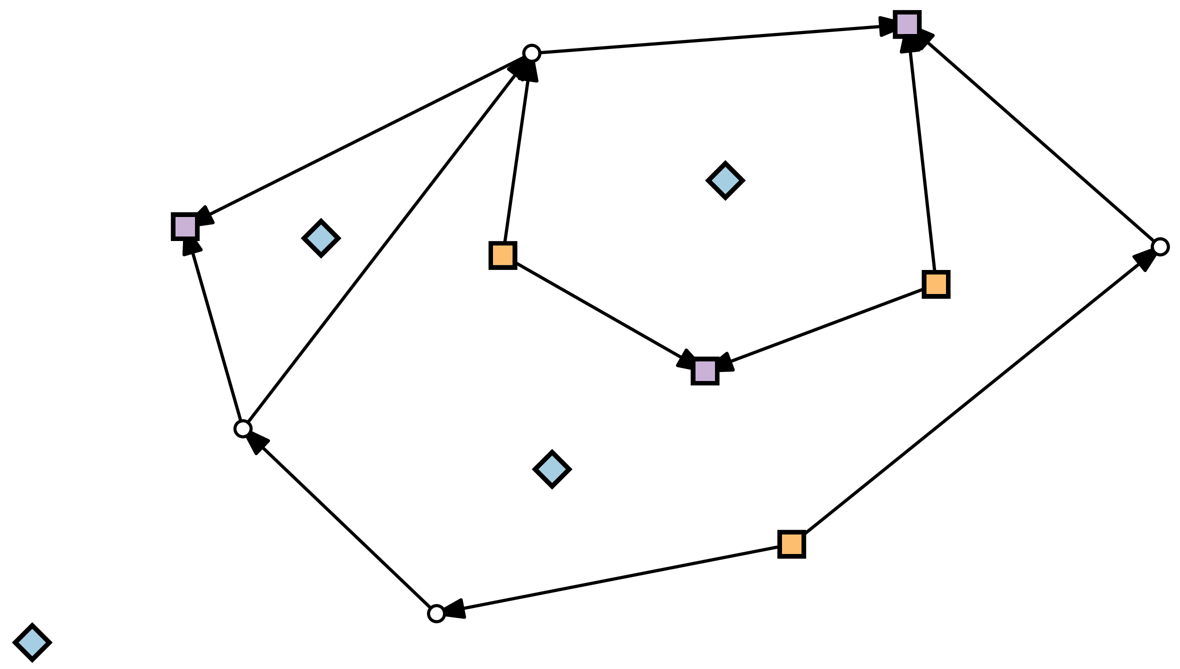
Finding a Consistent Assignment

Idea.
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

Flow network.
$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W = \{v \in V \mid v$ source or sink$\} \cup F$
- $E' =$
- $\ell(e) =$
- $u(e) =$
- $b(w) =$

Example.



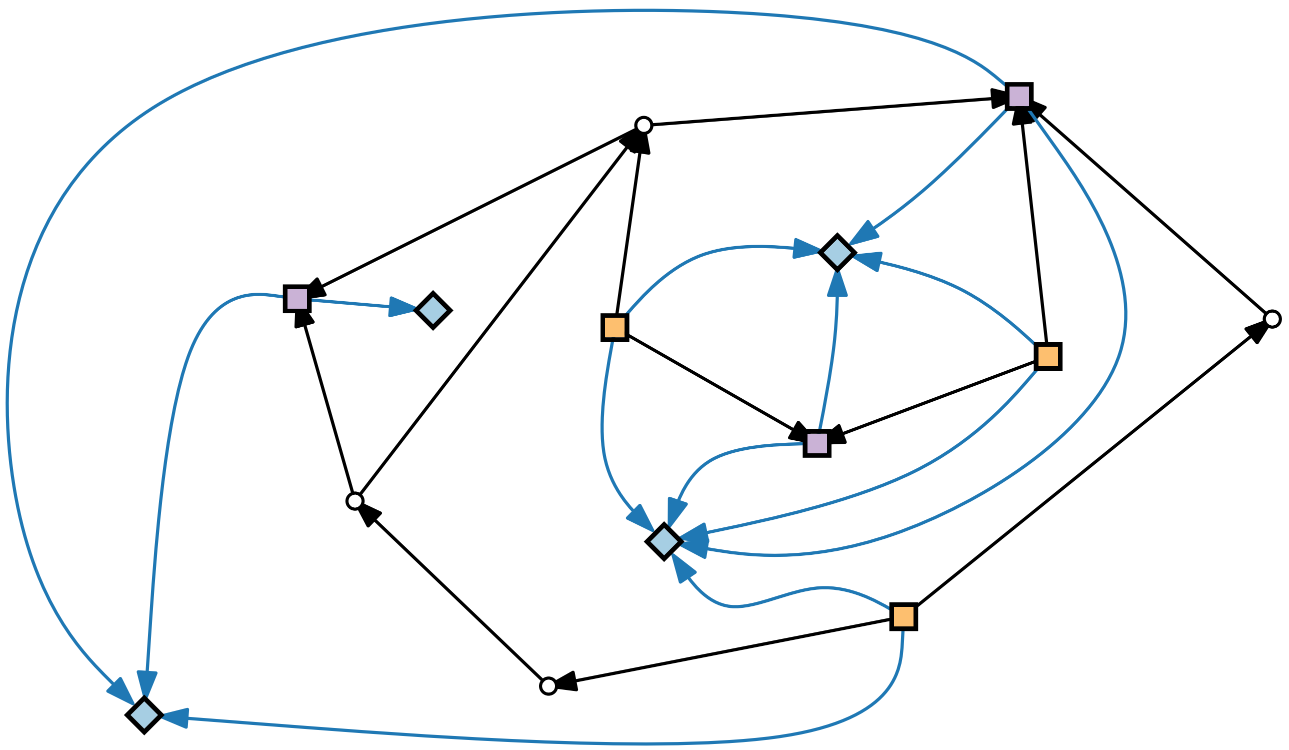
Finding a Consistent Assignment

Idea.
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

Flow network.
$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W = \{v \in V \mid v$ source or sink$\} \cup F$
- $E' = \{(v, f) \mid v$ incident to $f\}$
- $\ell(e) =$
- $u(e) =$
- $b(w) =$

Example.



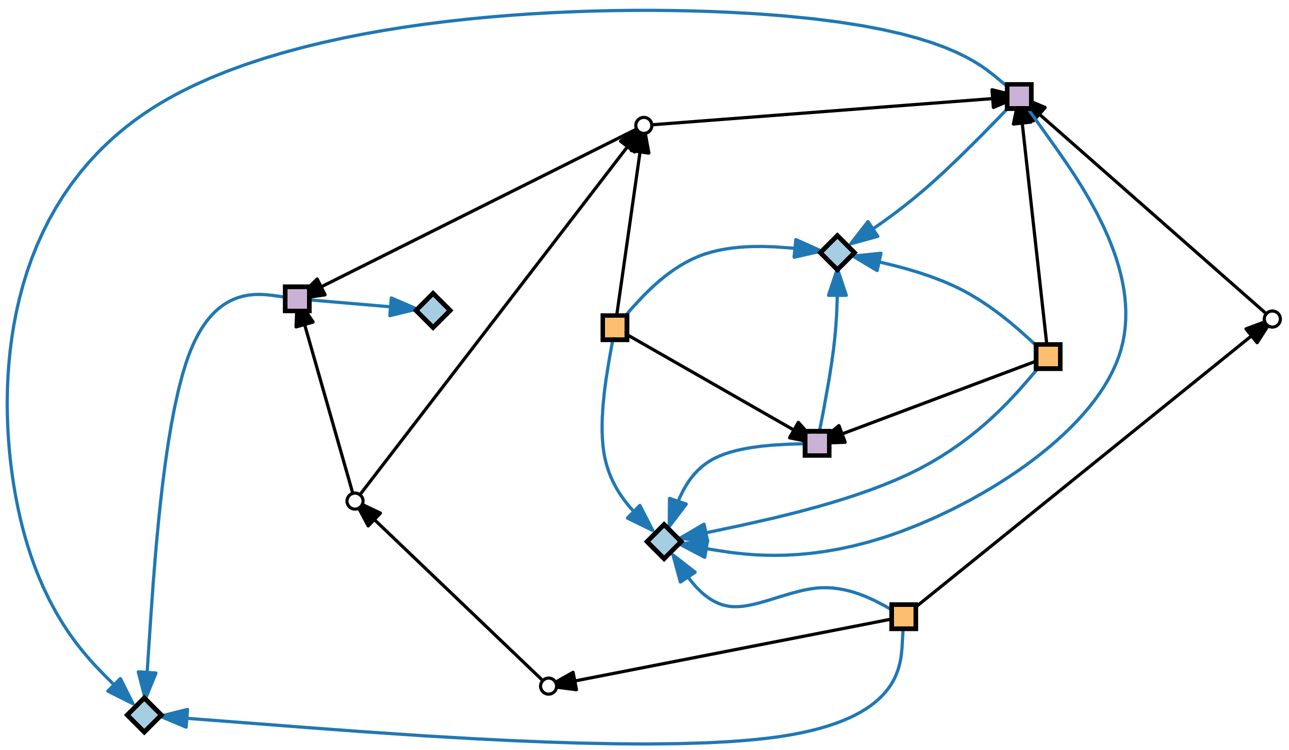
Finding a Consistent Assignment

Idea.
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

Flow network.
$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W = \{v \in V \mid v$ source or sink$\} \cup F$
- $E' = \{(v, f) \mid v$ incident to $f\}$
- $\ell(e) = 0 \; \forall e \in E'$
- $u(e) = 1 \; \forall e \in E'$
- $b(w) =$

Example.



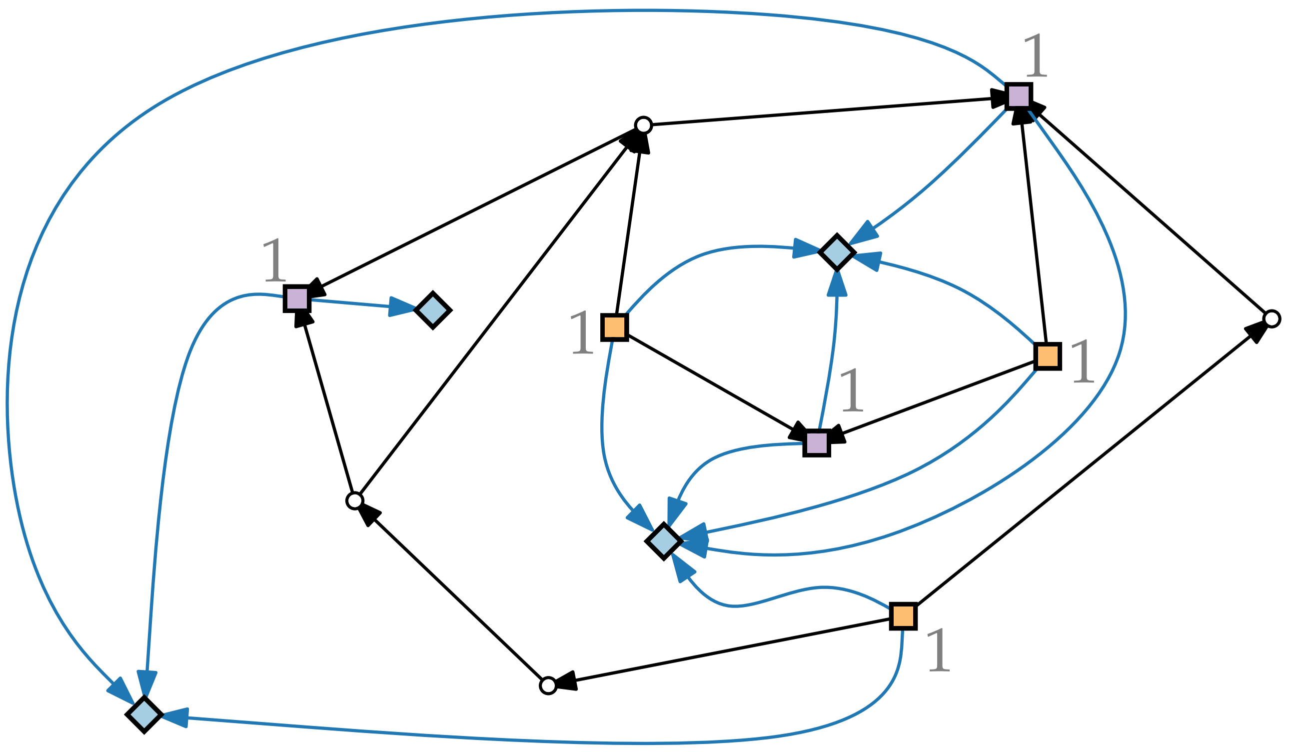
Finding a Consistent Assignment

Idea.
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

Flow network.
$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W = \{v \in V \mid v$ source or sink$\} \cup F$
- $E' = \{(v, f) \mid v$ incident to $f\}$
- $\ell(e) = 0 \ \forall e \in E'$
- $u(e) = 1 \ \forall e \in E'$
- $b(w) = \begin{cases} 1 & \forall w \in W \cap V \\ & \end{cases}$

Example.



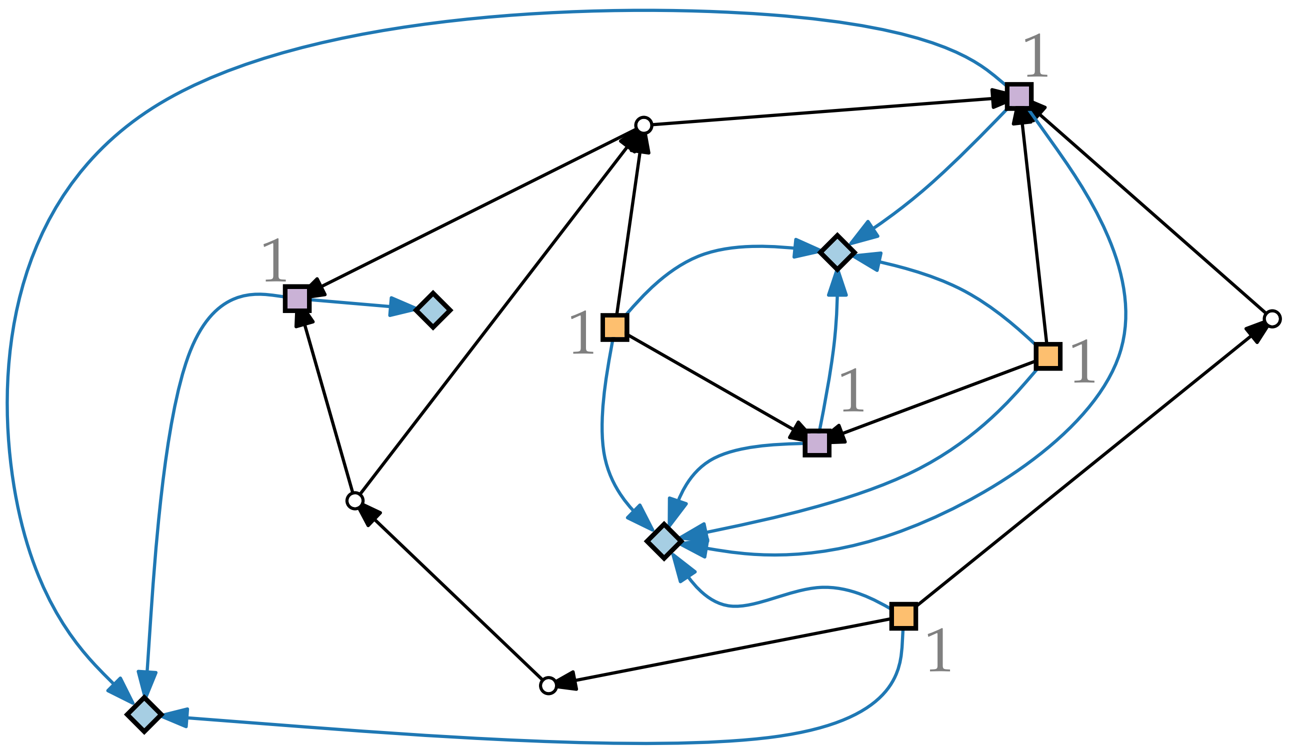
Finding a Consistent Assignment

Idea.
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

Flow network.
$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W = \{v \in V \mid v \text{ source or sink}\} \cup F$
- $E' = \{(v, f) \mid v \text{ incident to } f\}$
- $\ell(e) = 0 \ \forall e \in E'$
- $u(e) = 1 \ \forall e \in E'$
- $b(w) = \begin{cases} 1 & \forall w \in W \cap V \\ -(A(w) - 1) & \forall w \in F \setminus \{f_0\} \end{cases}$

Example.



Finding a Consistent Assignment

Idea.

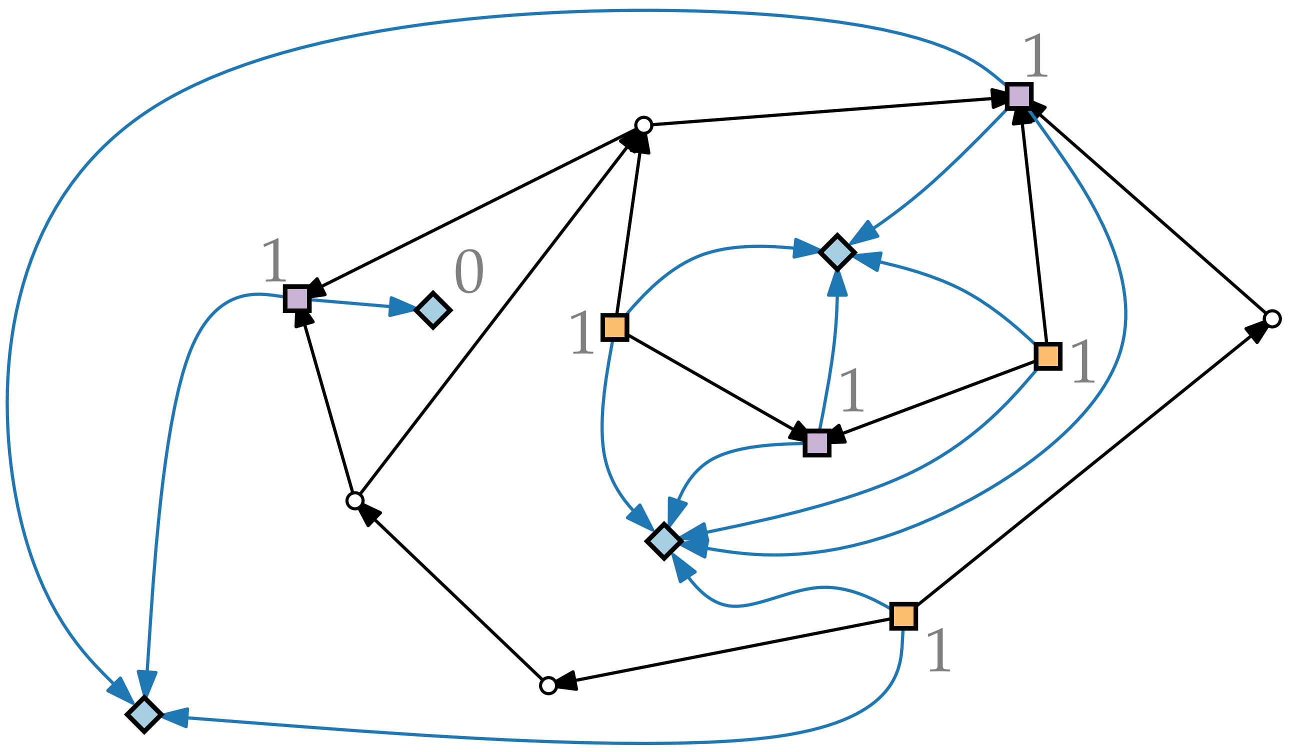
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

Flow network.

$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W = \{v \in V \mid v \text{ source or sink}\} \cup F$
- $E' = \{(v, f) \mid v \text{ incident to } f\}$
- $\ell(e) = 0 \; \forall e \in E'$
- $u(e) = 1 \; \forall e \in E'$
- $b(w) = \begin{cases} 1 & \forall w \in W \cap V \\ -(A(w) - 1) & \forall w \in F \setminus \{f_0\} \end{cases}$

Example.



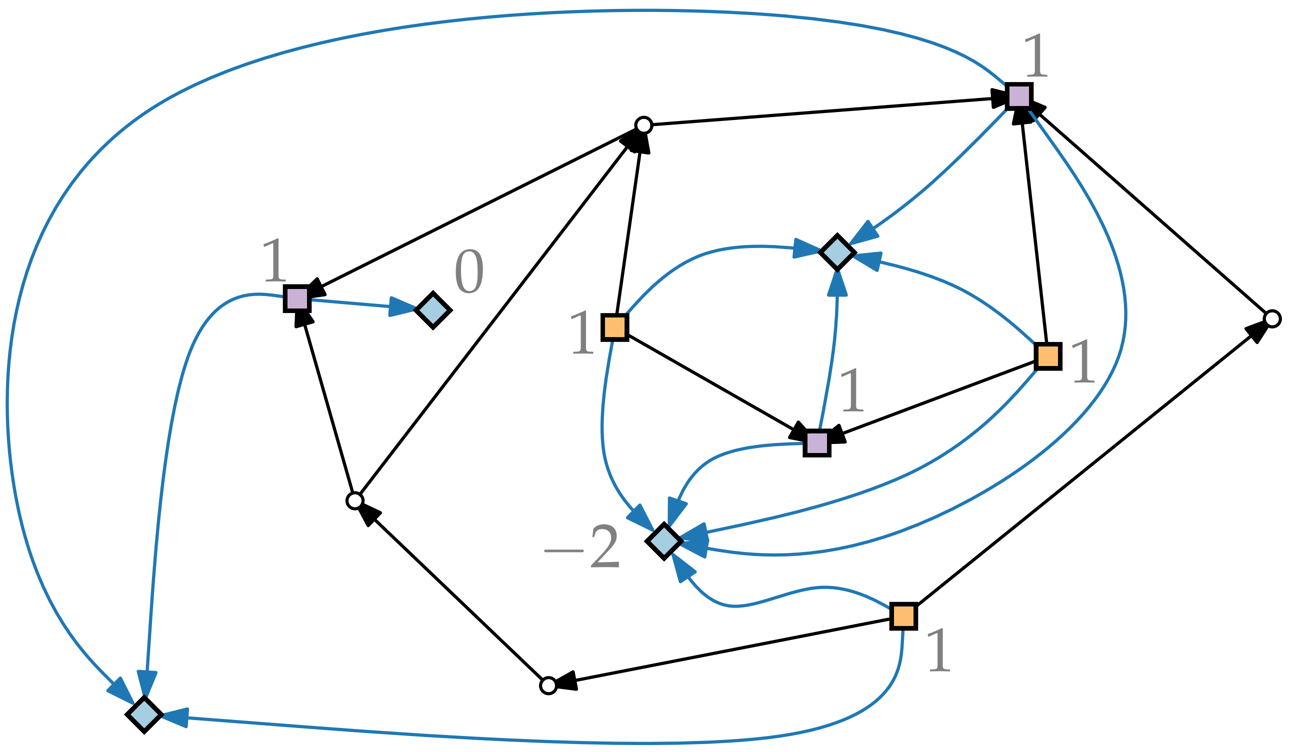
Finding a Consistent Assignment

Idea.
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

Flow network.
$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W = \{v \in V \mid v$ source or sink$\} \cup F$
- $E' = \{(v, f) \mid v$ incident to $f\}$
- $\ell(e) = 0 \; \forall e \in E'$
- $u(e) = 1 \; \forall e \in E'$
- $b(w) = \begin{cases} 1 & \forall w \in W \cap V \\ -(A(w) - 1) & \forall w \in F \setminus \{f_0\} \end{cases}$

Example.



Finding a Consistent Assignment

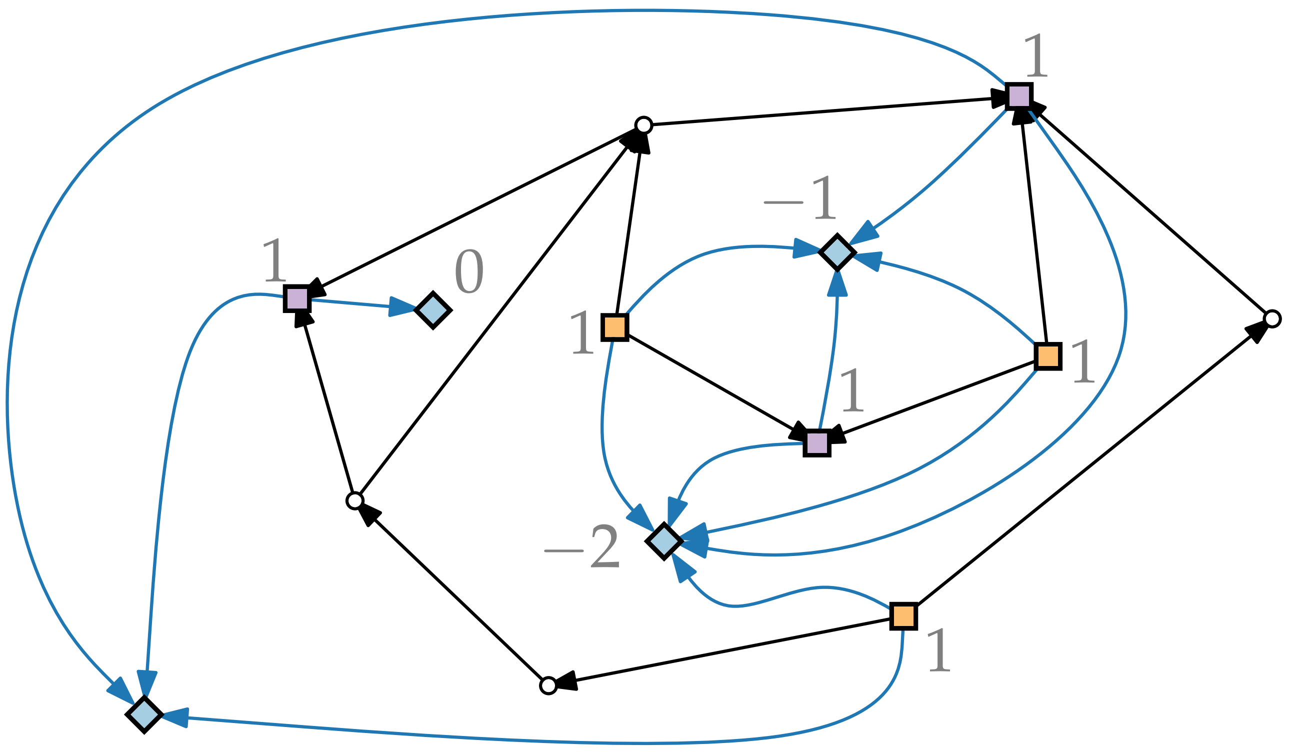
Idea.
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

Flow network.

$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W = \{v \in V \mid v$ source or sink$\} \cup F$
- $E' = \{(v, f) \mid v$ incident to $f\}$
- $\ell(e) = 0 \ \forall e \in E'$
- $u(e) = 1 \ \forall e \in E'$
- $b(w) = \begin{cases} 1 & \forall w \in W \cap V \\ -(A(w) - 1) & \forall w \in F \setminus \{f_0\} \end{cases}$

Example.



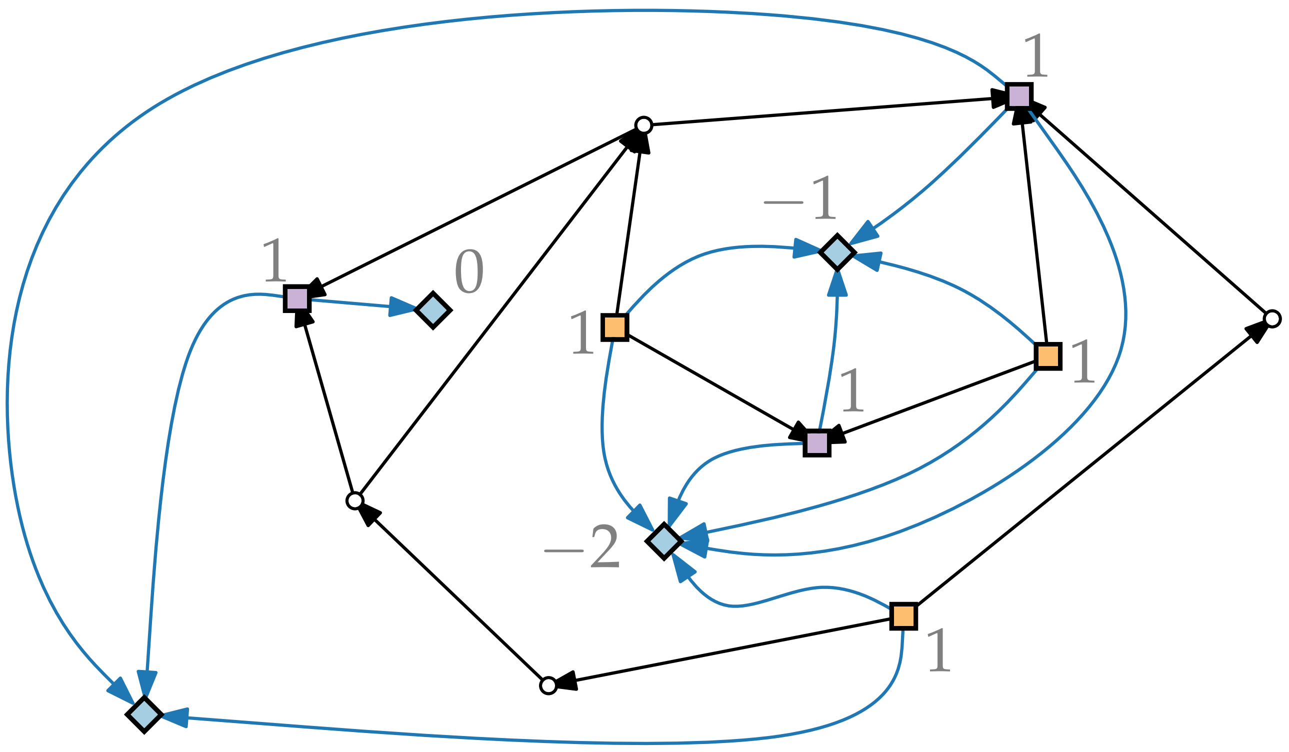
Finding a Consistent Assignment

Idea.
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

Flow network.
$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W = \{v \in V \mid v$ source or sink$\} \cup F$
- $E' = \{(v, f) \mid v$ incident to $f\}$
- $\ell(e) = 0 \ \forall e \in E'$
- $u(e) = 1 \ \forall e \in E'$
- $b(w) = \begin{cases} 1 & \forall w \in W \cap V \\ -(A(w) - 1) & \forall w \in F \setminus \{f_0\} \\ -(A(w) + 1) & w = f_0 \end{cases}$

Example.



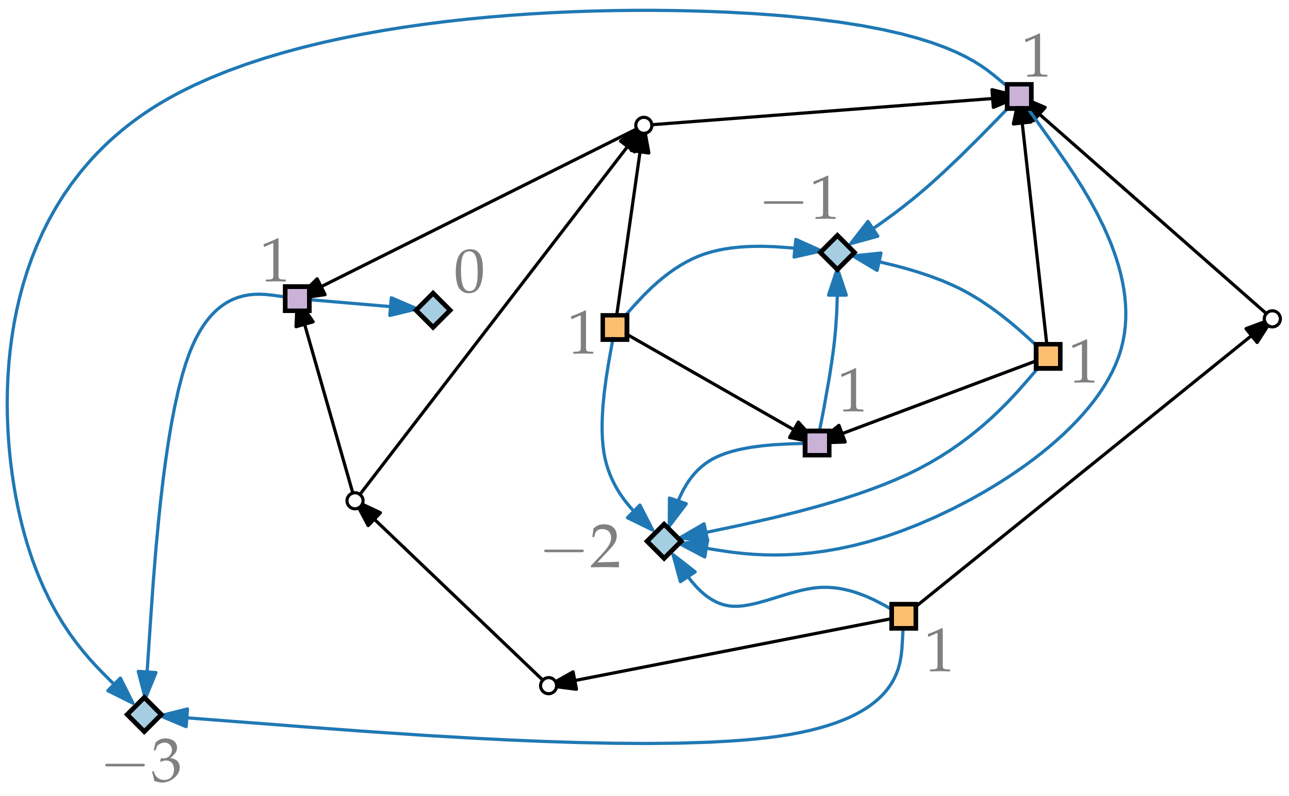
Finding a Consistent Assignment

Idea.
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

Flow network.
$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W = \{v \in V \mid v$ source or sink$\} \cup F$
- $E' = \{(v, f) \mid v$ incident to $f\}$
- $\ell(e) = 0 \ \forall e \in E'$
- $u(e) = 1 \ \forall e \in E'$
- $b(w) = \begin{cases} 1 & \forall w \in W \cap V \\ -(A(w) - 1) & \forall w \in F \setminus \{f_0\} \\ -(A(w) + 1) & w = f_0 \end{cases}$

Example.



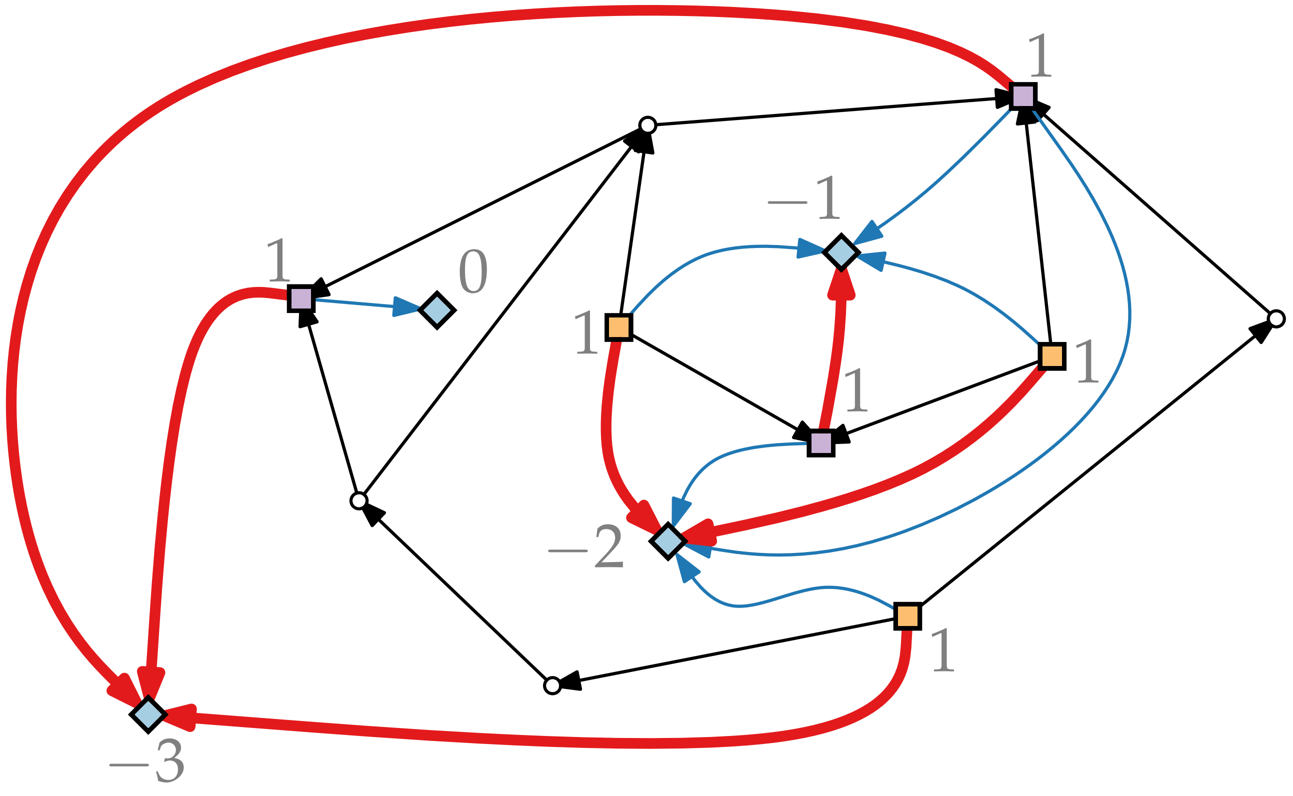
Finding a Consistent Assignment

Idea.
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

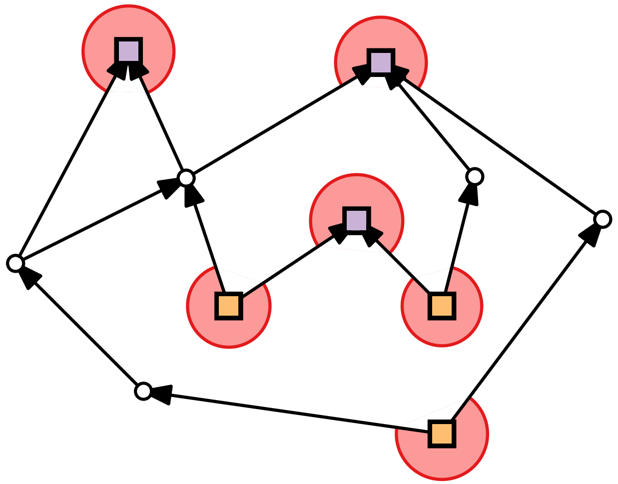
Flow network.
$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W = \{v \in V \mid v \text{ source or sink}\} \cup F$
- $E' = \{(v, f) \mid v \text{ incident to } f\}$
- $\ell(e) = 0 \ \forall e \in E'$
- $u(e) = 1 \ \forall e \in E'$
- $b(w) = \begin{cases} 1 & \forall w \in W \cap V \\ -(A(w) - 1) & \forall w \in F \setminus \{f_0\} \\ -(A(w) + 1) & w = f_0 \end{cases}$

Example.



Finding a Consistent Assignment

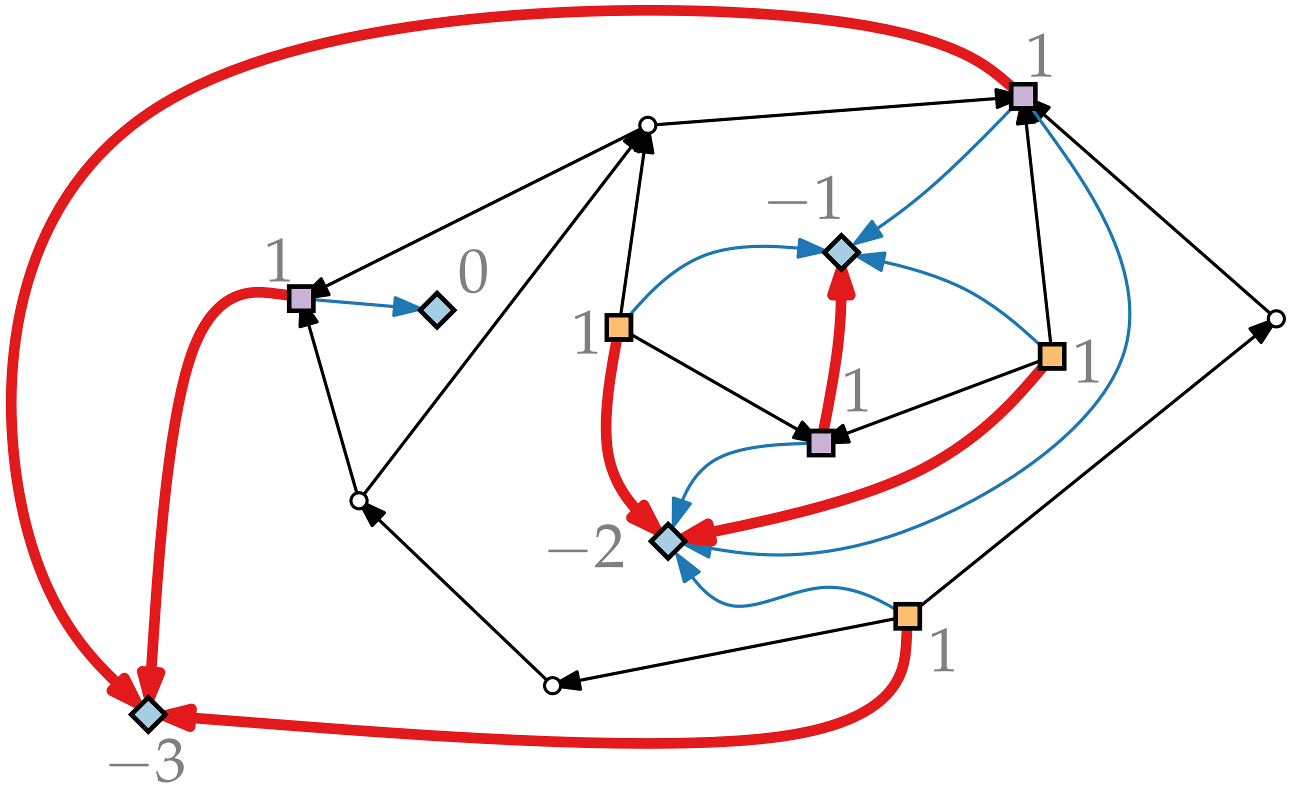


Idea.
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

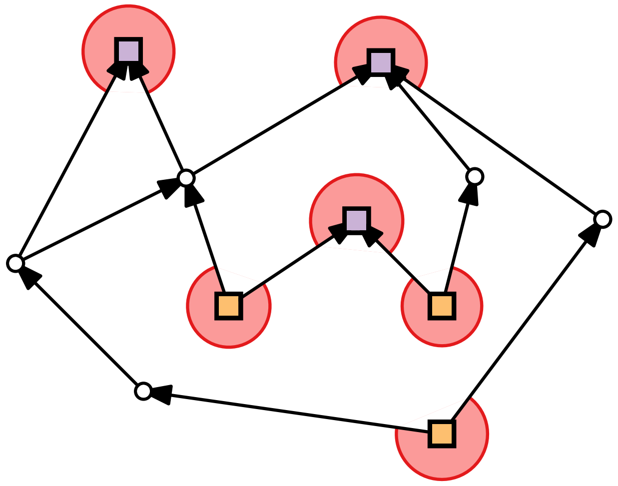
Flow network.
$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W = \{v \in V \mid v \text{ source or sink}\} \cup F$
- $E' = \{(v, f) \mid v \text{ incident to } f\}$
- $\ell(e) = 0 \ \forall e \in E'$
- $u(e) = 1 \ \forall e \in E'$
- $b(w) = \begin{cases} 1 & \forall w \in W \cap V \\ -(A(w) - 1) & \forall w \in F \setminus \{f_0\} \\ -(A(w) + 1) & w = f_0 \end{cases}$

Example.



Finding a Consistent Assignment

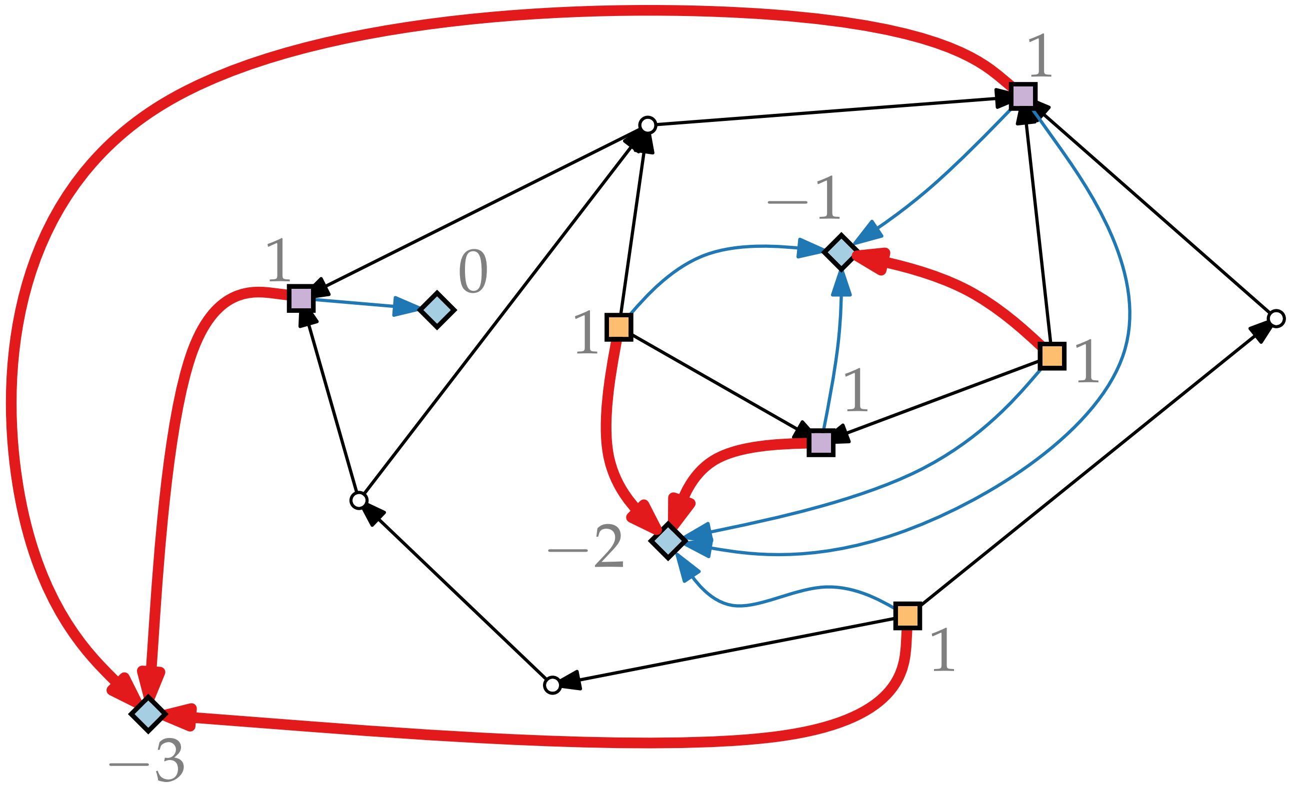


Idea.
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

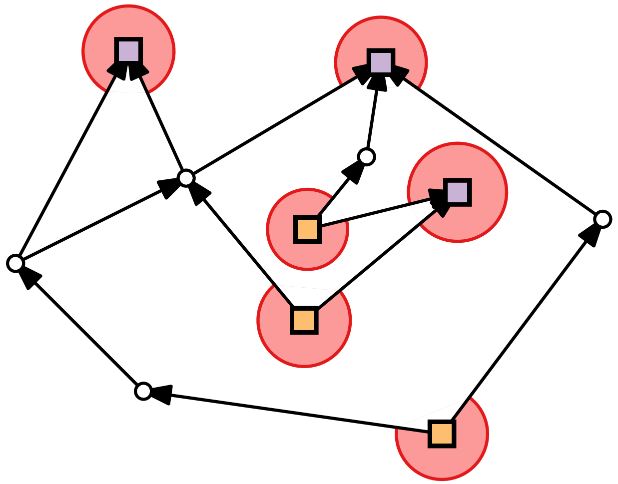
Flow network.
$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W = \{v \in V \mid v$ source or sink$\} \cup F$
- $E' = \{(v, f) \mid v$ incident to $f\}$
- $\ell(e) = 0 \ \forall e \in E'$
- $u(e) = 1 \ \forall e \in E'$
- $b(w) = \begin{cases} 1 & \forall w \in W \cap V \\ -(A(w) - 1) & \forall w \in F \setminus \{f_0\} \\ -(A(w) + 1) & w = f_0 \end{cases}$

Example.



Finding a Consistent Assignment

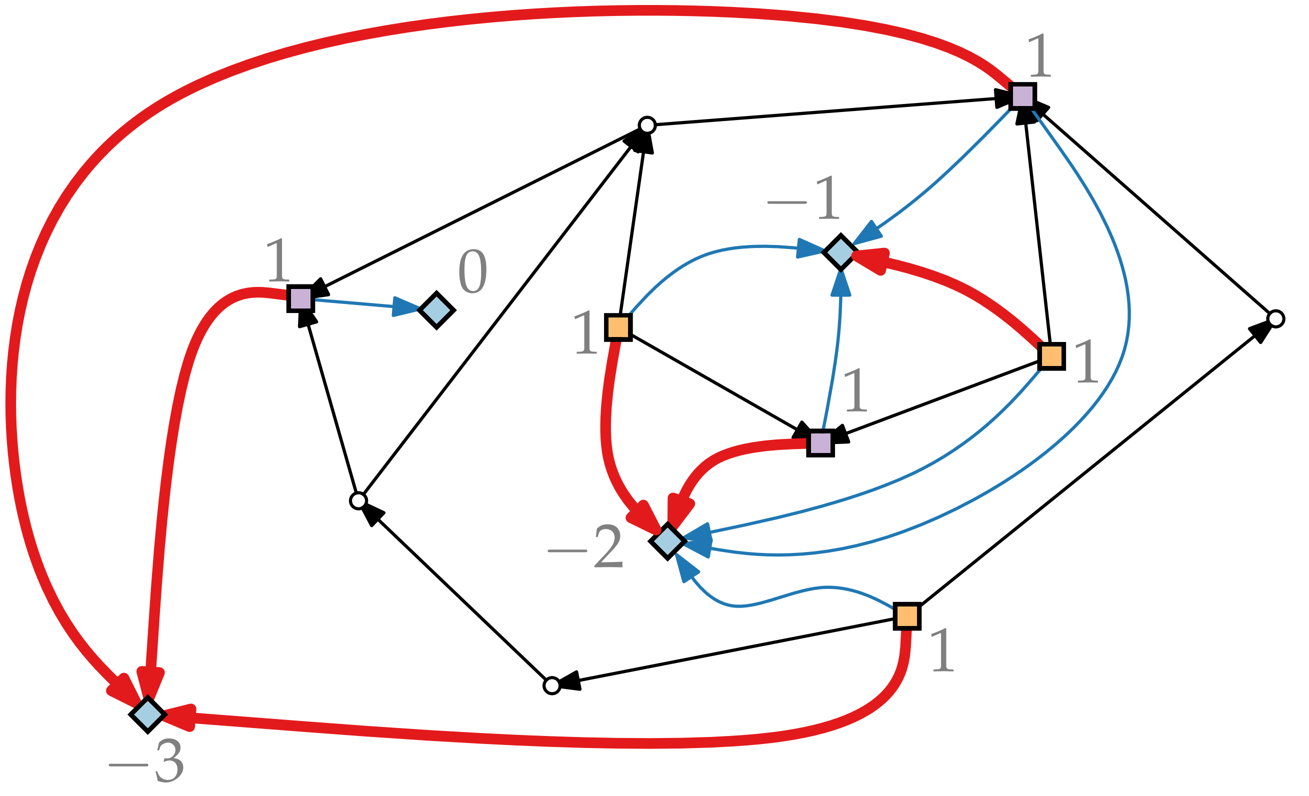


Idea.
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

Flow network.
$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

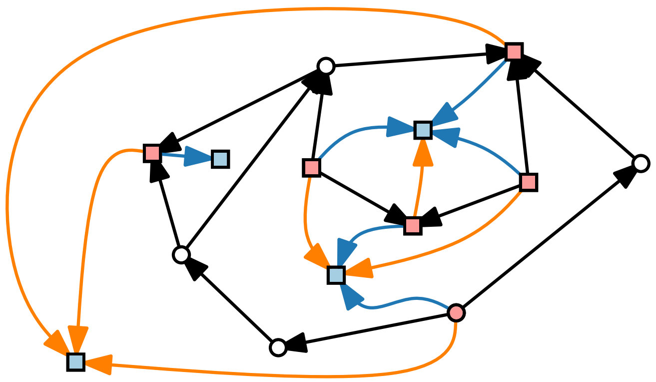
- $W = \{v \in V \mid v$ source or sink$\} \cup F$
- $E' = \{(v, f) \mid v$ incident to $f\}$
- $\ell(e) = 0 \; \forall e \in E'$
- $u(e) = 1 \; \forall e \in E'$
- $b(w) = \begin{cases} 1 & \forall w \in W \cap V \\ -(A(w) - 1) & \forall w \in F \setminus \{f_0\} \\ -(A(w) + 1) & w = f_0 \end{cases}$

Example.

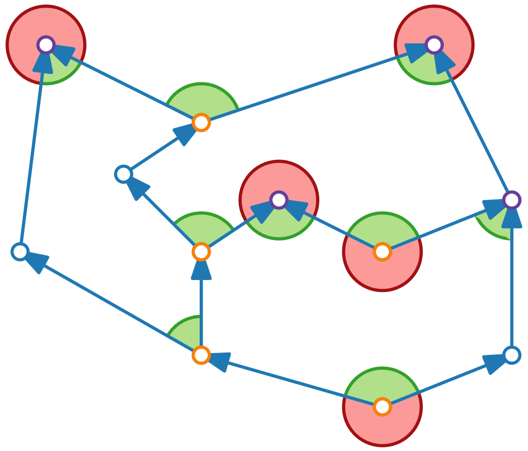


Universität Trier

Visualization of Graphs



Lecture 7:
Upward Planar Drawings



Part IV:
Testing Algorithm

Philipp Kindermann

Result Characterization

Theorem 3.
Let $G = (V, E)$ be an acyclic plane digraph with embedding given by F, f_0.

bimodal and there exists consistent assignment Φ.

Result Characterization

Theorem 3.
Let $G = (V, E)$ be an acyclic plane digraph with embedding given by F, f_0.
Then G is upward planar (respecting F, f_0) if and only if G is bimodal and there exists consistent assignment Φ.

Result Characterization

Theorem 3.
Let $G = (V, E)$ be an acyclic plane digraph with embedding given by F, f_0.
Then G is upward planar (respecting F, f_0) if and only if G is bimodal and there exists consistent assignment Φ.

Proof.
$\Rightarrow$: As constructed before.

Result Characterization

Theorem 3.
Let $G = (V, E)$ be an acyclic plane digraph with embedding given by F, f_0.
Then G is upward planar (respecting F, f_0) if and only if G is bimodal and there exists consistent assignment Φ.

Proof.

$\Rightarrow$: As constructed before.

$\Leftarrow$: Idea:

- Construct planar st-digraph that is supergraph of G.

Result Characterization

Theorem 3.
Let $G = (V, E)$ be an acyclic plane digraph with embedding given by F, f_0.
Then G is upward planar (respecting F, f_0) if and only if G is bimodal and there exists consistent assignment Φ.

Proof.

$\Rightarrow$: As constructed before.

$\Leftarrow$: Idea:

- Construct planar st-digraph that is supergraph of G.
- Apply equivalence from Theorem 1.

Result Characterization

Theorem 3.
Let $G = (V, E)$ be an acyclic plane digraph with embedding given by F, f_0.
Then G is upward planar (respecting F, f_0) if and only if G is bimodal and there exists consistent assignment Φ.

Proof.

$\Rightarrow$: As constructed before.

$\Leftarrow$: Idea:

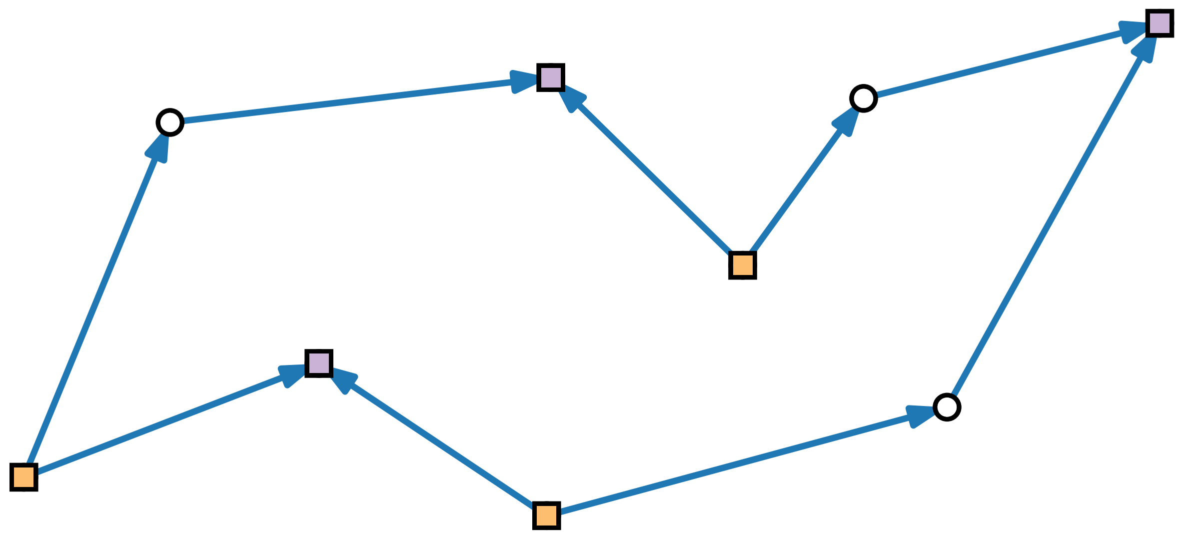
- Construct planar st-digraph that is supergraph of G.
- Apply equivalence from Theorem 1.

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
[...] G is upward planar
$\Leftrightarrow$ G is the spanning subgraph of a planar st-digraph.

Refinement Algorithm – $\Phi, F, f_0 \rightarrow$ st-digraph

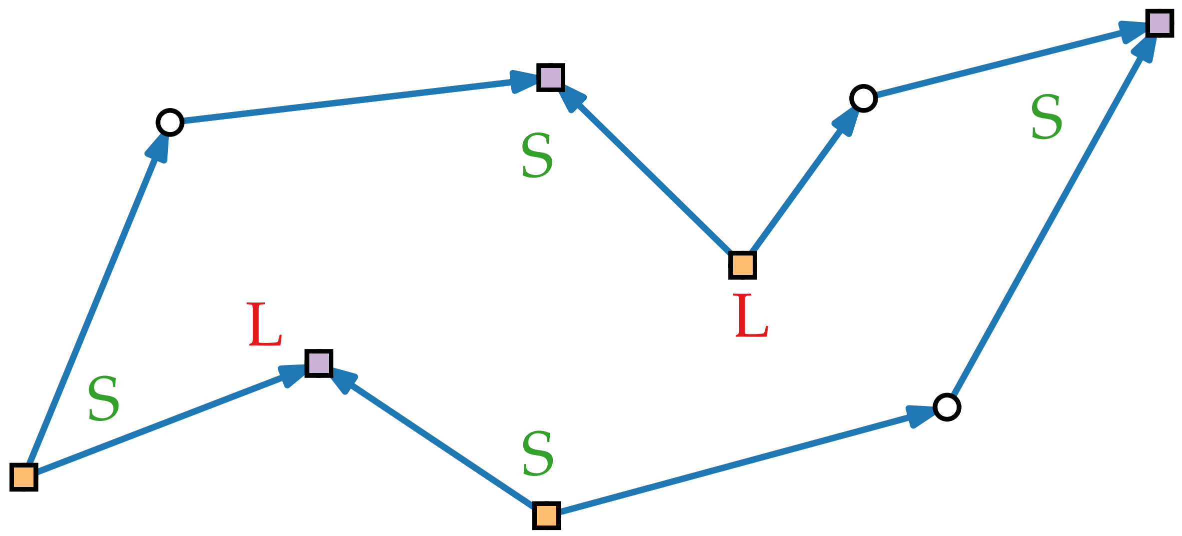
Refinement Algorithm – $\Phi, F, f_0 \rightarrow$ st-digraph

Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.



Refinement Algorithm – $\Phi, F, f_0 \rightarrow$ st-digraph

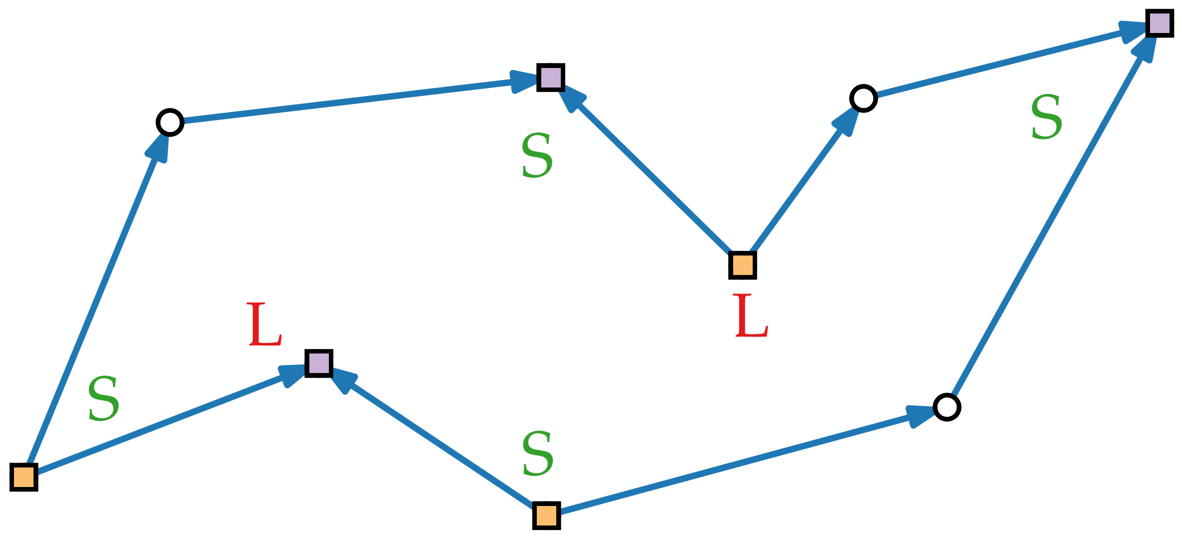
Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.



Refinement Algorithm – $\Phi, F, f_0 \rightarrow$ st-digraph

Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.

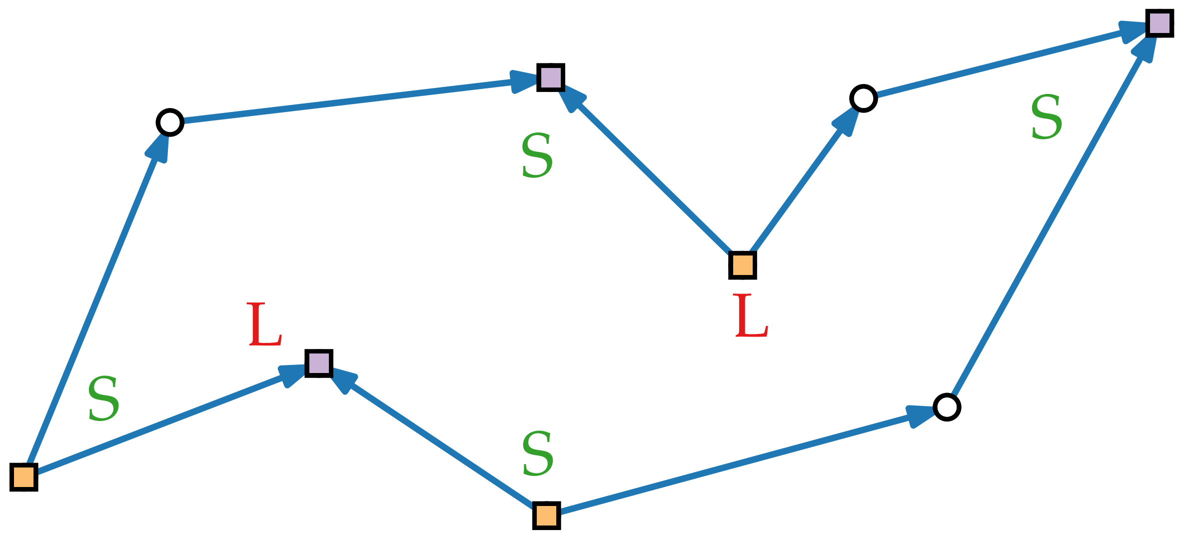
- Goal: Add edges to break large angles (sources and sinks).



Refinement Algorithm – $\Phi, F, f_0 \rightarrow$ st-digraph

Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.

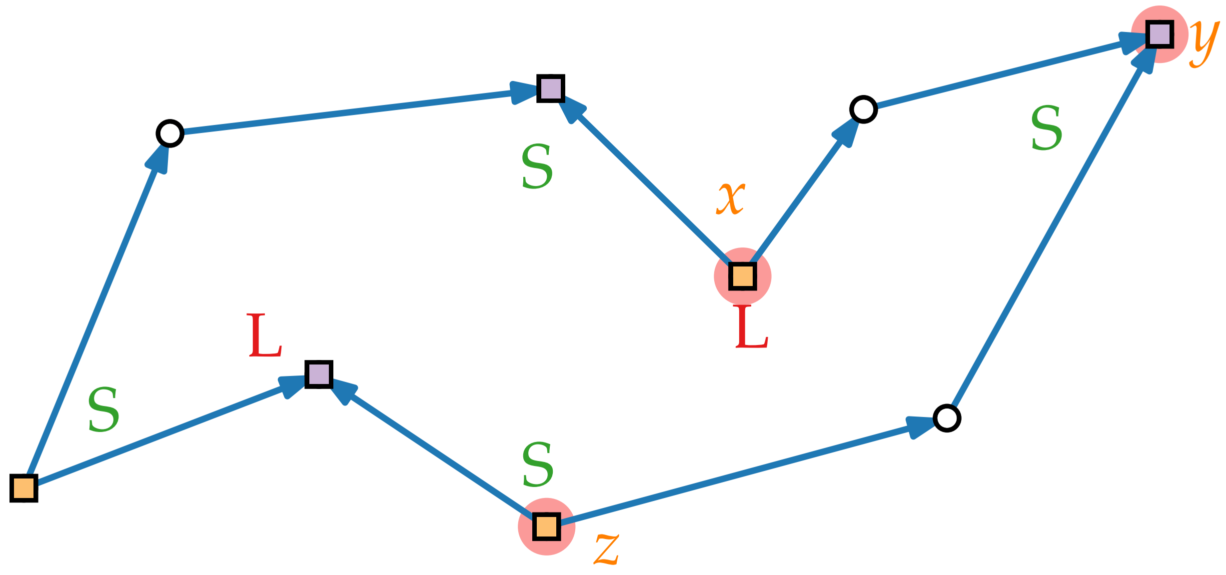
- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:



Refinement Algorithm – $\Phi, F, f_0 \to$ st-digraph

Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.

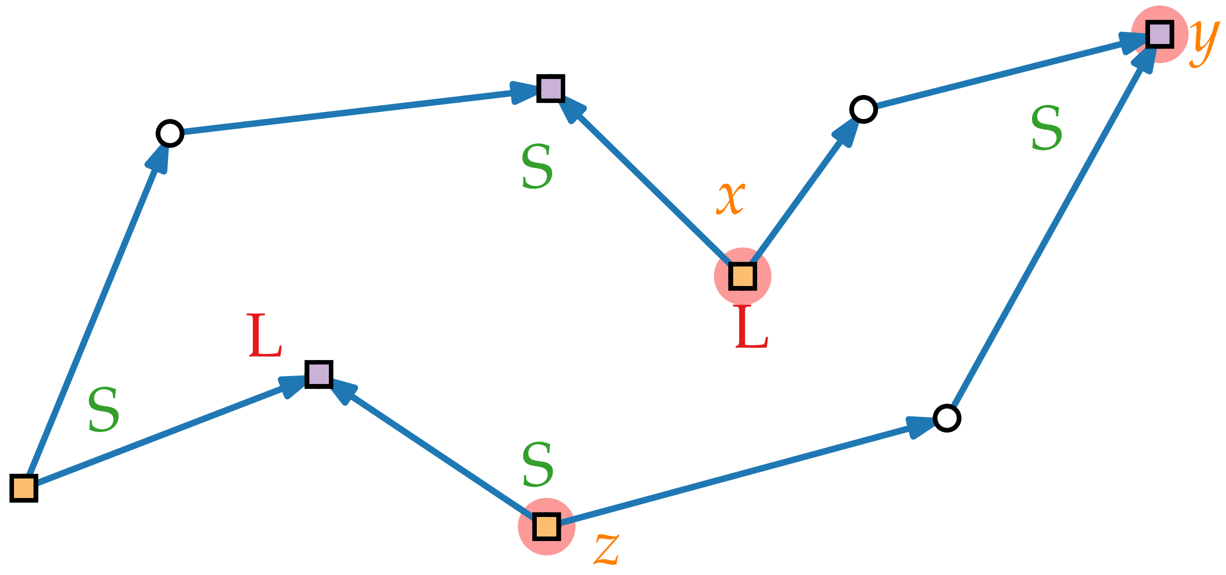
- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:



Refinement Algorithm – $\Phi, F, f_0 \to$ st-digraph

Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.

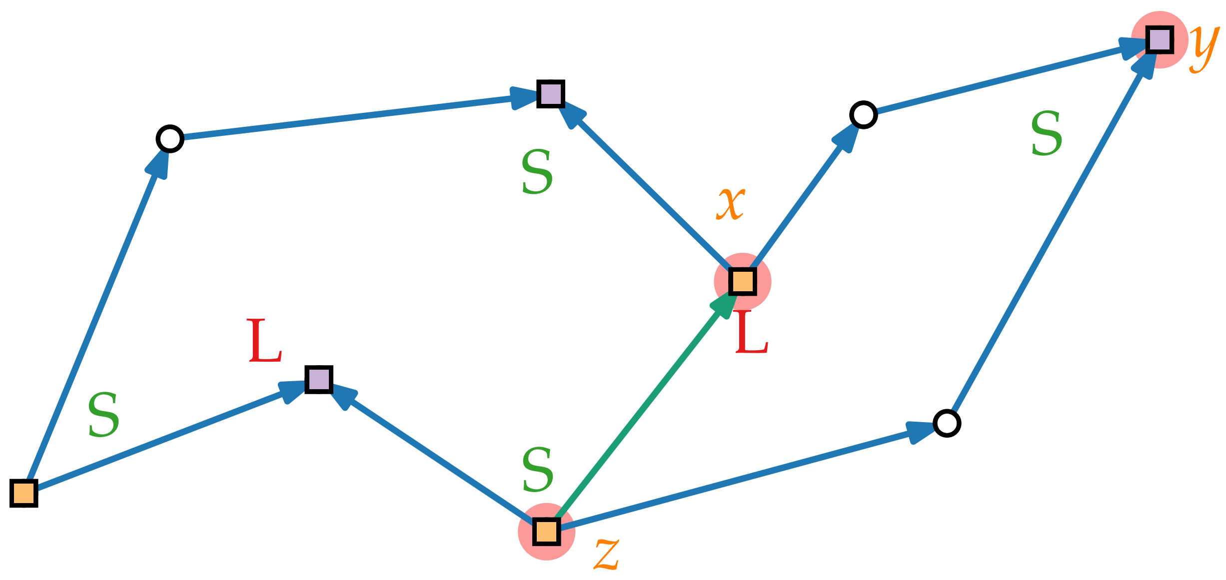
- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:
- x source $\Rightarrow$ insert edge (z, x)



Refinement Algorithm – $\Phi, F, f_0 \to$ st-digraph

Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.

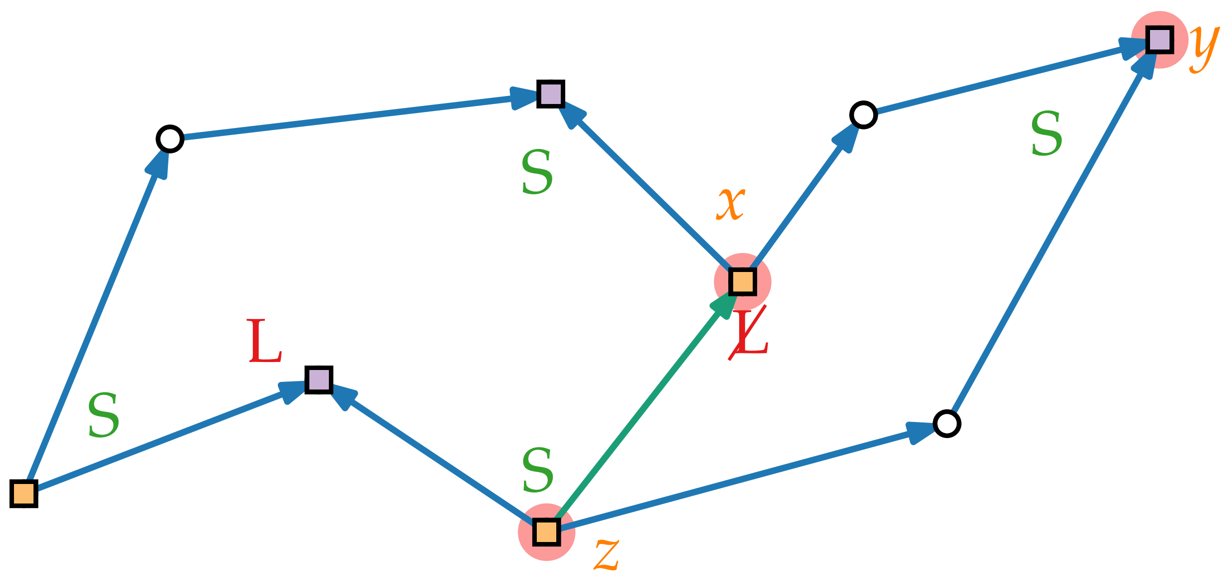
- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:
- x source $\Rightarrow$ insert edge (z, x)



Refinement Algorithm – $\Phi, F, f_0 \to$ st-digraph

Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.

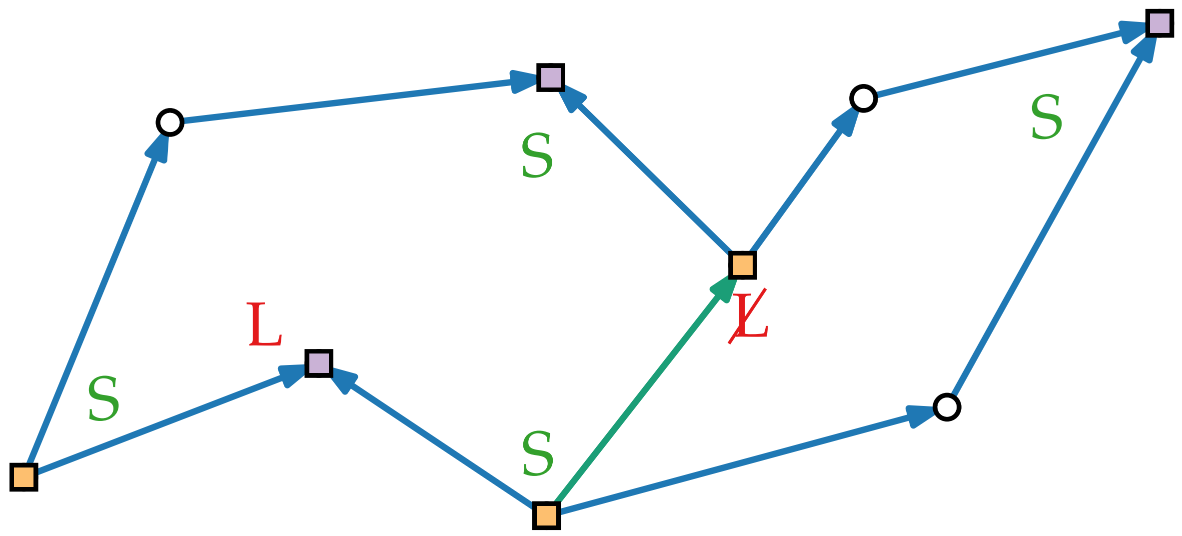
- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:
- x source $\Rightarrow$ insert edge (z, x)



Refinement Algorithm – $\Phi, F, f_0 \rightarrow$ st-digraph

Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.

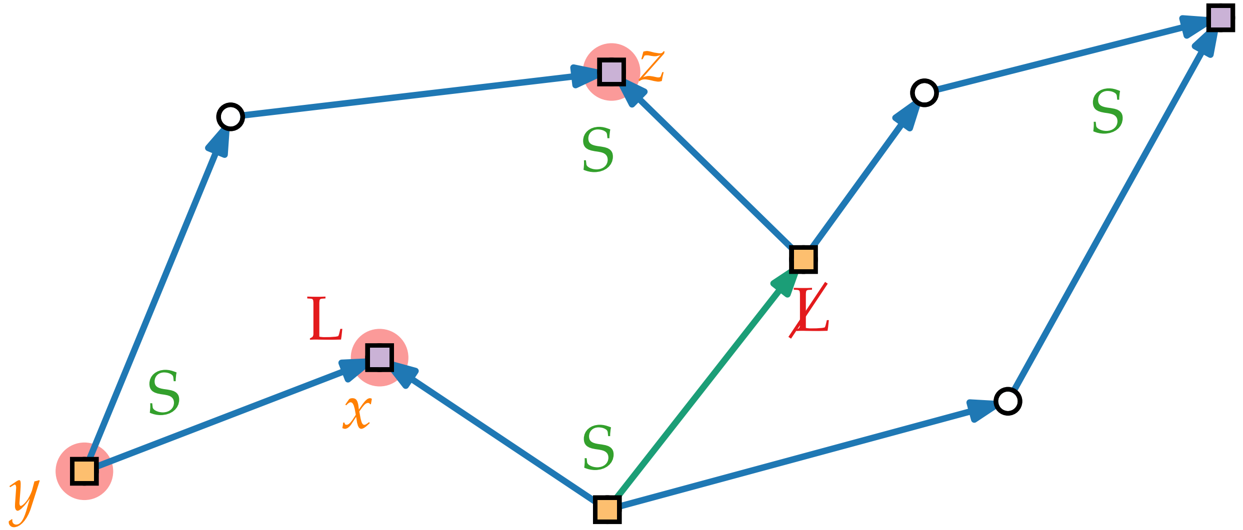
- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:
- x source $\Rightarrow$ insert edge (z, x)



Refinement Algorithm – $\Phi, F, f_0 \to$ st-digraph

Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.

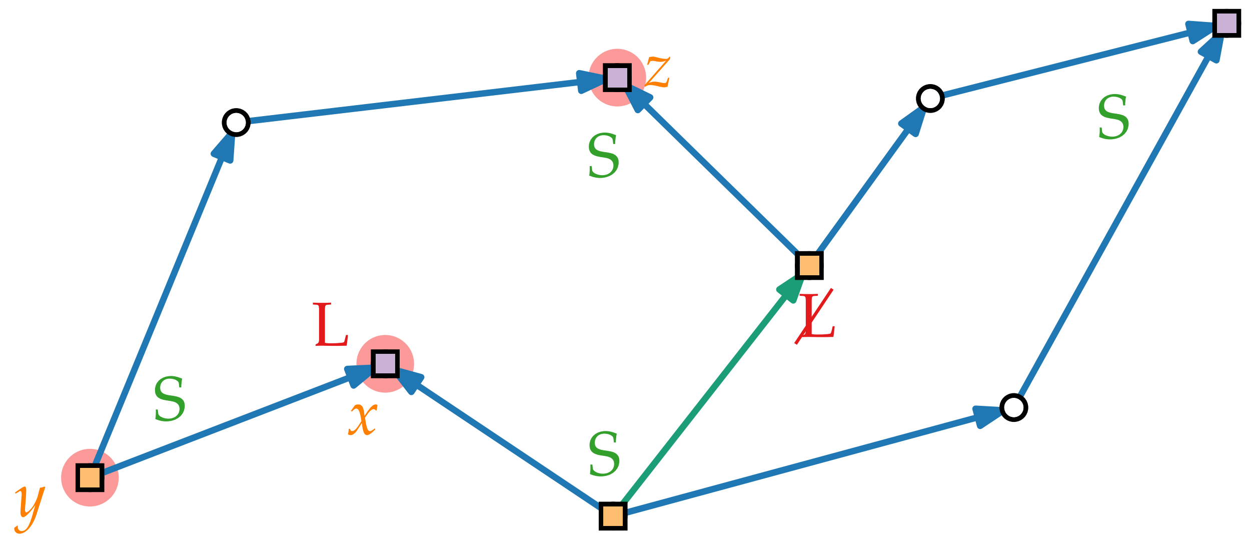
- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:
- x source $\Rightarrow$ insert edge (z, x)



Refinement Algorithm – $\Phi, F, f_0 \rightarrow$ st-digraph

Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.

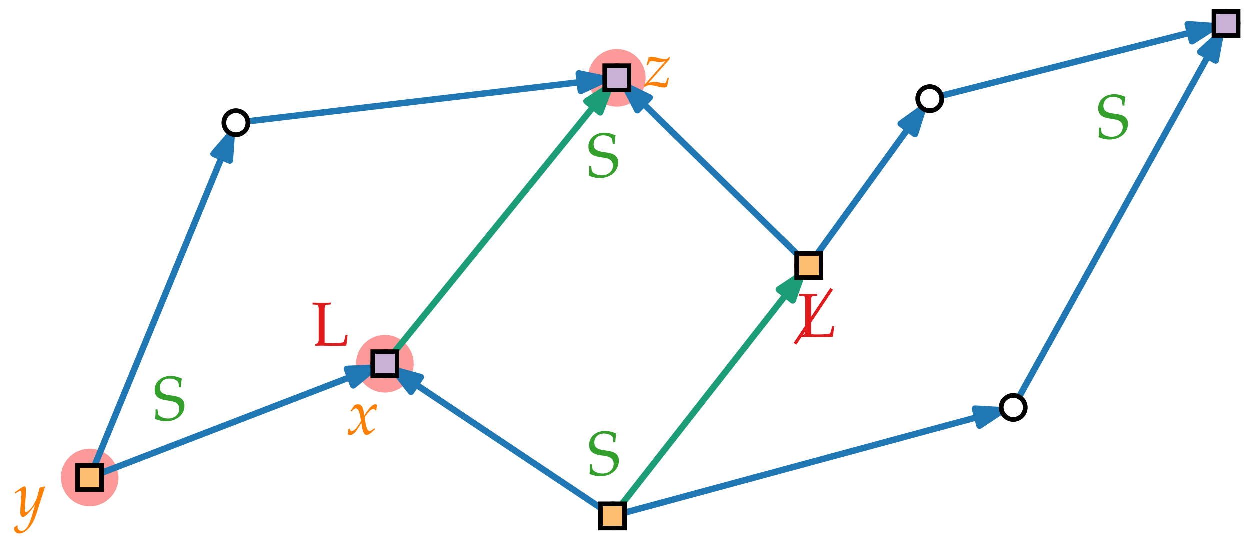
- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:
- x source $\Rightarrow$ insert edge (z, x)
- x sink $\Rightarrow$ insert edge (x, z).



Refinement Algorithm – $\Phi, F, f_0 \rightarrow$ st-digraph

Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.

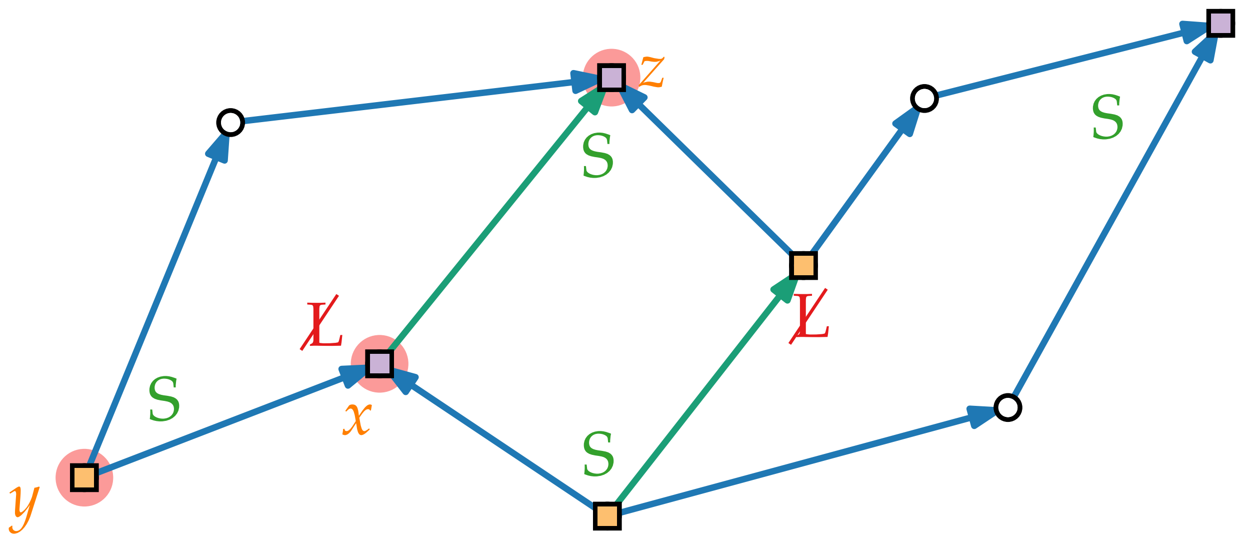
- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:
- x source $\Rightarrow$ insert edge (z, x)
- x sink $\Rightarrow$ insert edge (x, z).



Refinement Algorithm – $\Phi, F, f_0 \rightarrow$ st-digraph

Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.

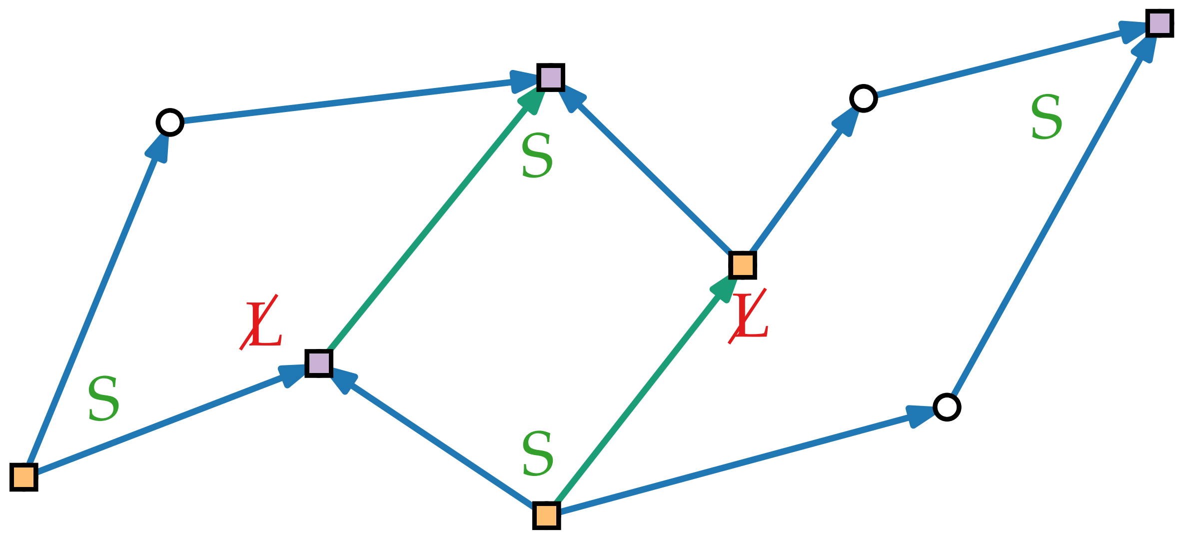
- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:
- x source $\Rightarrow$ insert edge (z, x)
- x sink $\Rightarrow$ insert edge (x, z).



Refinement Algorithm – $\Phi, F, f_0 \rightarrow$ st-digraph

Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.

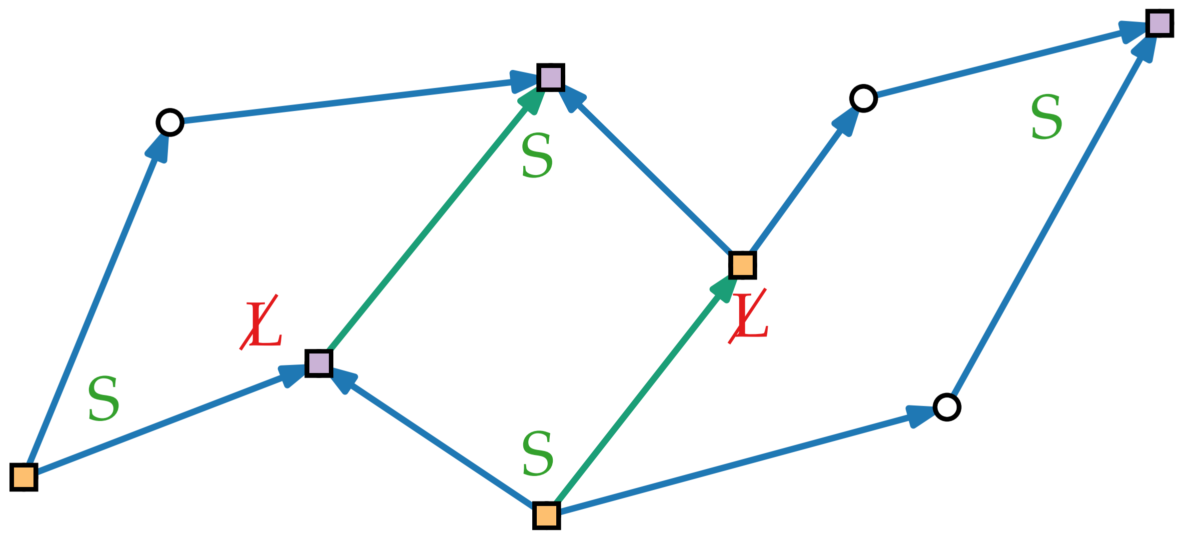
- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:
- x source $\Rightarrow$ insert edge (z, x)
- x sink $\Rightarrow$ insert edge (x, z).



Refinement Algorithm – $\Phi, F, f_0 \to$ st-digraph

Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.

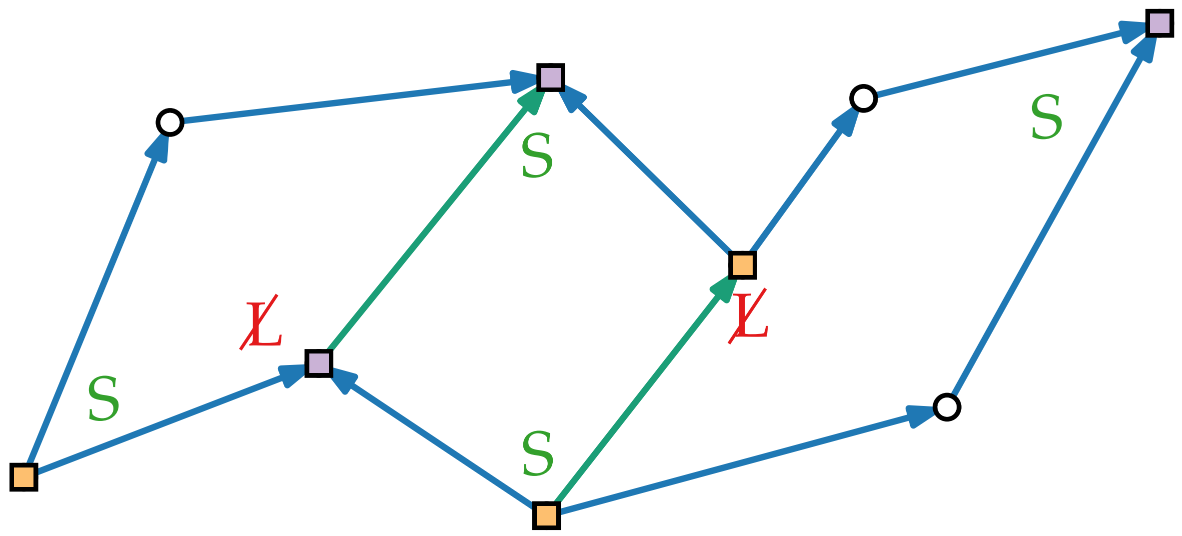
- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:
- x source $\Rightarrow$ insert edge (z, x)
- x sink $\Rightarrow$ insert edge (x, z).
- Refine outer face f_0.



Refinement Algorithm – $\Phi, F, f_0 \rightarrow$ st-digraph

Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.

- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:
- x source $\Rightarrow$ insert edge (z, x)
- x sink $\Rightarrow$ insert edge (x, z).
- Refine outer face f_0.

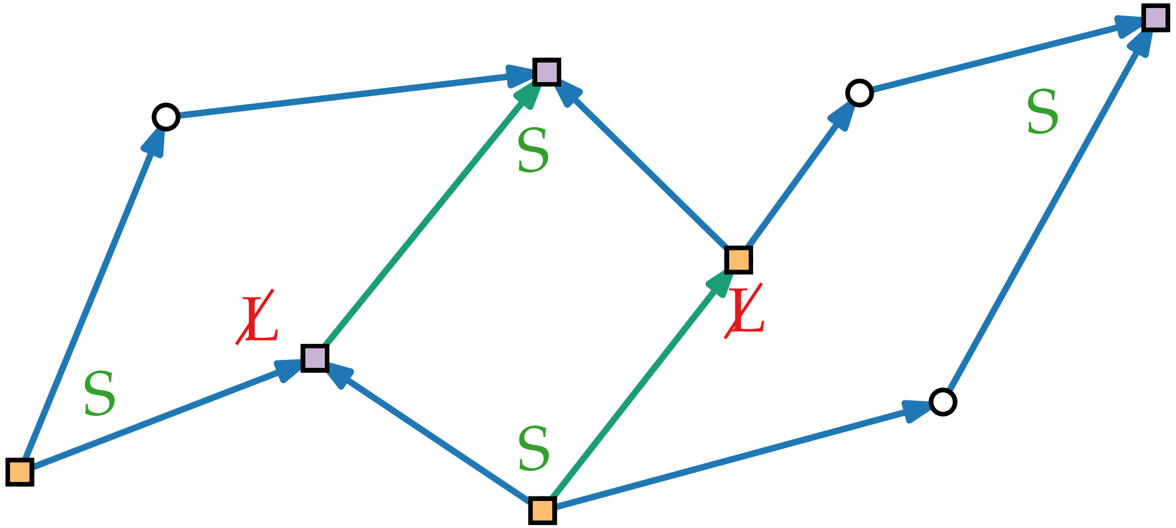


- Refine all faces. $\Rightarrow$ G is contained in a planar st-digraph.

Refinement Algorithm – $\Phi, F, f_0 \rightarrow$ st-digraph

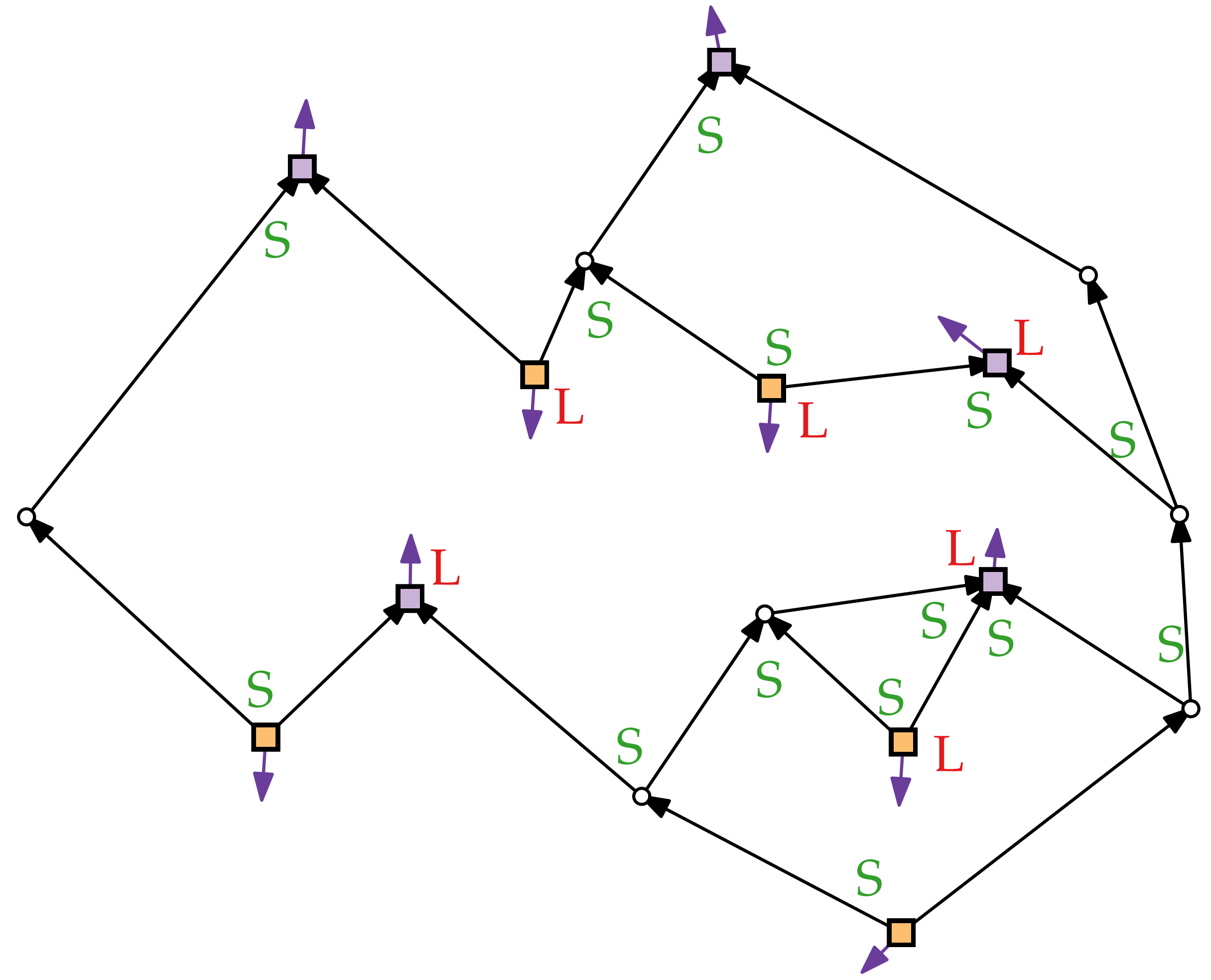
Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.

- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:
- x source $\Rightarrow$ insert edge (z, x)
- x sink $\Rightarrow$ insert edge (x, z).
- Refine outer face f_0.

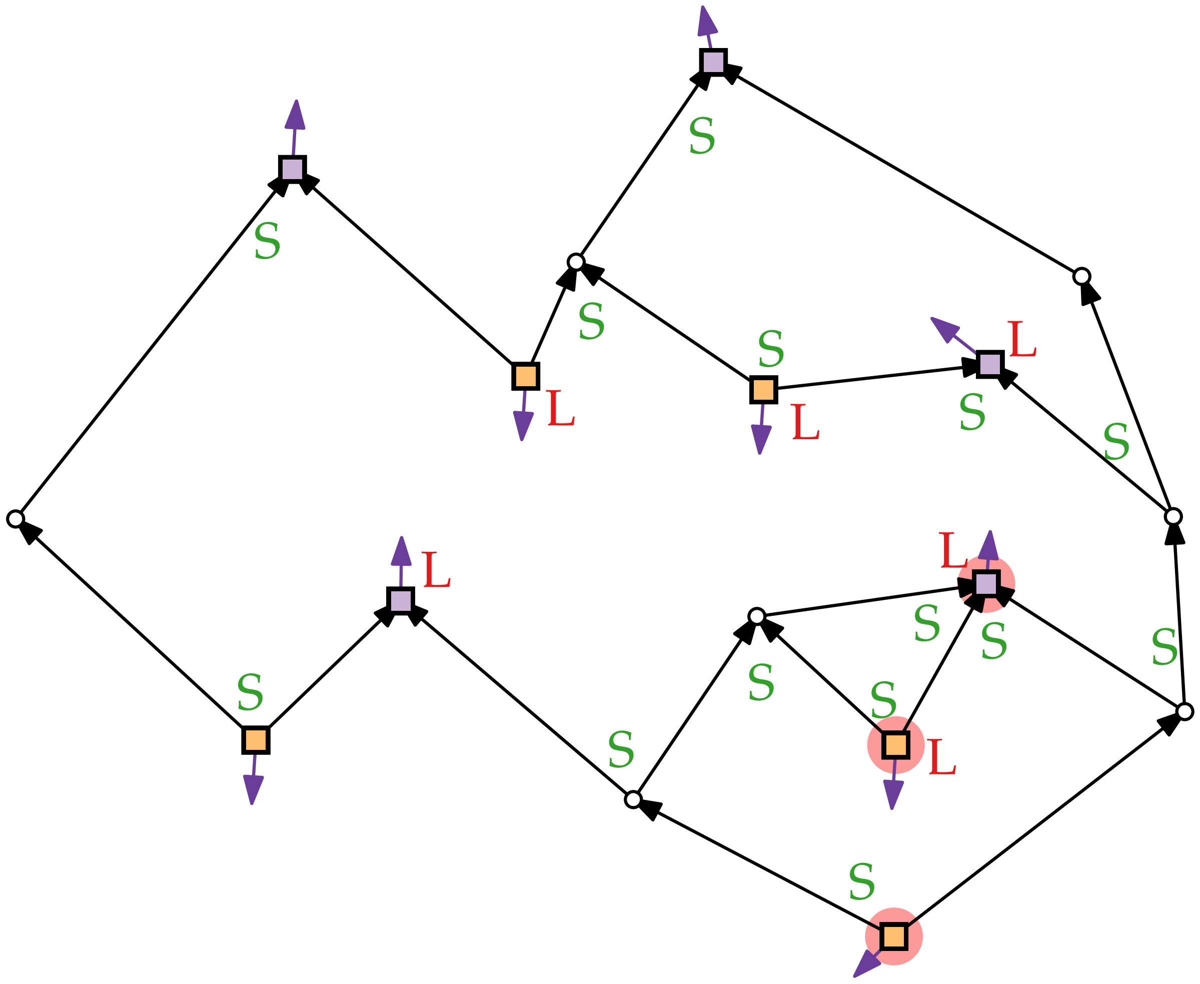


- Refine all faces. $\Rightarrow$ G is contained in a planar st-digraph.
- Planarity, acyclicity, bimodality are invariants under construction.

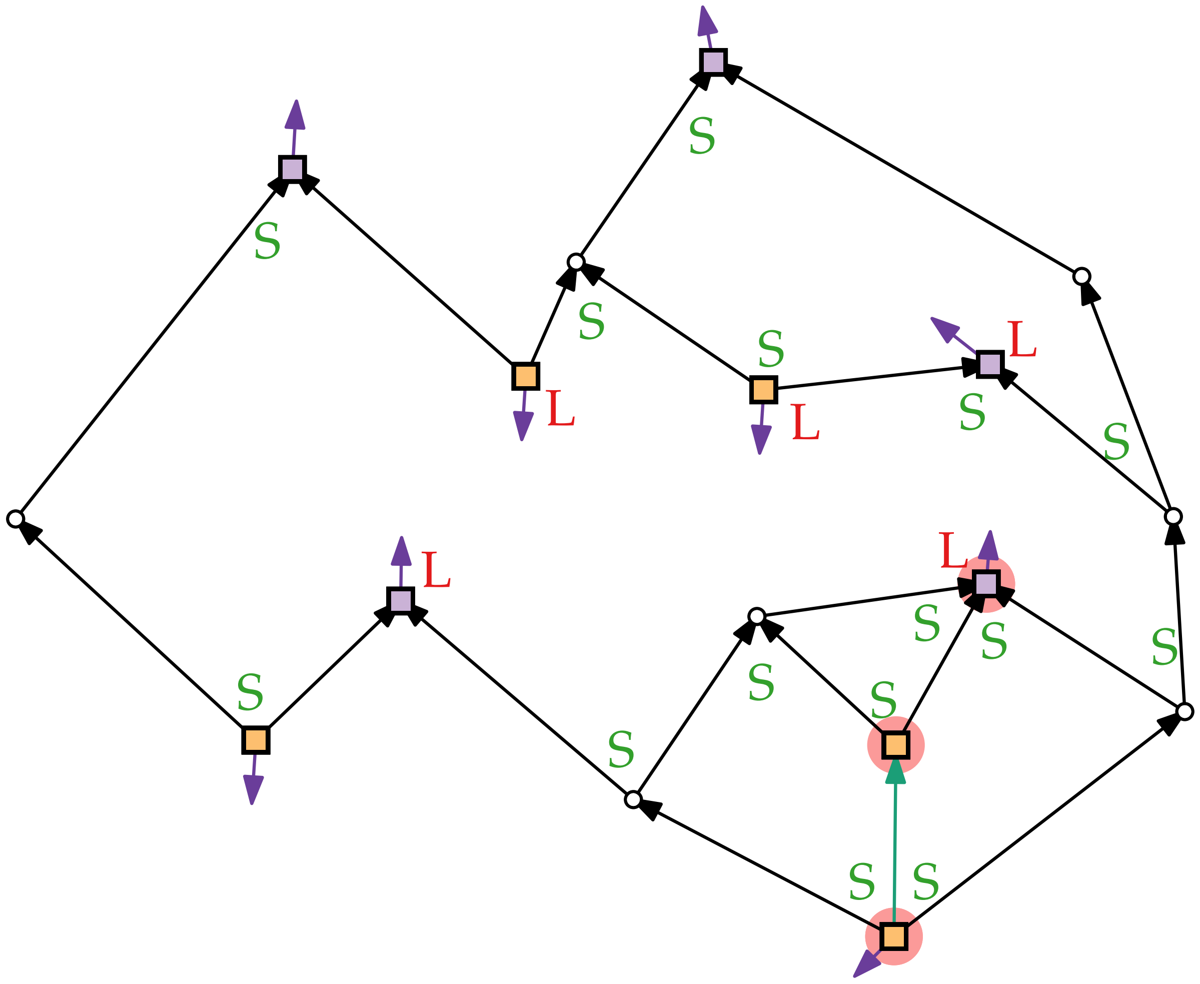
Refinement Example



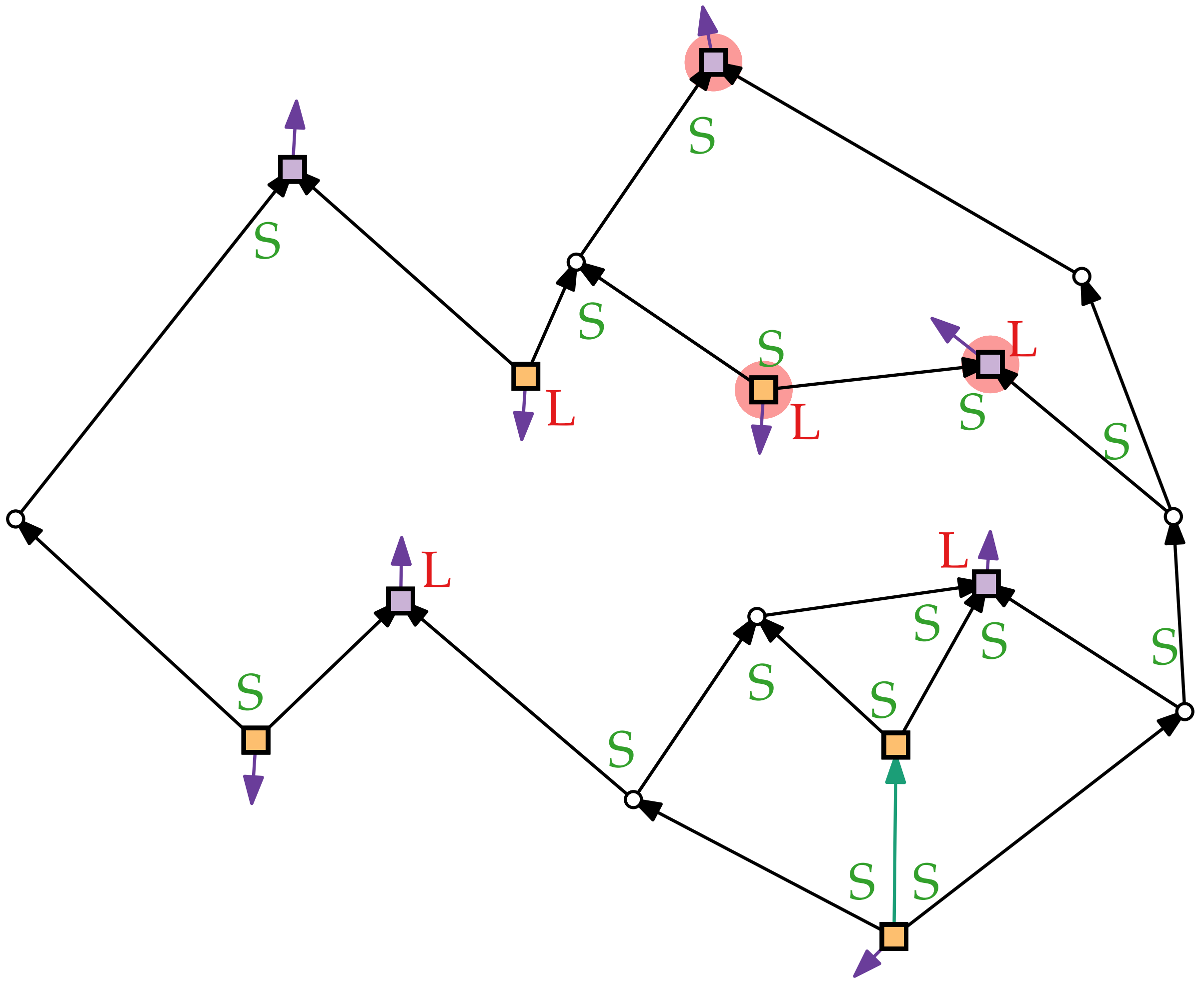
Refinement Example



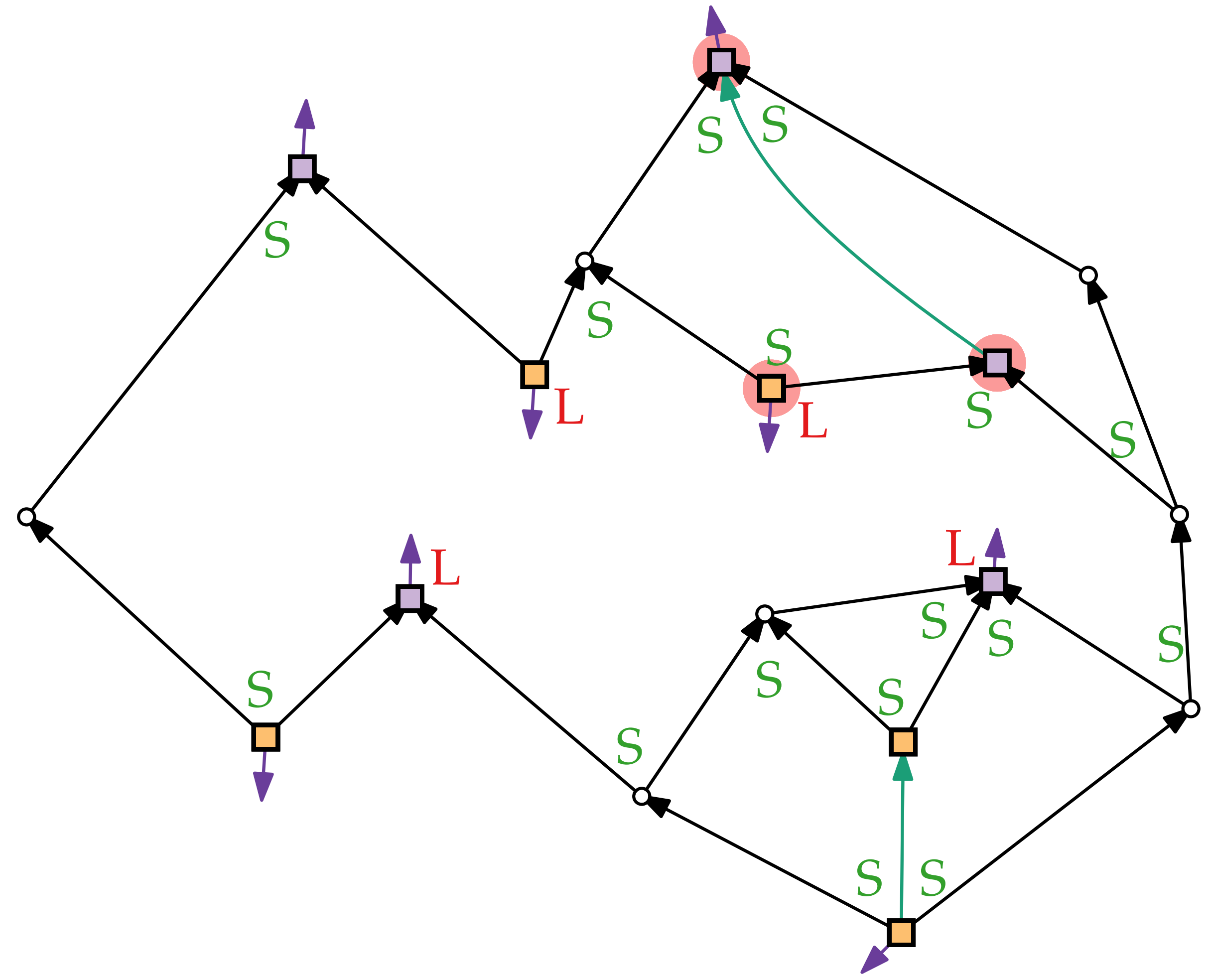
Refinement Example



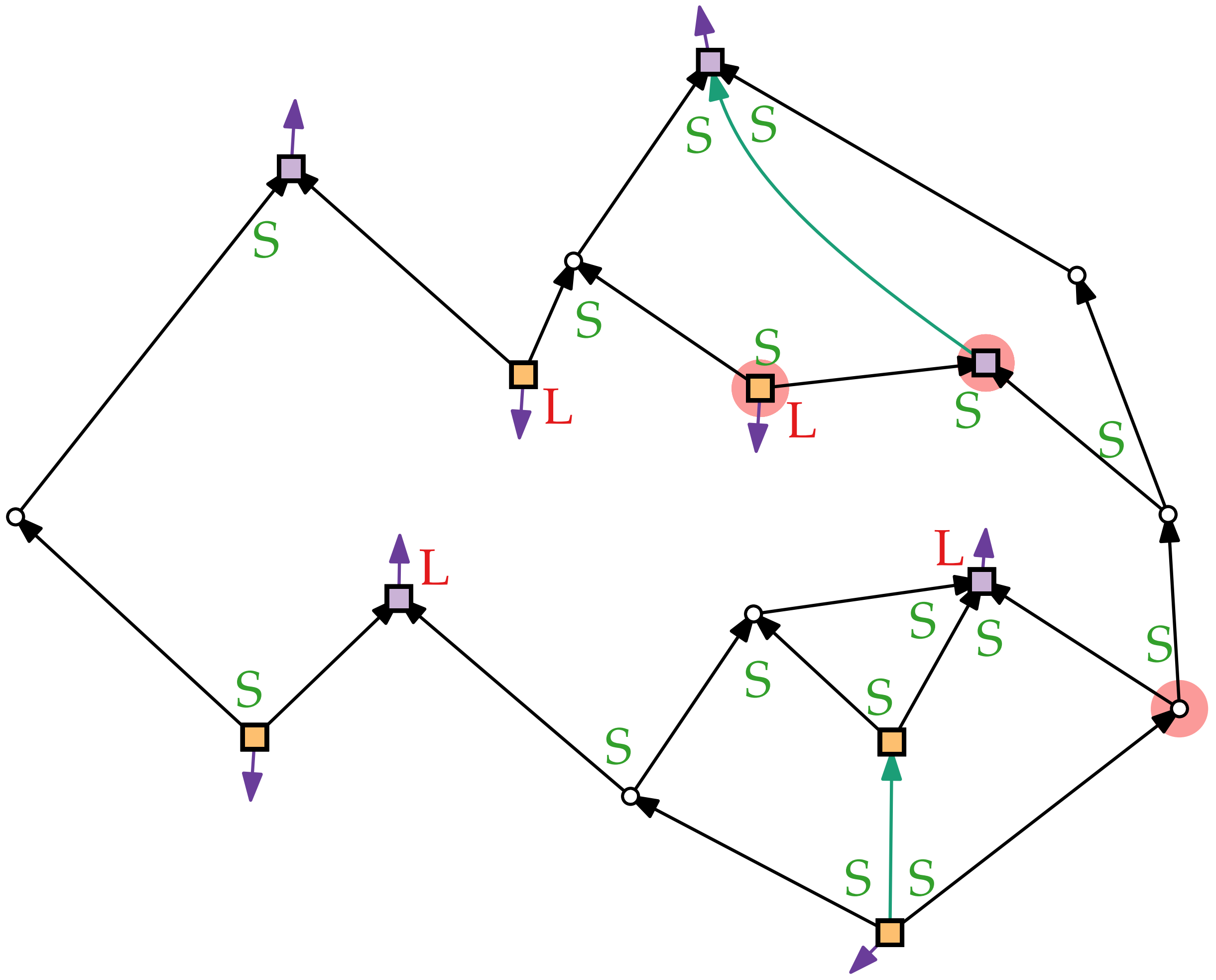
Refinement Example



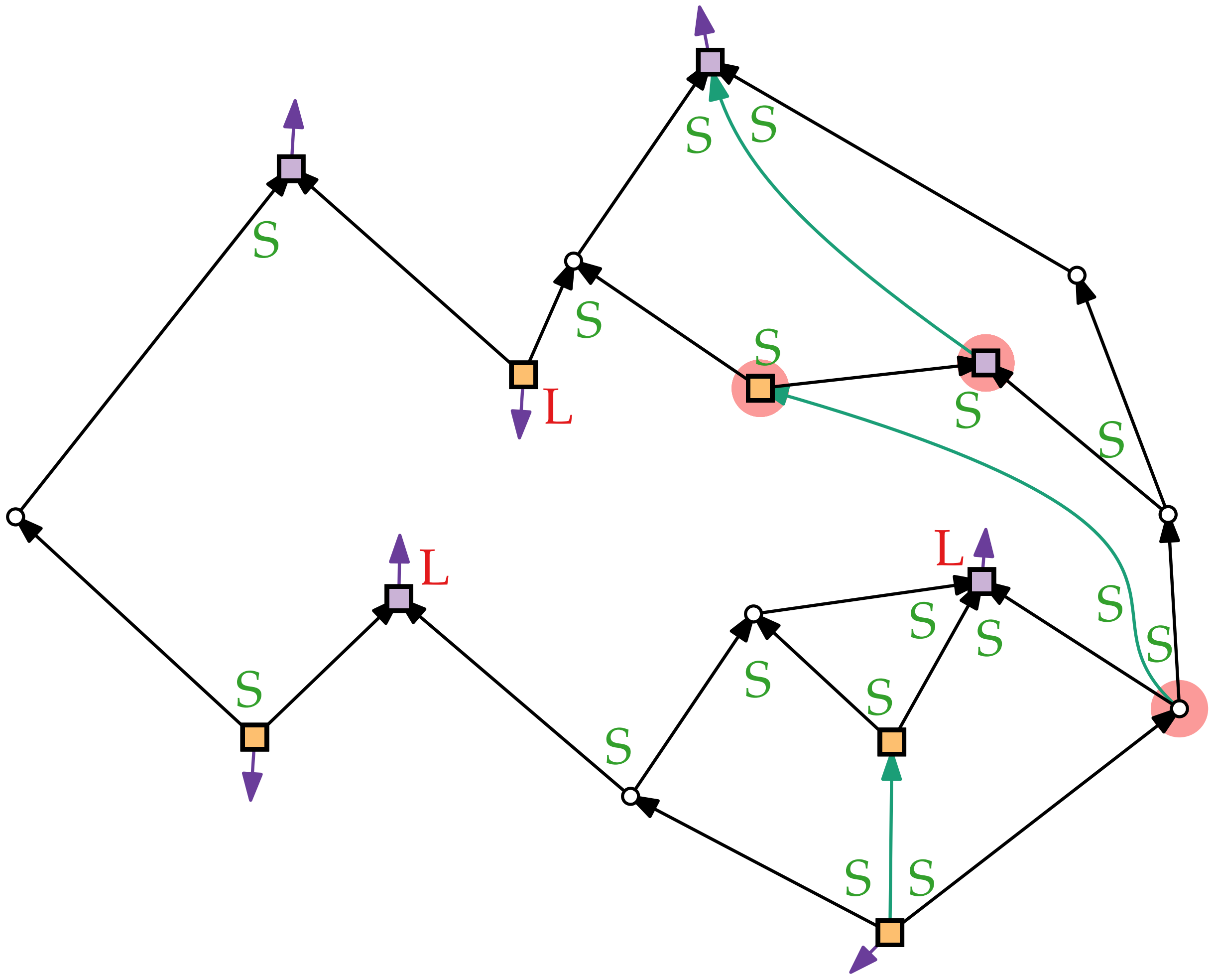
Refinement Example



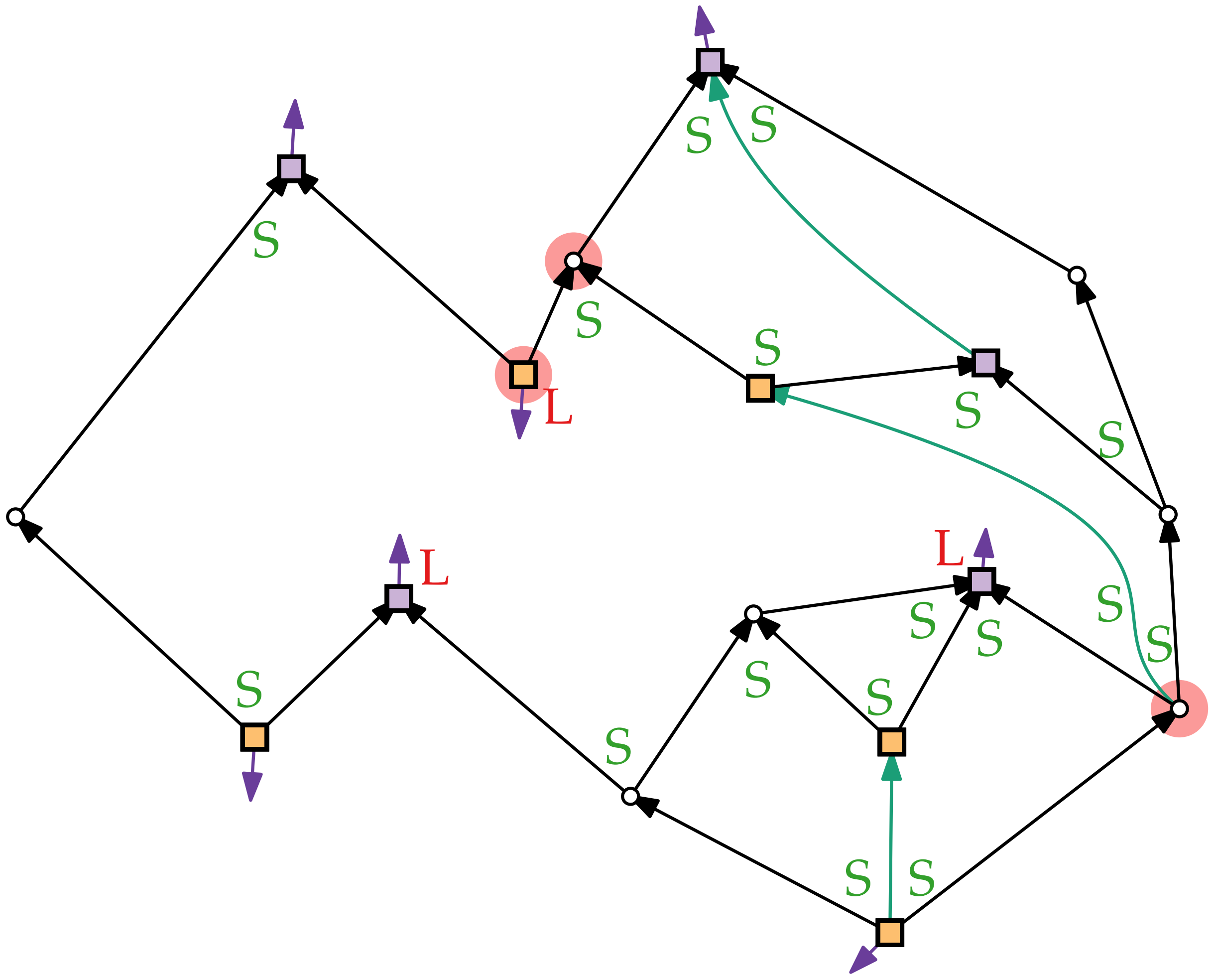
Refinement Example



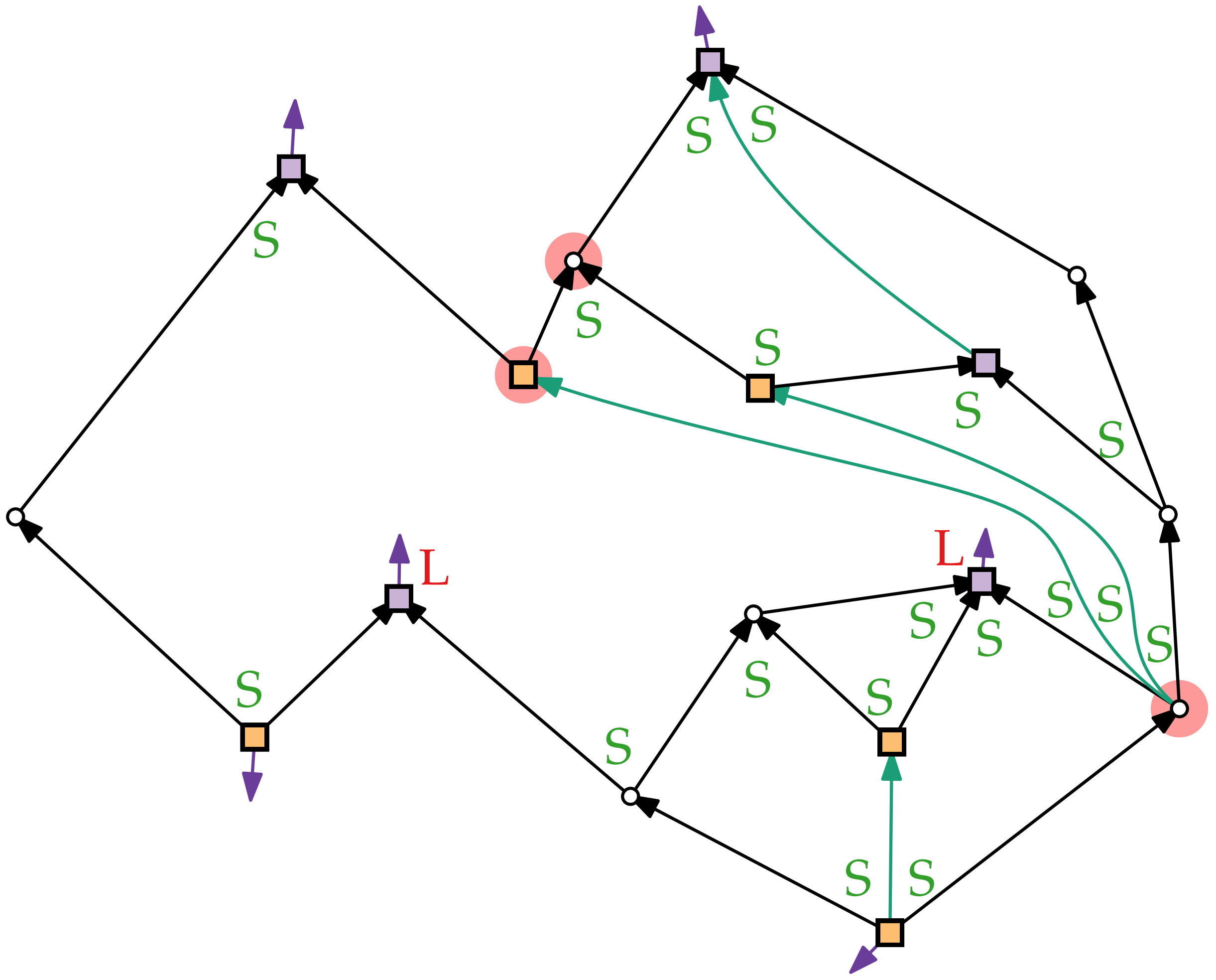
Refinement Example



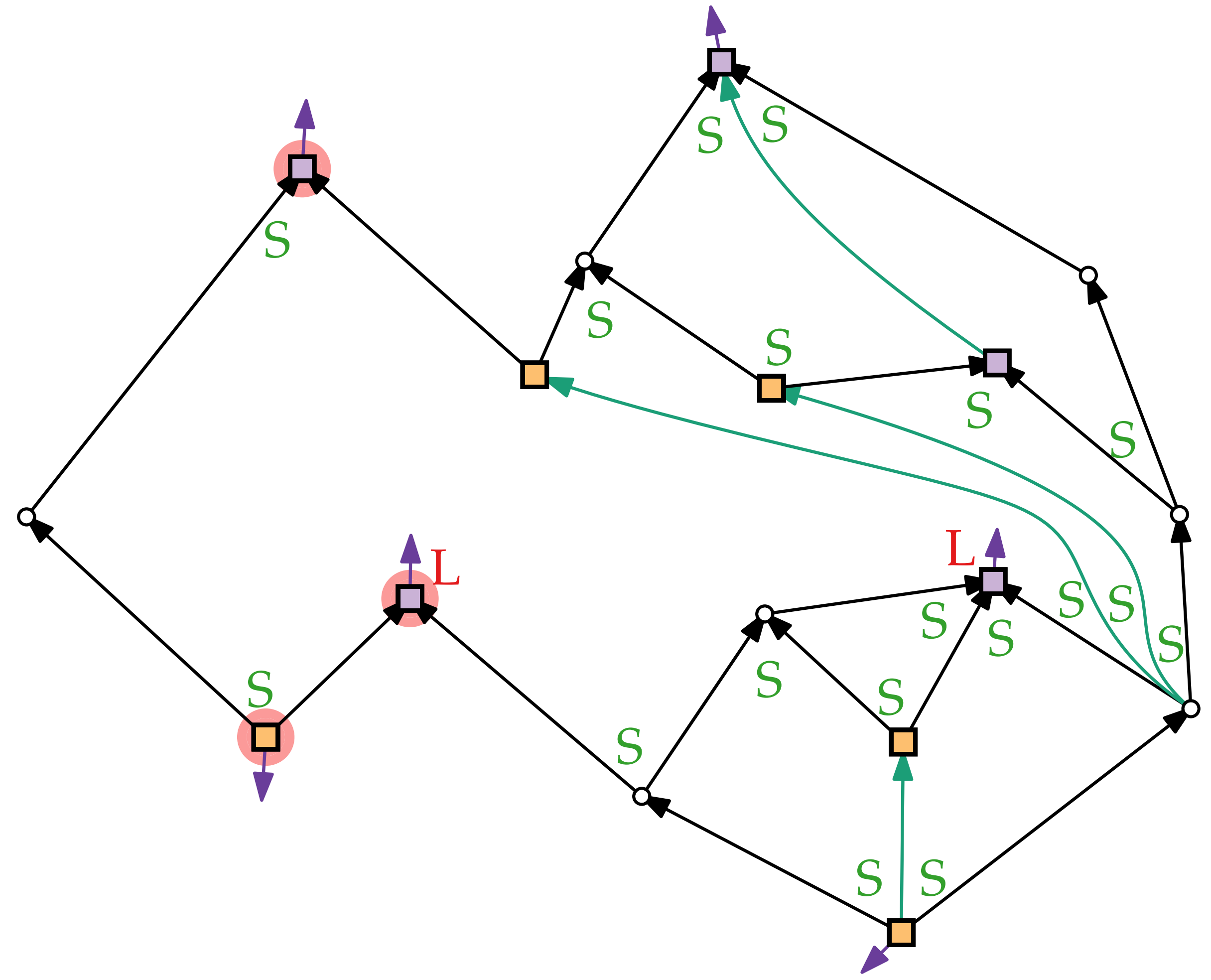
Refinement Example



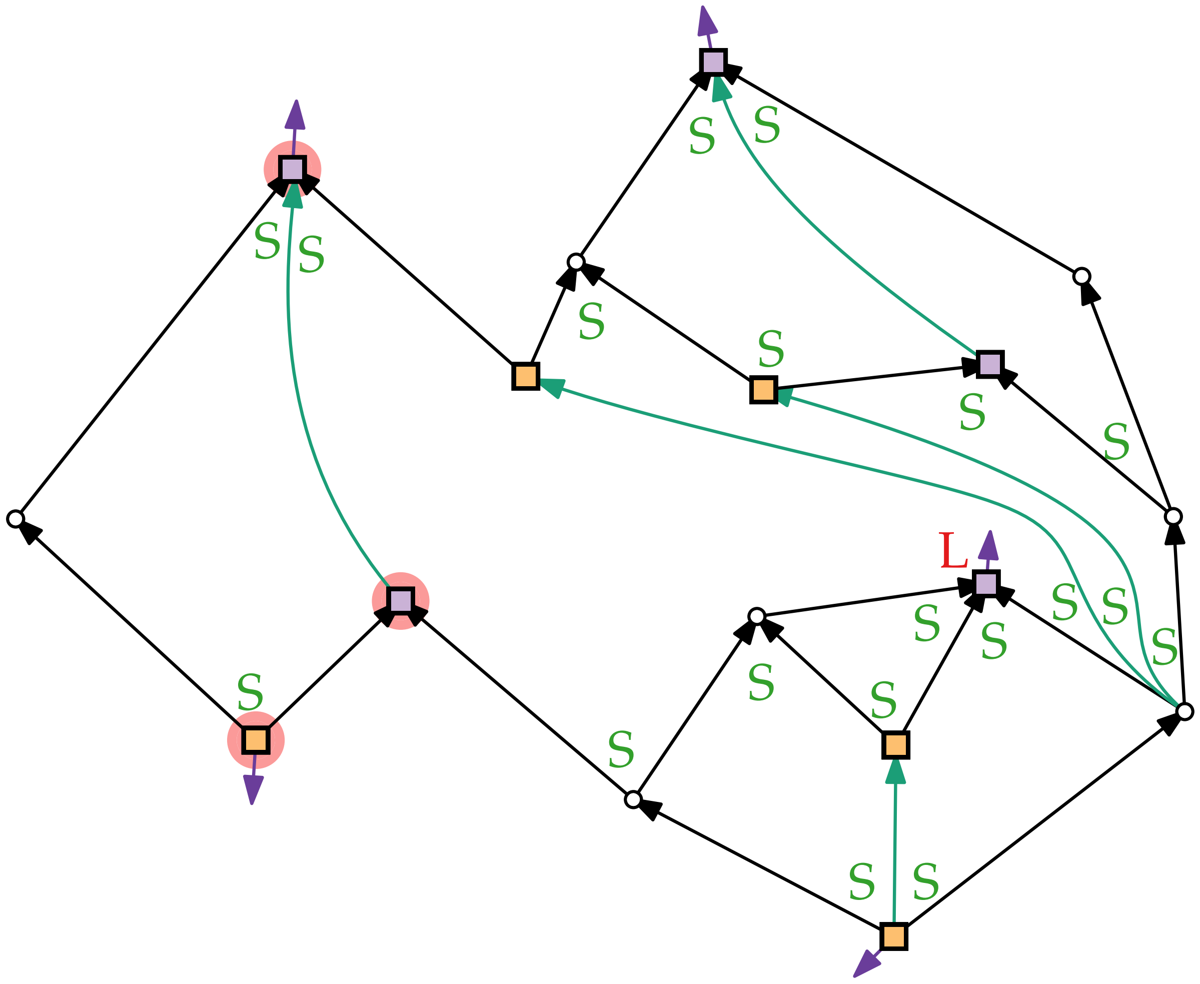
Refinement Example



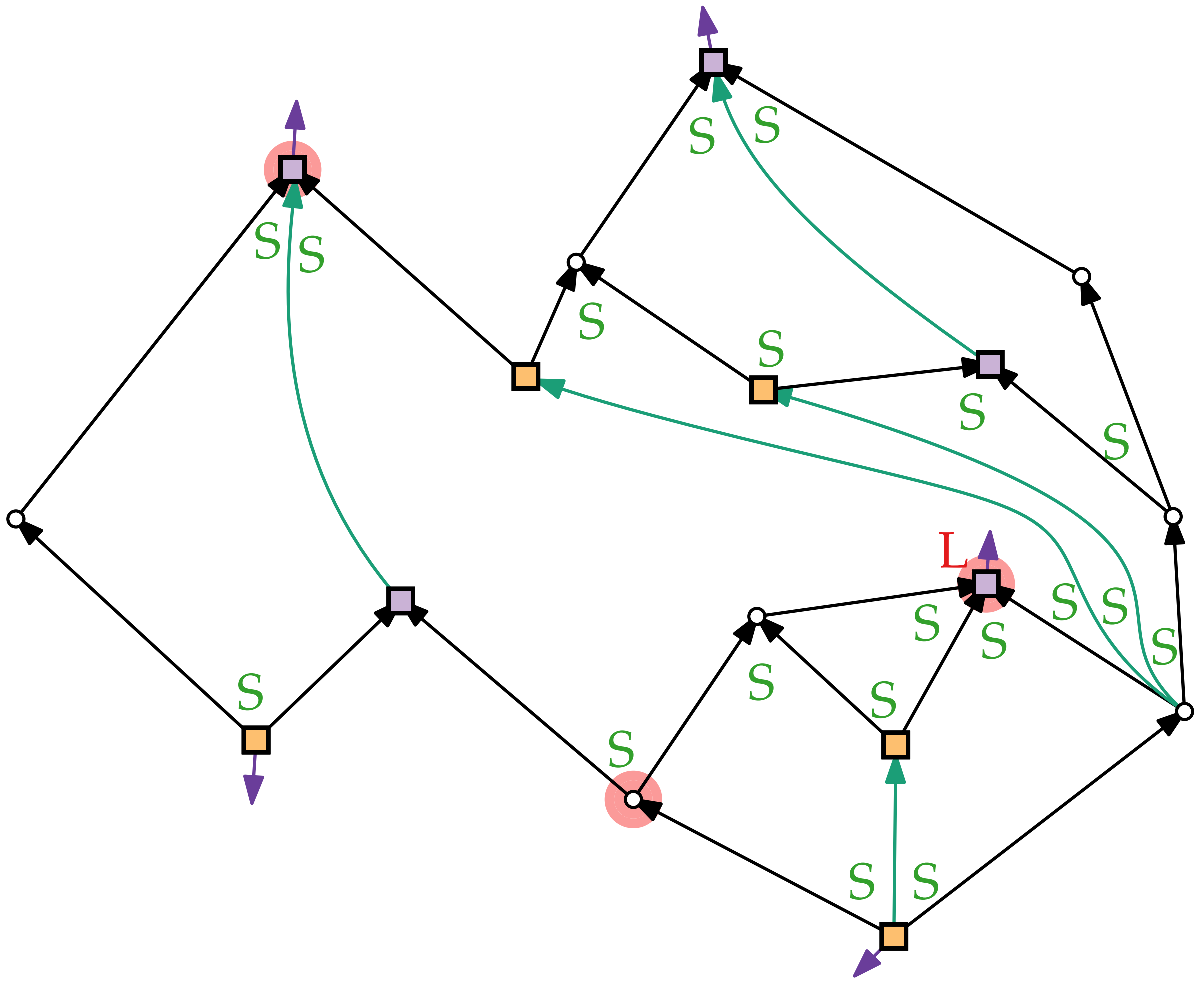
Refinement Example



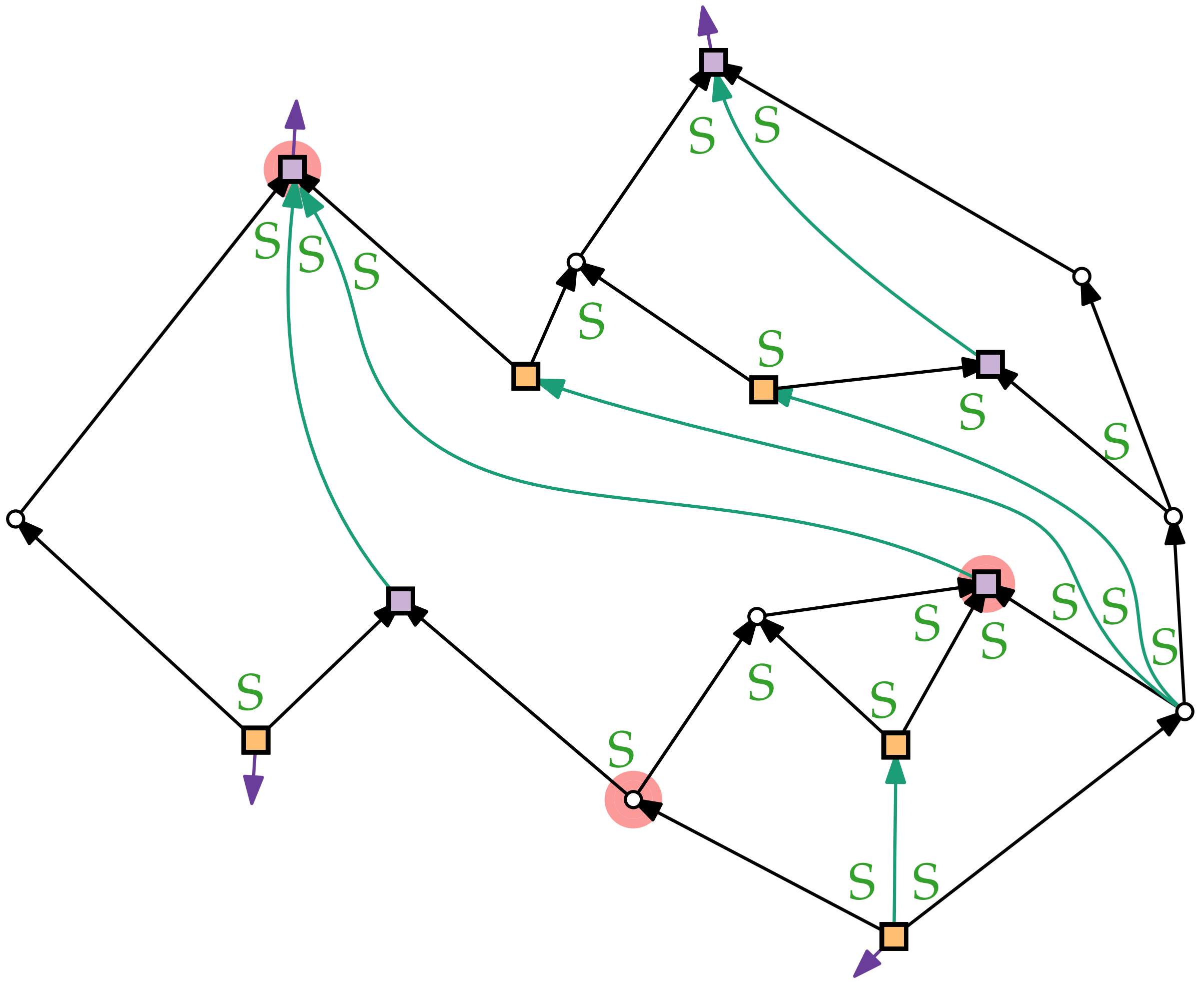
Refinement Example



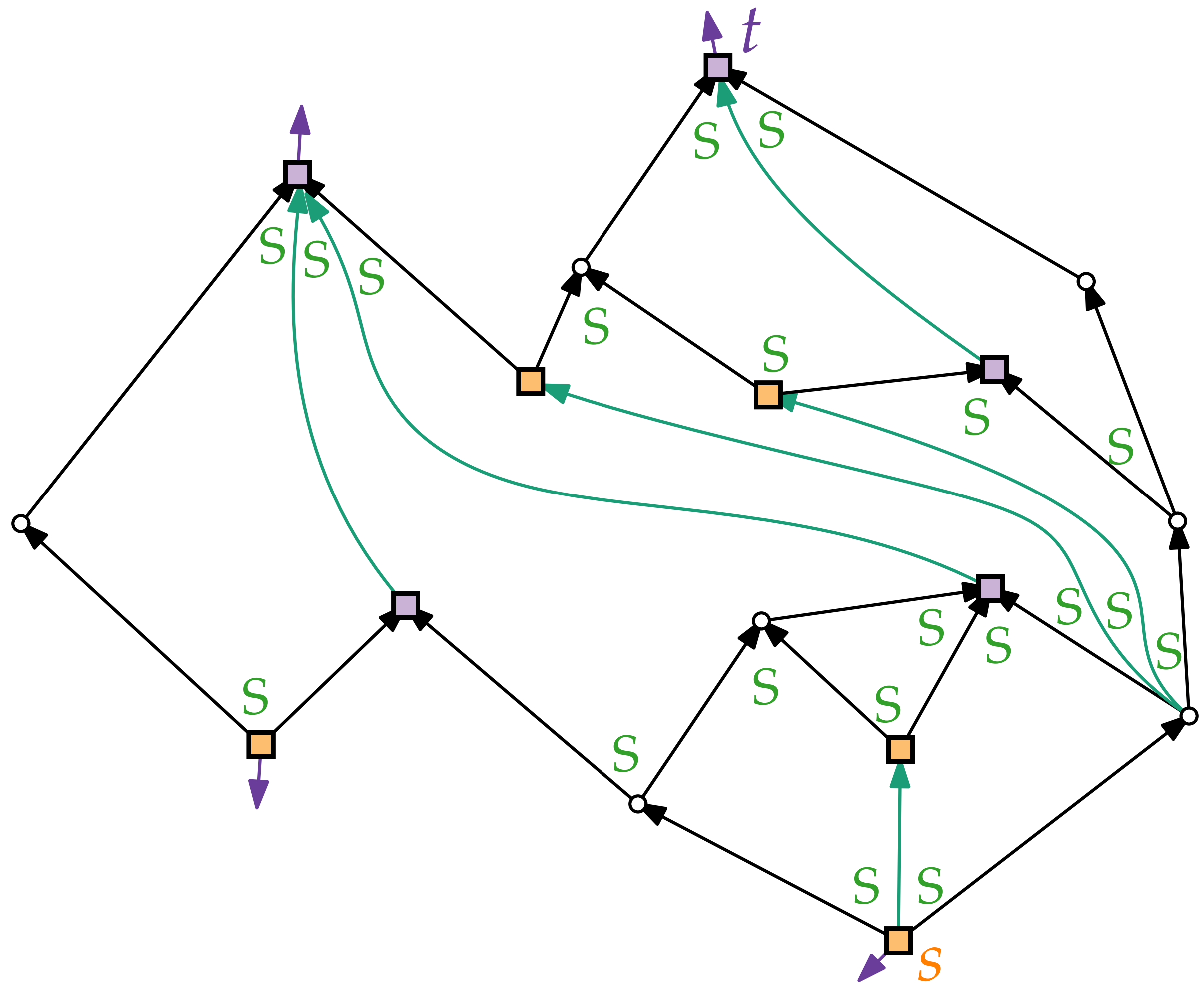
Refinement Example



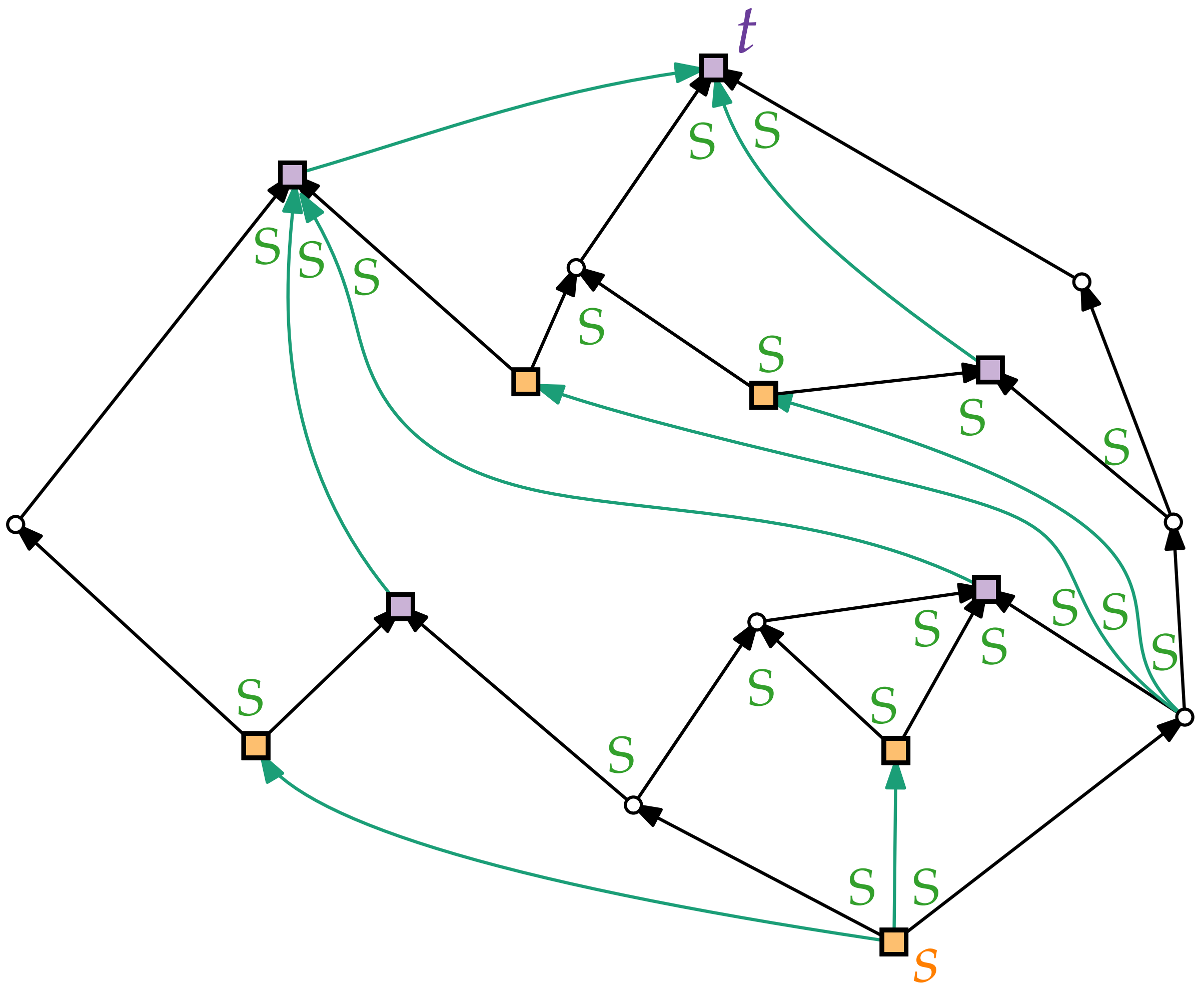
Refinement Example



Refinement Example



Refinement Example



Result Upward Planarity Test

Theorem 2. [Bertolazzi et al., 1994]
For a *combinatorially embedded* planar digraph G it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Result Upward Planarity Test

Theorem 2. [Bertolazzi et al., 1994]
For a *combinatorially embedded* planar digraph G it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Proof.

- Test for bimodality.

Result Upward Planarity Test

Theorem 2. [Bertolazzi et al., 1994]
For a *combinatorially embedded* planar digraph G it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Proof.

- Test for bimodality.
- Test for a consistent assignment Φ (via flow network).

Result Upward Planarity Test

Theorem 2. [Bertolazzi et al., 1994]
For a *combinatorially embedded* planar digraph G it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Proof.

- Test for bimodality.
- Test for a consistent assignment Φ (via flow network).
- If G bimodal and Φ exists, refine G to plane st-digraph H.

Result Upward Planarity Test

Theorem 2. [Bertolazzi et al., 1994]
For a *combinatorially embedded* planar digraph G it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Proof.

- Test for bimodality.
- Test for a consistent assignment Φ (via flow network).
- If G bimodal and Φ exists, refine G to plane st-digraph H.
- Draw H upward planar.

Result Upward Planarity Test

Theorem 2. [Bertolazzi et al., 1994]
For a *combinatorially embedded* planar digraph G it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Proof.

- Test for bimodality.
- Test for a consistent assignment Φ (via flow network).
- If G bimodal and Φ exists, refine G to plane st-digraph H.
- Draw H upward planar.
- Deleted edges added in refinement step.

Discussion

- There exist fixed-parameter tractable algorithms to test upward planarity of general digraphs with the parameter being the number of triconnected components.
  [Healy, Lynch 2005, Didimo et al. 2009]

Discussion

- There exist fixed-parameter tractable algorithms to test upward planarity of general digraphs with the parameter being the number of triconnected components.
  [Healy, Lynch 2005, Didimo et al. 2009]

- Finding assignment in Theorem 2 can be sped up to $\mathcal{O}(n + r^{1.5})$ where $r = \#$ sources / sinks.
  [Abbasi, Healy, Rextin 2010]

Discussion

- There exist fixed-parameter tractable algorithms to test upward planarity of general digraphs with the parameter being the number of triconnected components.
  [Healy, Lynch 2005, Didimo et al. 2009]

- Finding assignment in Theorem 2 can be sped up to $\mathcal{O}(n + r^{1.5})$ where $r = \#$ sources / sinks.
  [Abbasi, Healy, Rextin 2010]

- Many related concepts have been studied: quasi-planarity, upward drawings of mixed graphs, upward planarity on cylinder/torus, ...