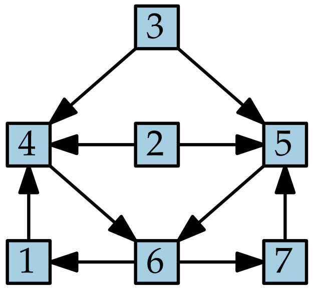
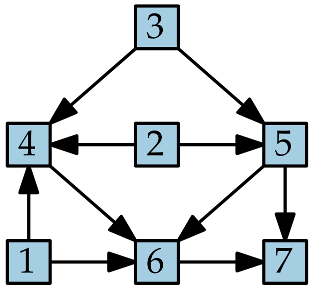
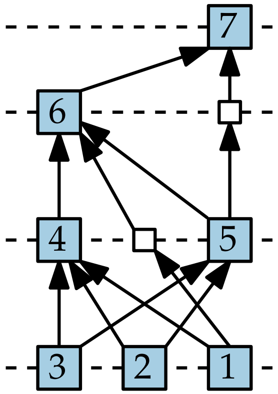
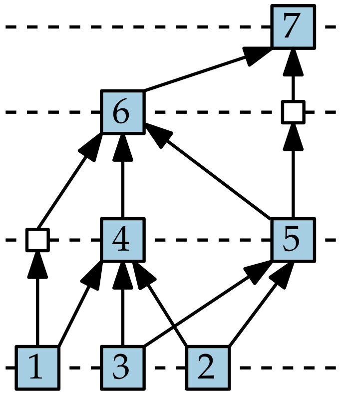
Universität Trier

# Visualization of Graphs

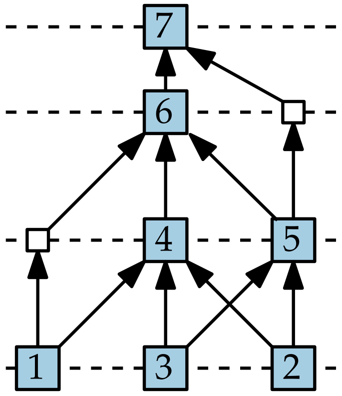


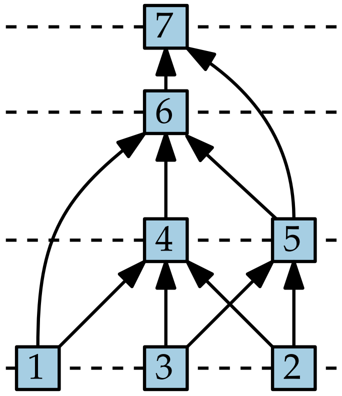






## Lecture 8: Hierarchical Layouts: Sugiyama Framework

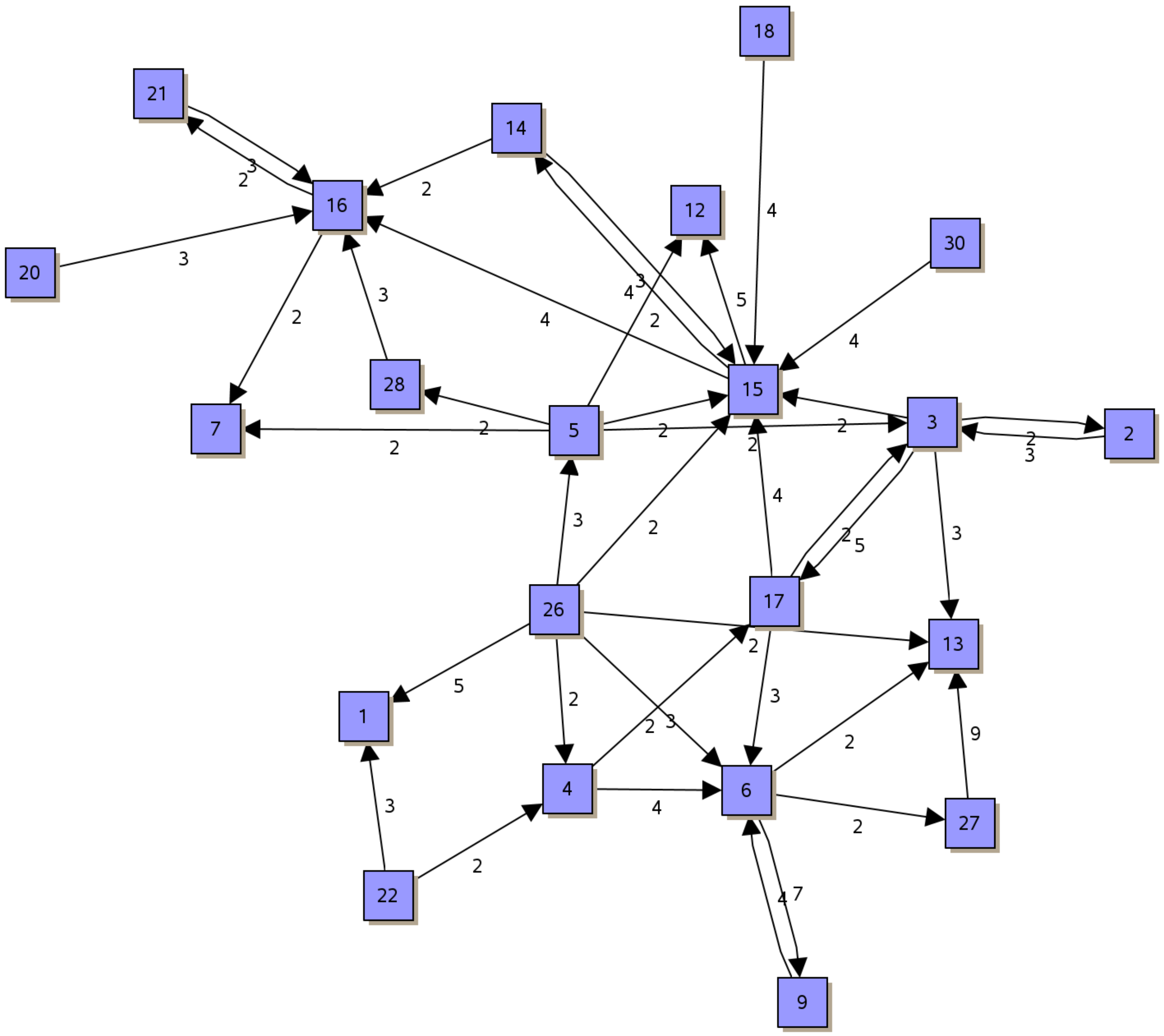




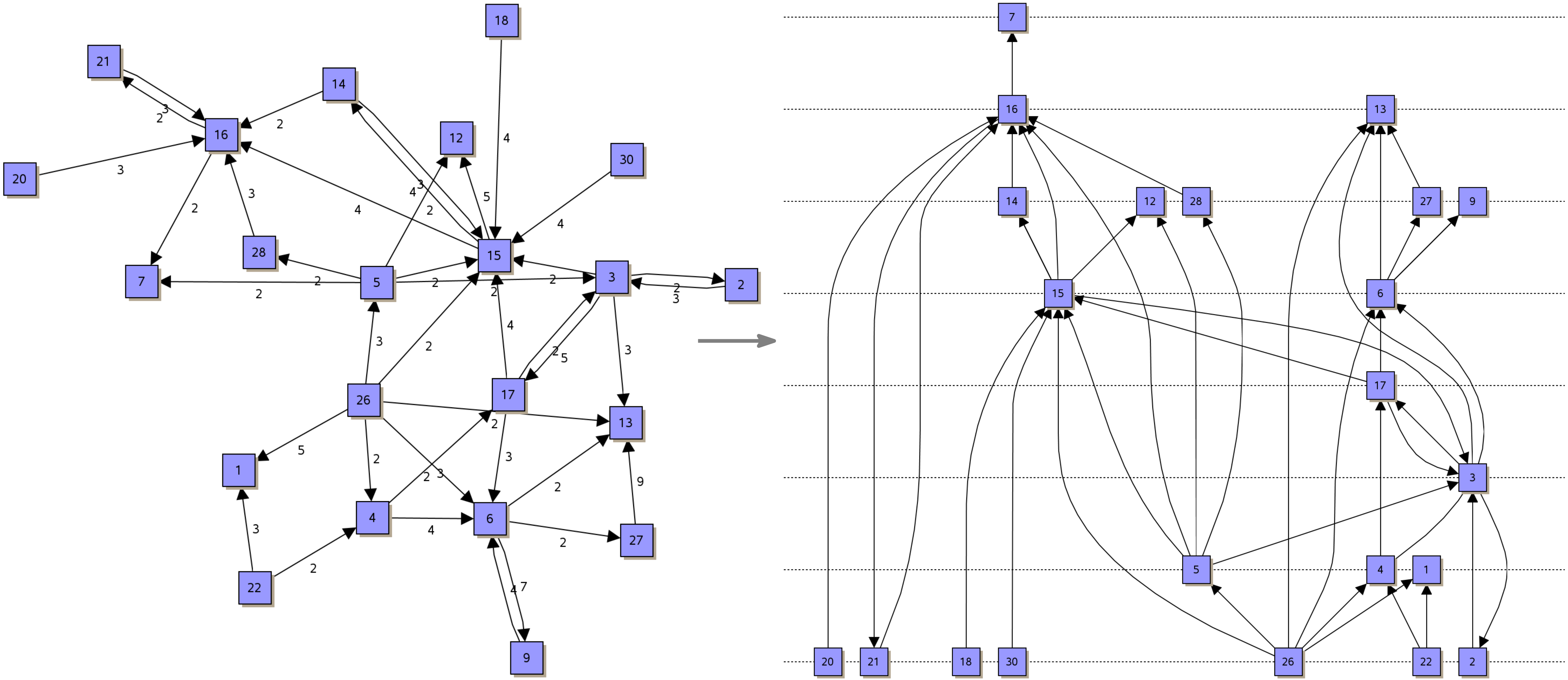
### Part I: The Framework

Philipp Kindermann

# Hierarchical Drawings – Motivation



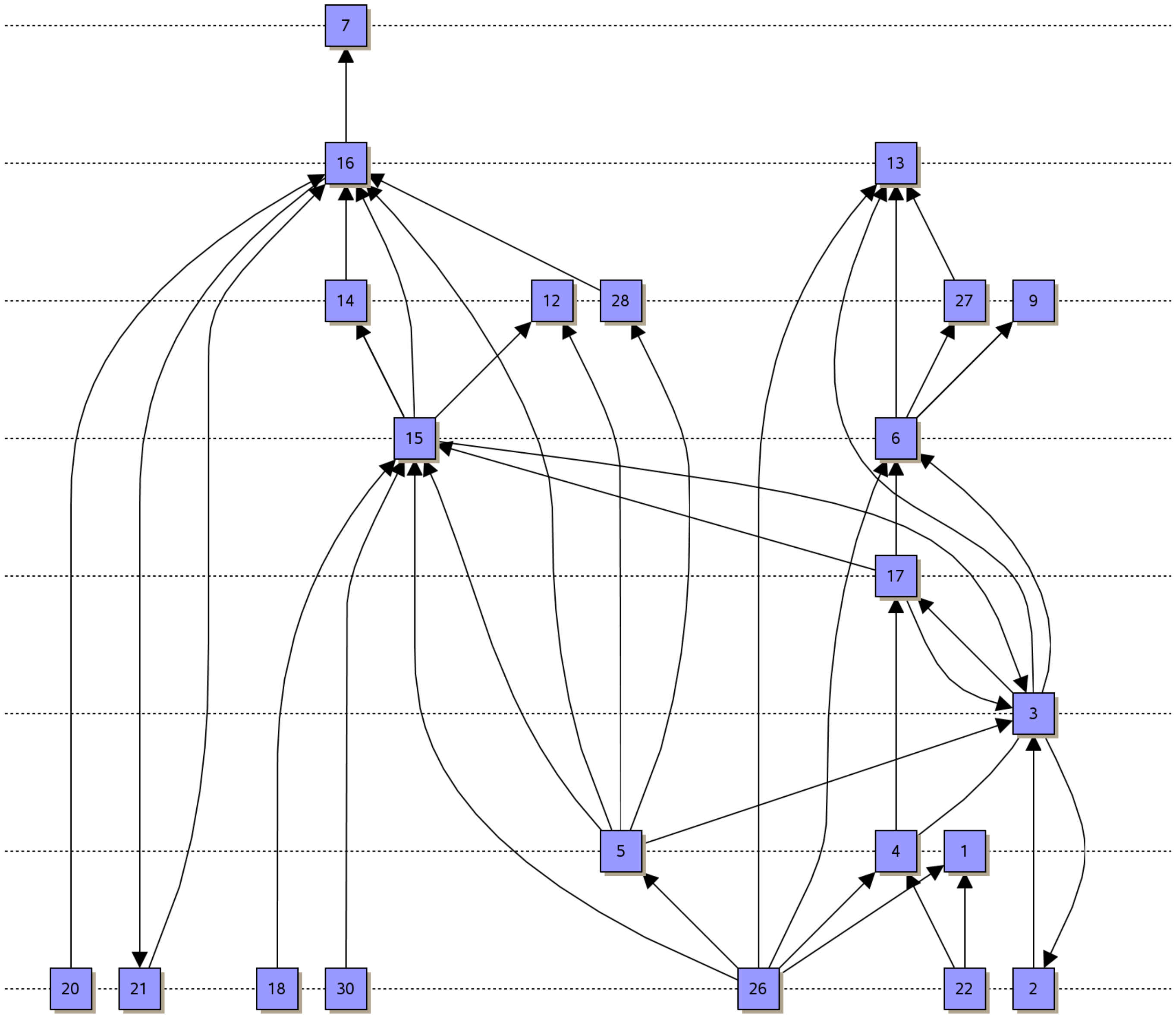
# Hierarchical Drawings – Motivation



# Hierarchical Drawing

## Problem Statement.

- Input: digraph $G = (V, E)$
- Output: drawing of $G$ that "closely" reproduces the hierarchical properties of $G$



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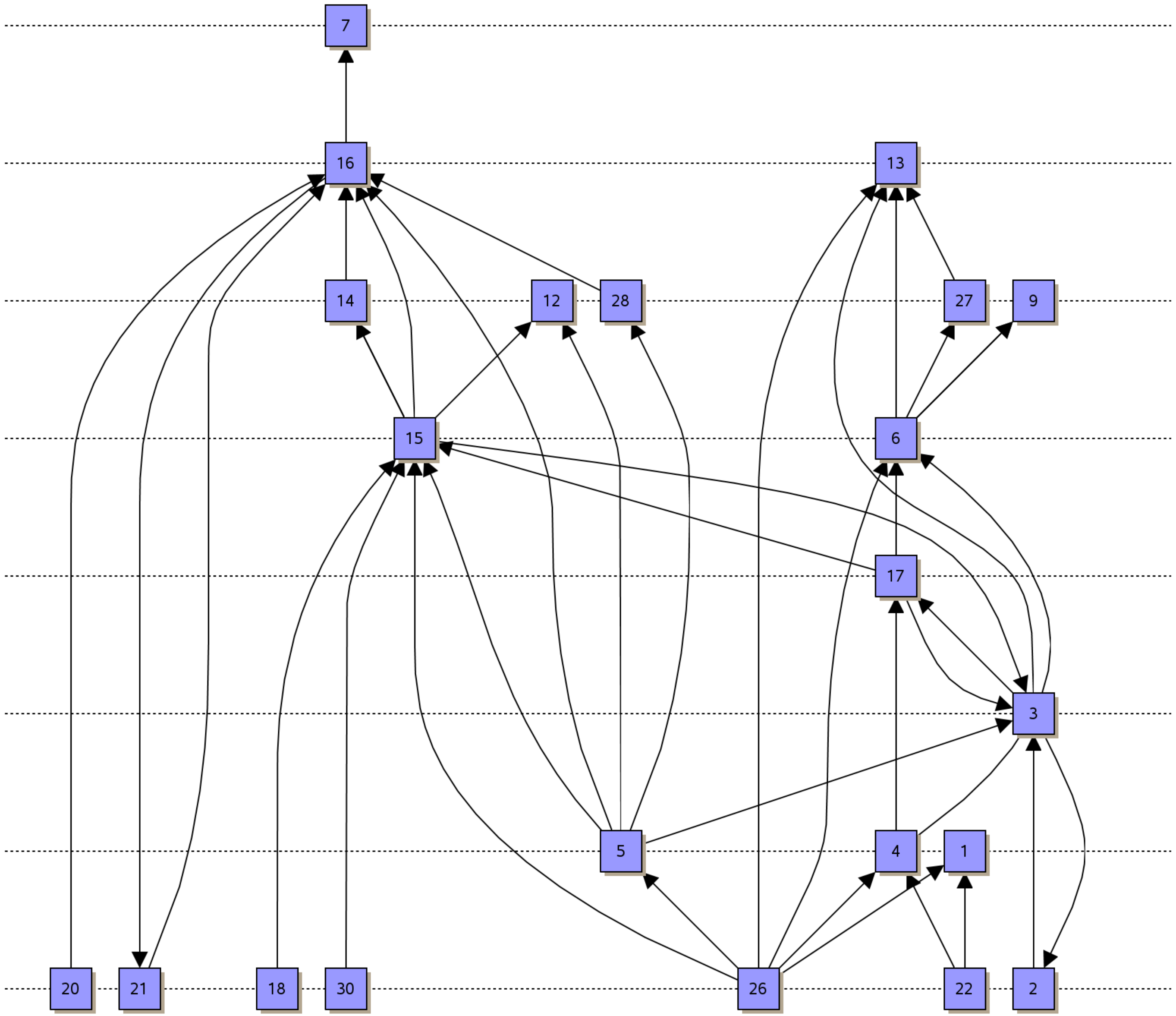


## Desirable Properties.

# Hierarchical Drawing

## Problem Statement.

- Input: digraph $G = (V, E)$
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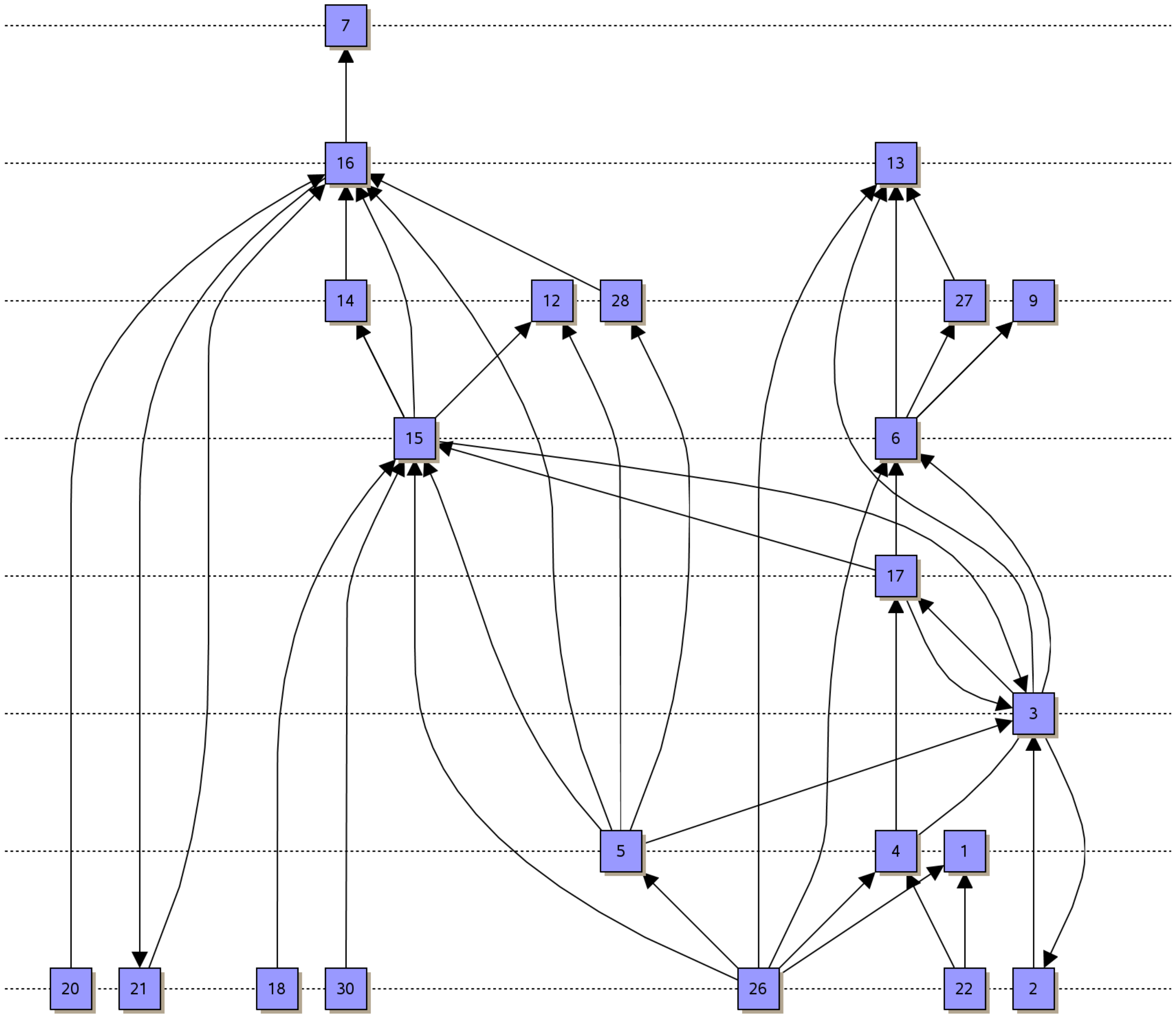
## Desirable Properties.

- vertices occur on (few) horizontal lines



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- Input: digraph $G = (V, E)$
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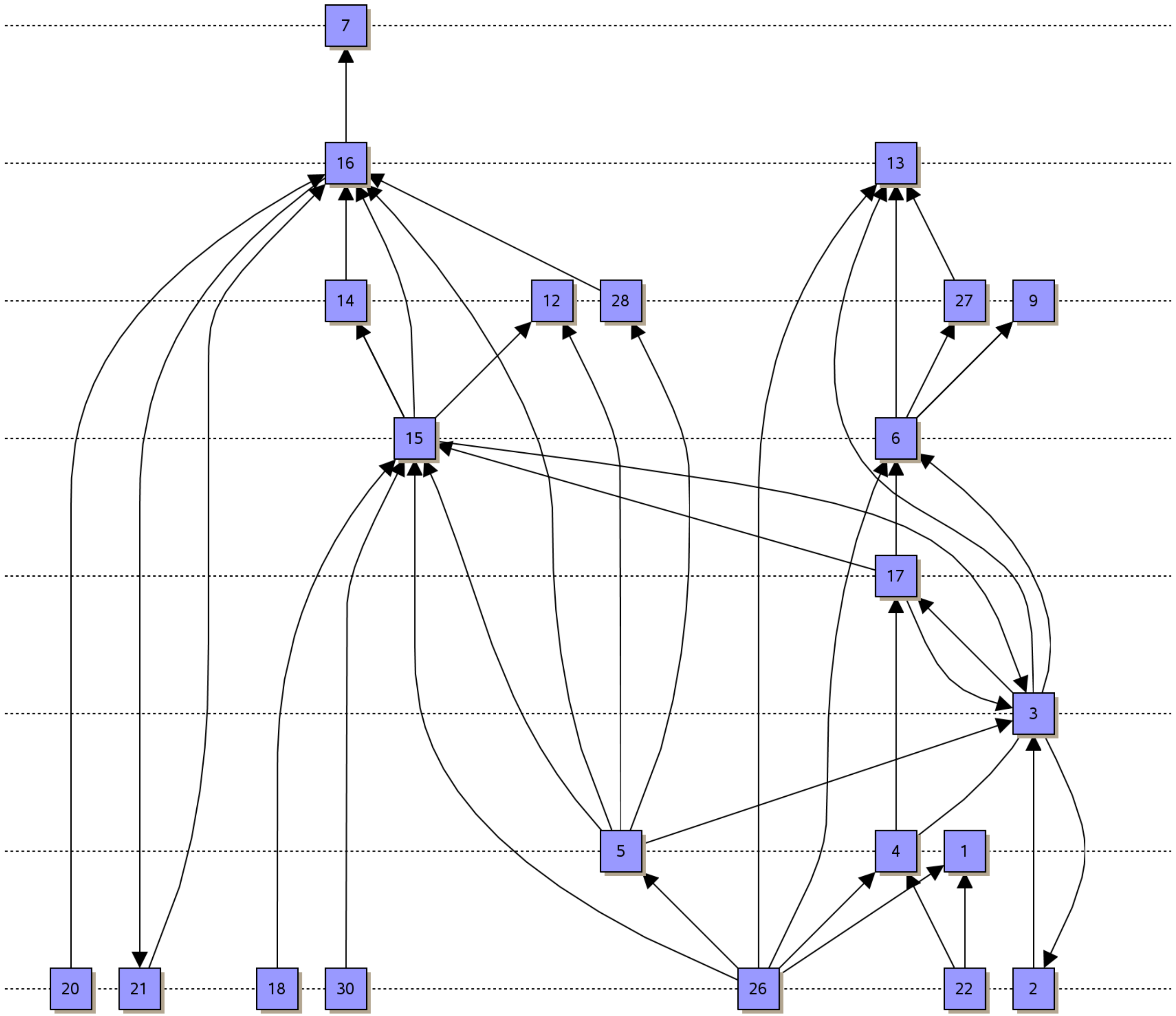
## Desirable Properties.

- vertices occur on (few) horizontal lines
- edges directed upwards



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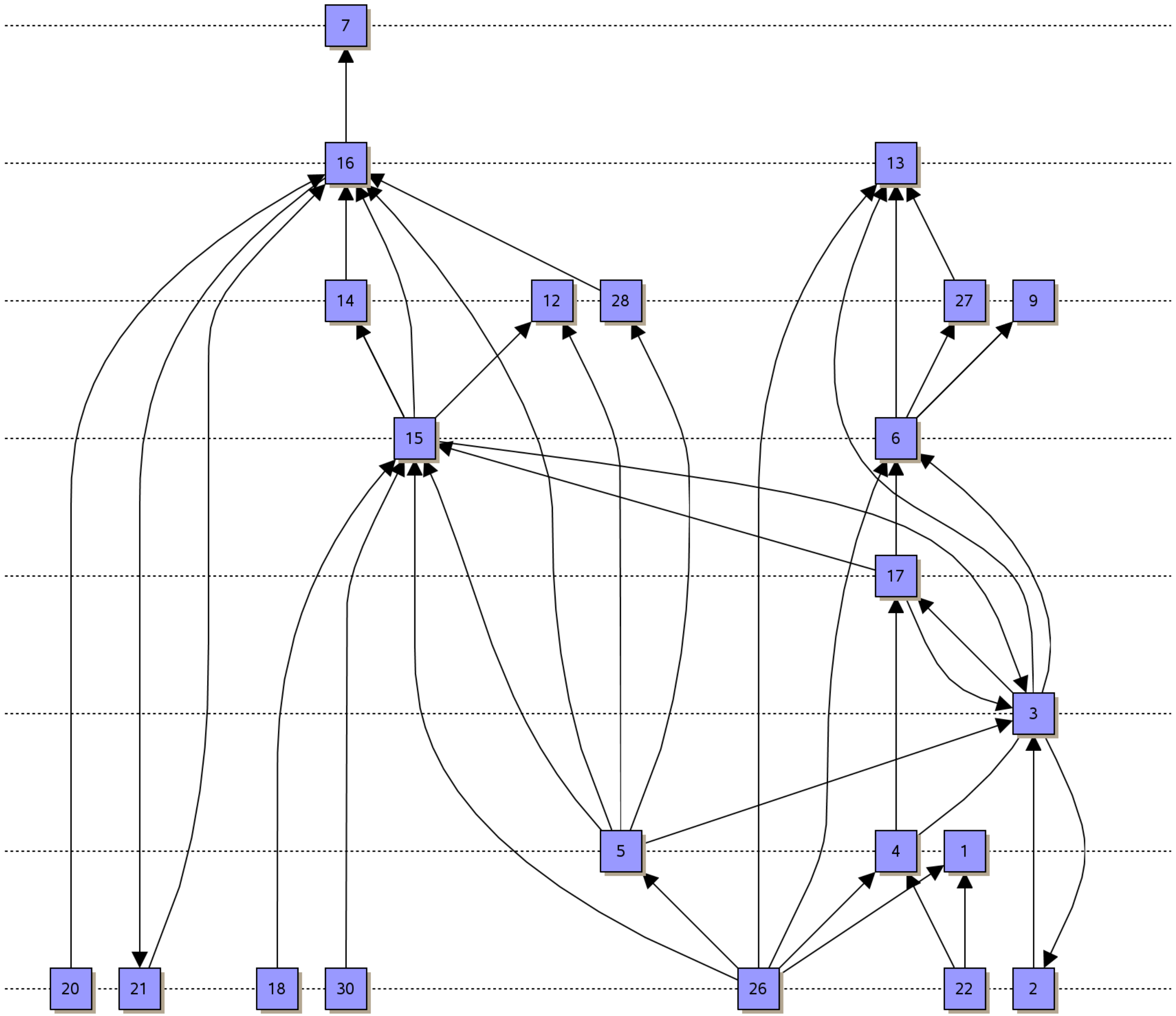


## Desirable Properties.

- vertices occur on (few) horizontal lines
- edges directed upwards
- edge crossings minimized

# Hierarchical Drawing

## Problem Statement.

- Input: digraph $G = (V, E)$
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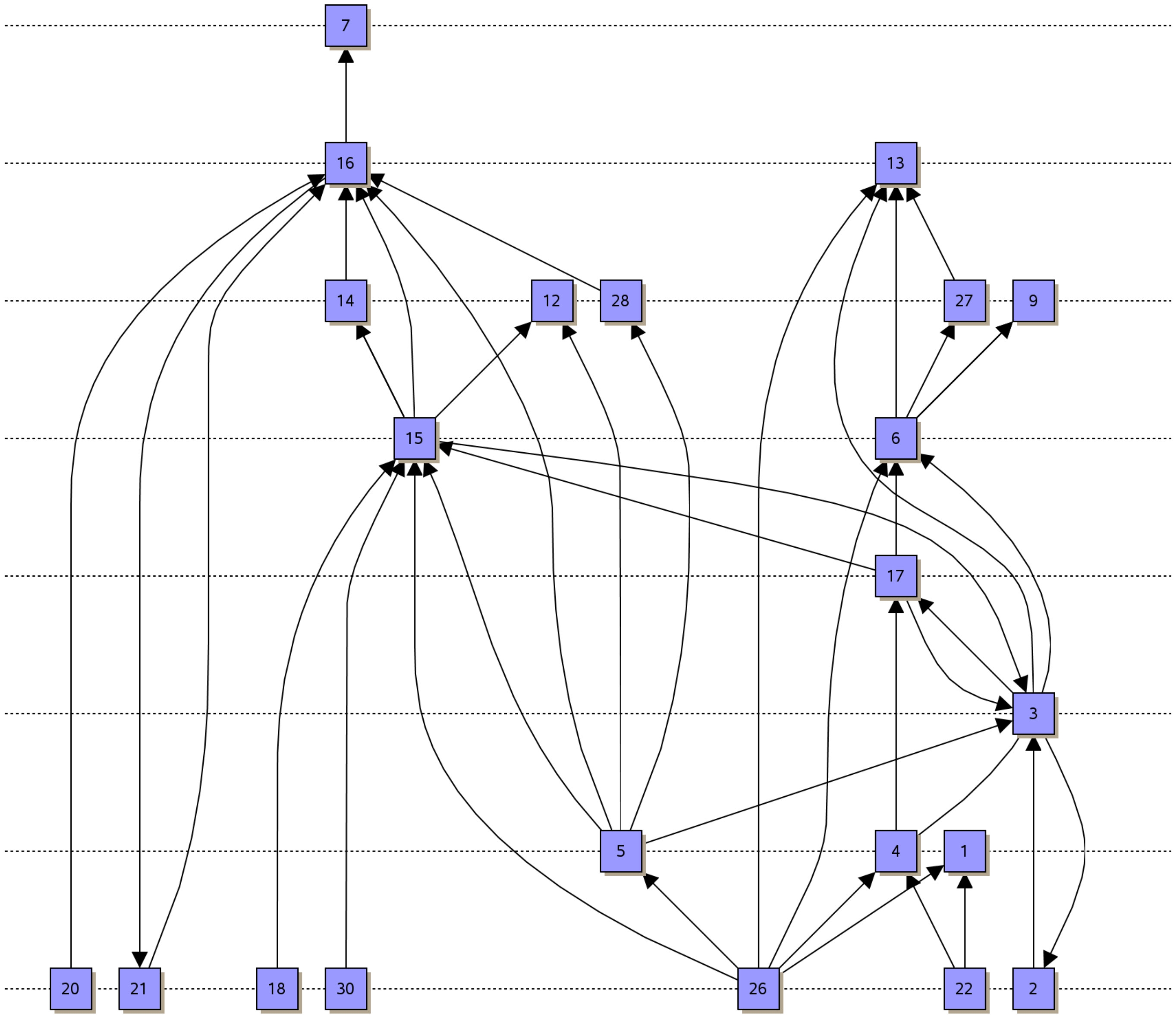
## Desirable Properties.

- vertices occur on (few) horizontal lines
- edges directed upwards
- edge crossings minimized
- edges as short as possible



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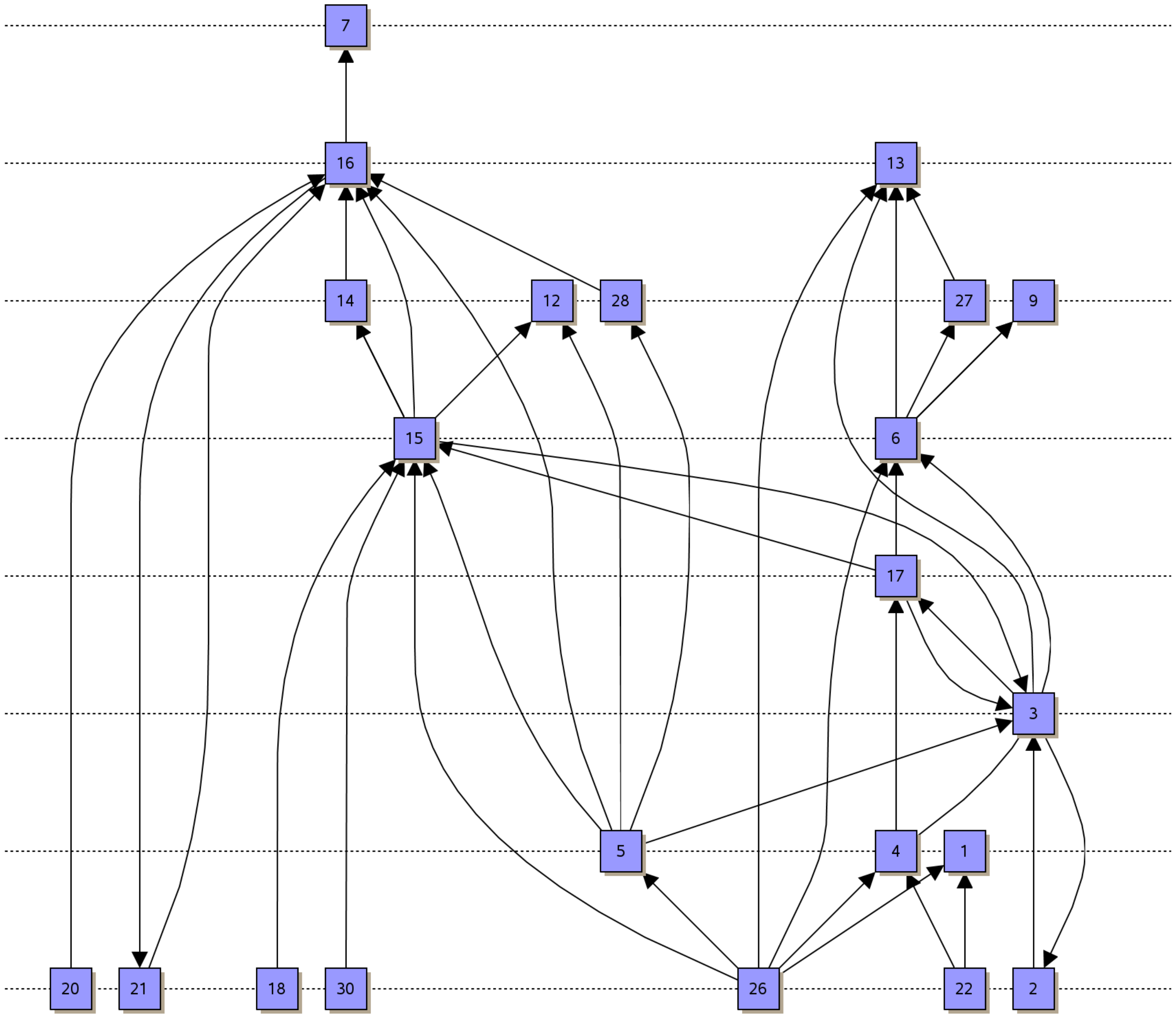
- vertices evenly spaced



# Hierarchical Drawing

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## Desirable Properties.

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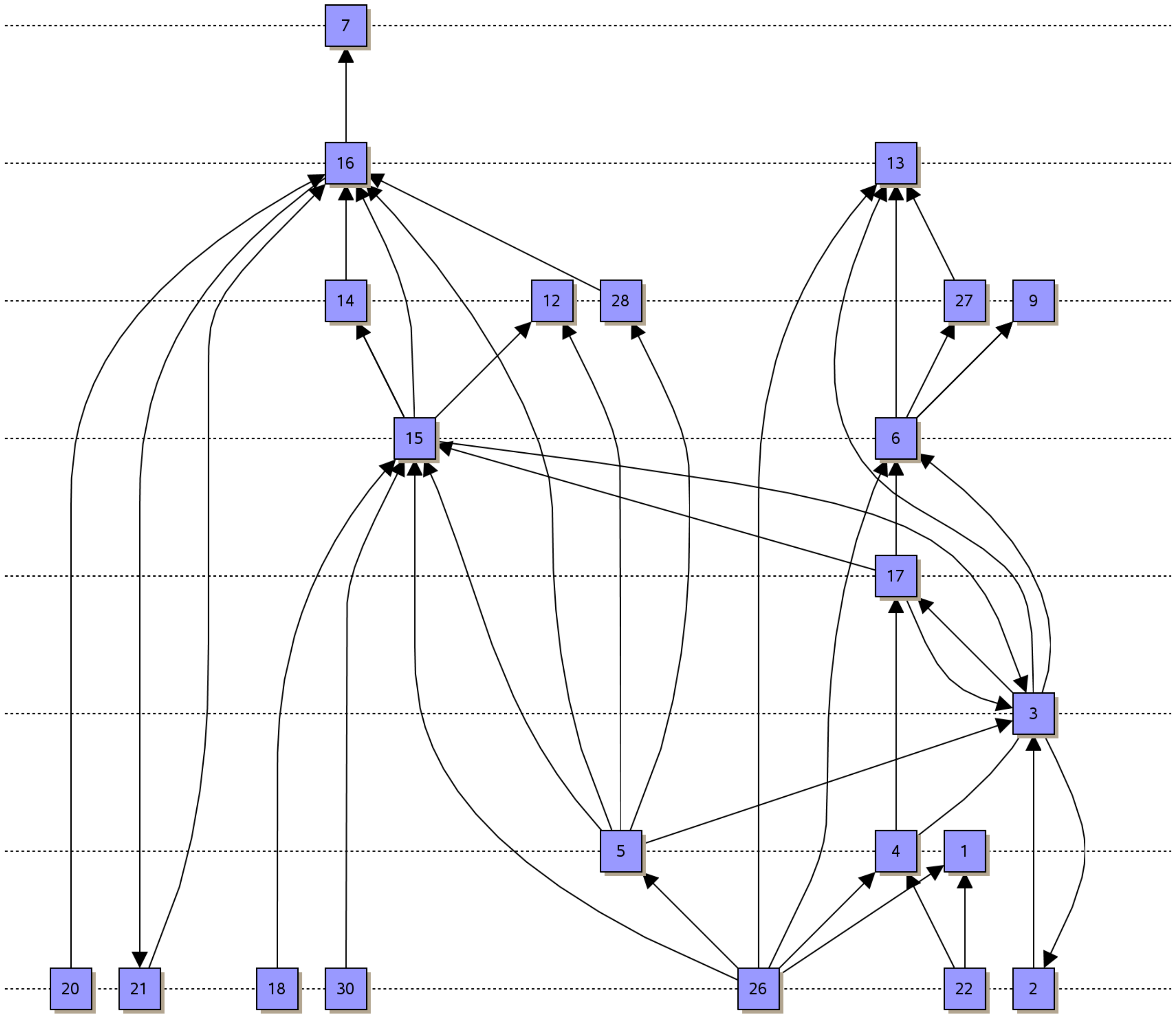
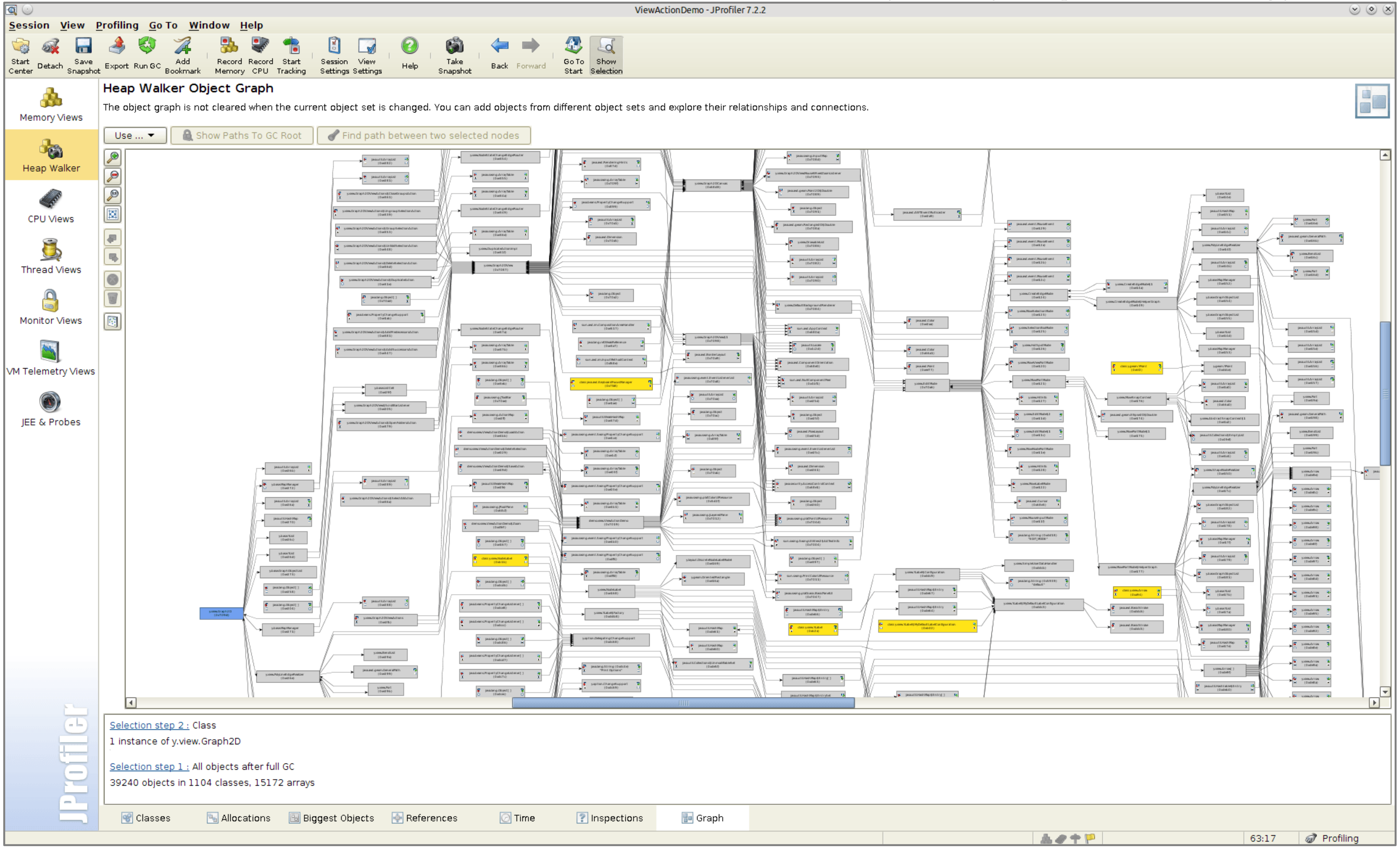
- edges directed upwards
- edge crossings minimized
- edges as short as possible
- vertices evenly spaced

Criteria can be contradictory!



# Hierarchical Drawing – Applications

yEd Gallery: Java profiler JProfiler using yFiles



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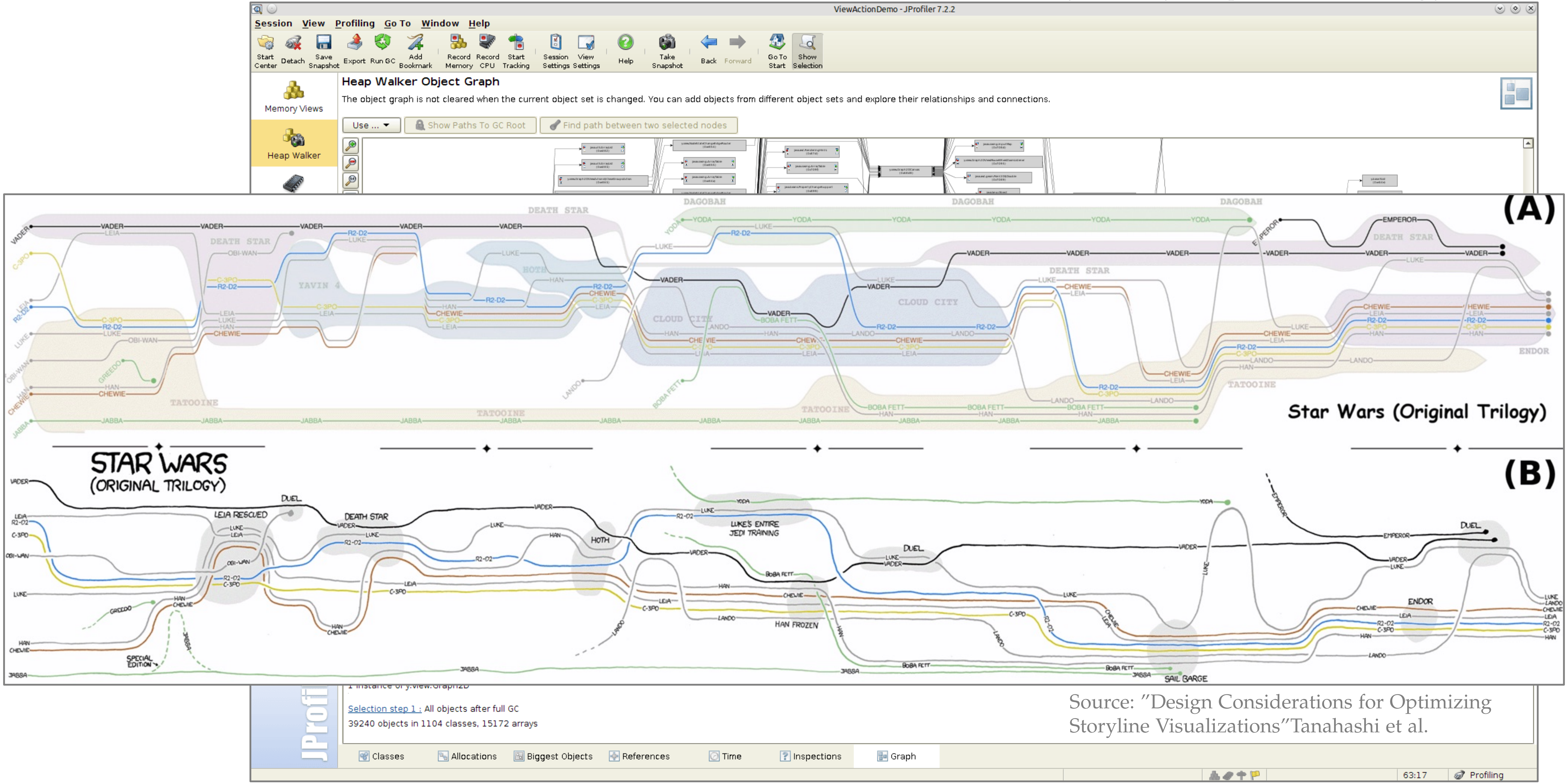
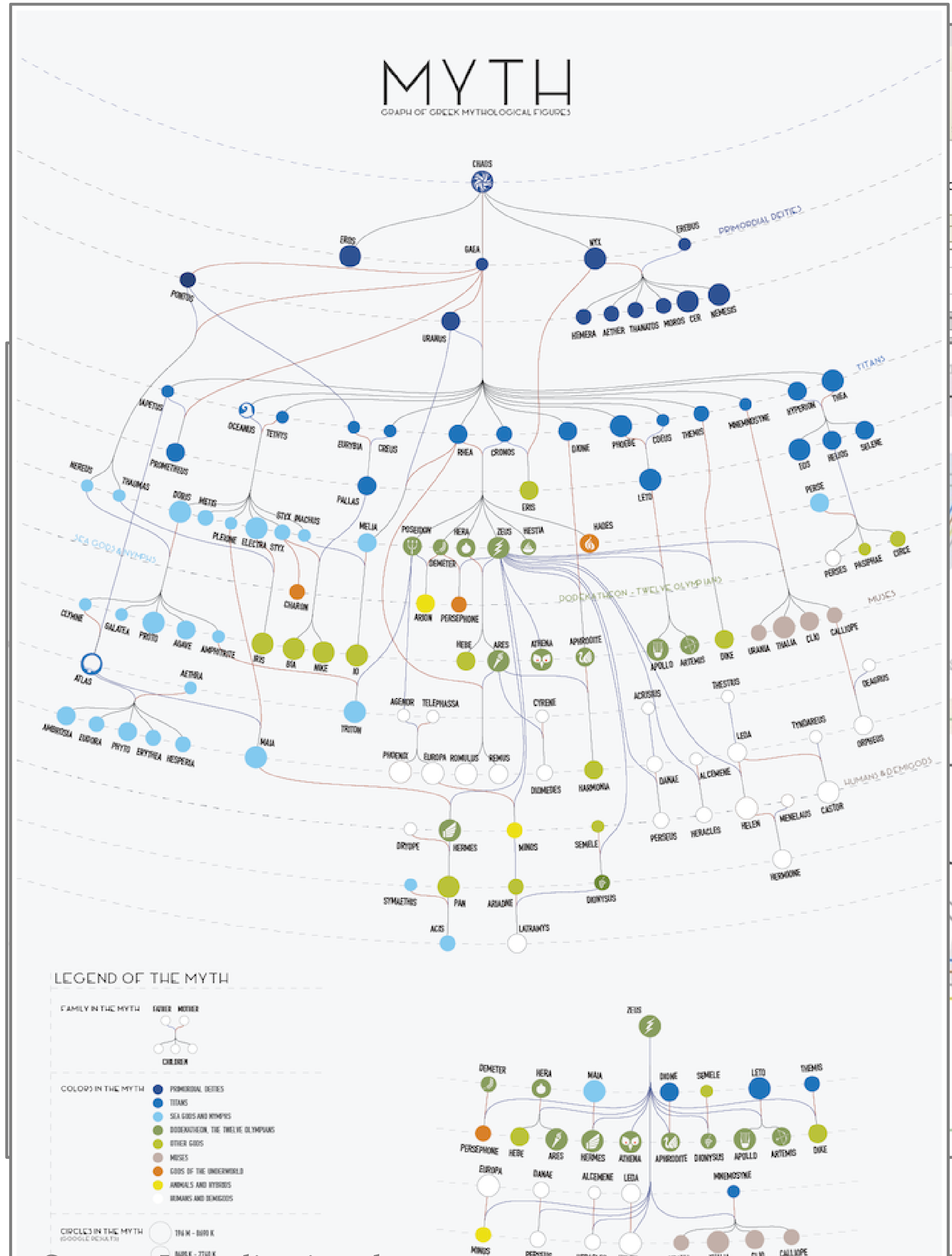


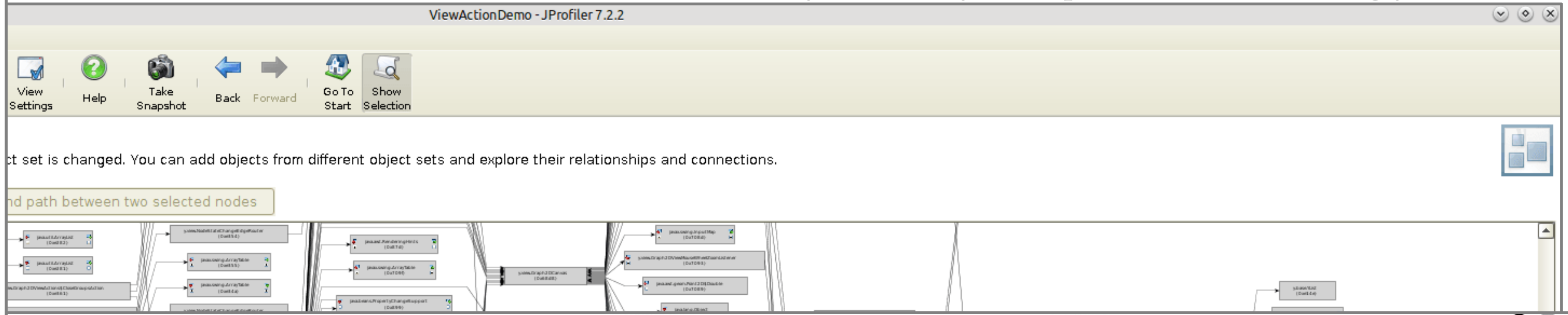

Source: "Design Considerations for Optimizing Storyline Visualizations" Tanahashi et al.

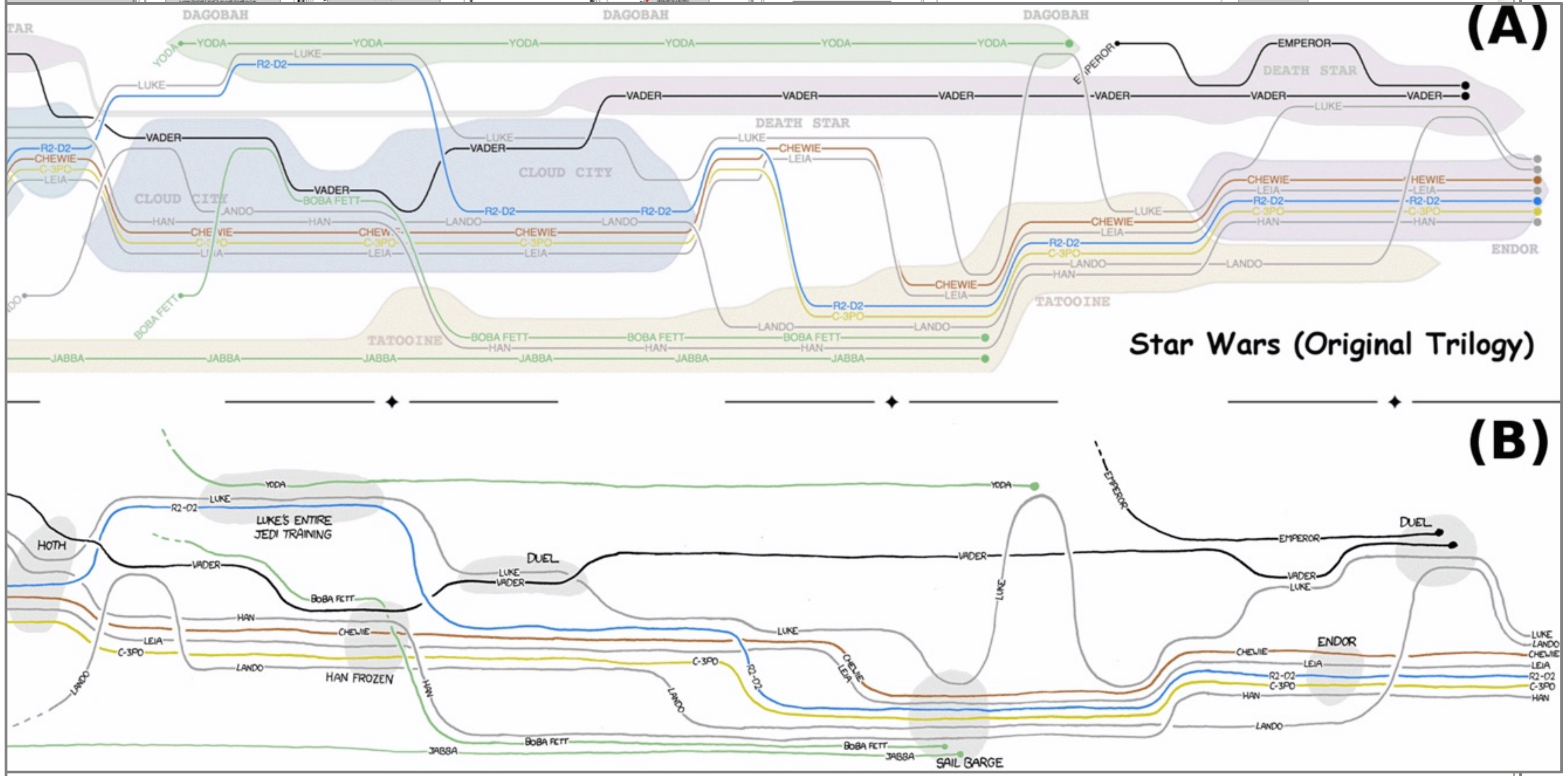
# Hierarchical Drawing – Applications




Source: Visualization that won the Graph Drawing Contest 2016. Klawitter & Mchedlidze

yEd Gallery: Java profiler JProfiler using yFiles

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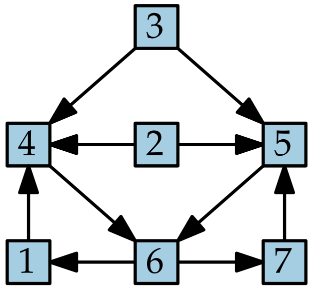
# Classical Approach – Sugiyama Framework

[Sugiyama, Tagawa, Toda ’81]

Input



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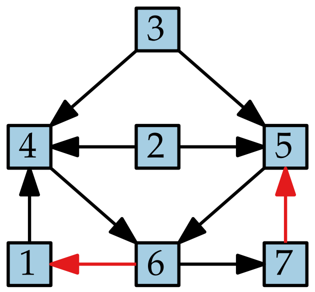

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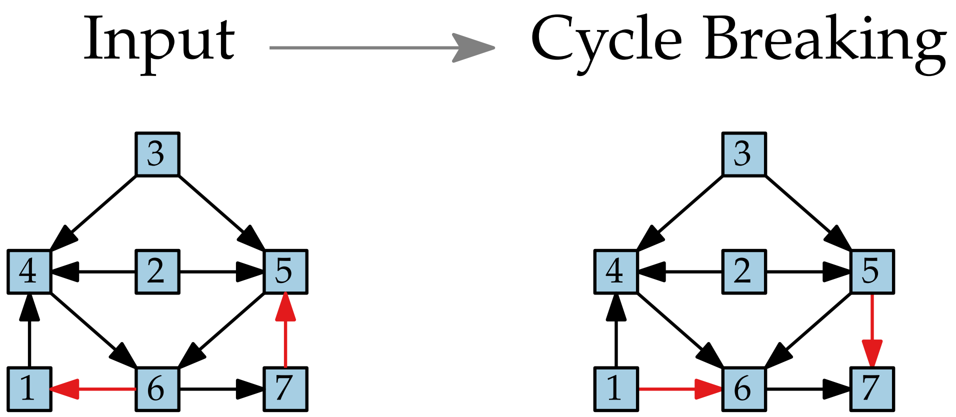

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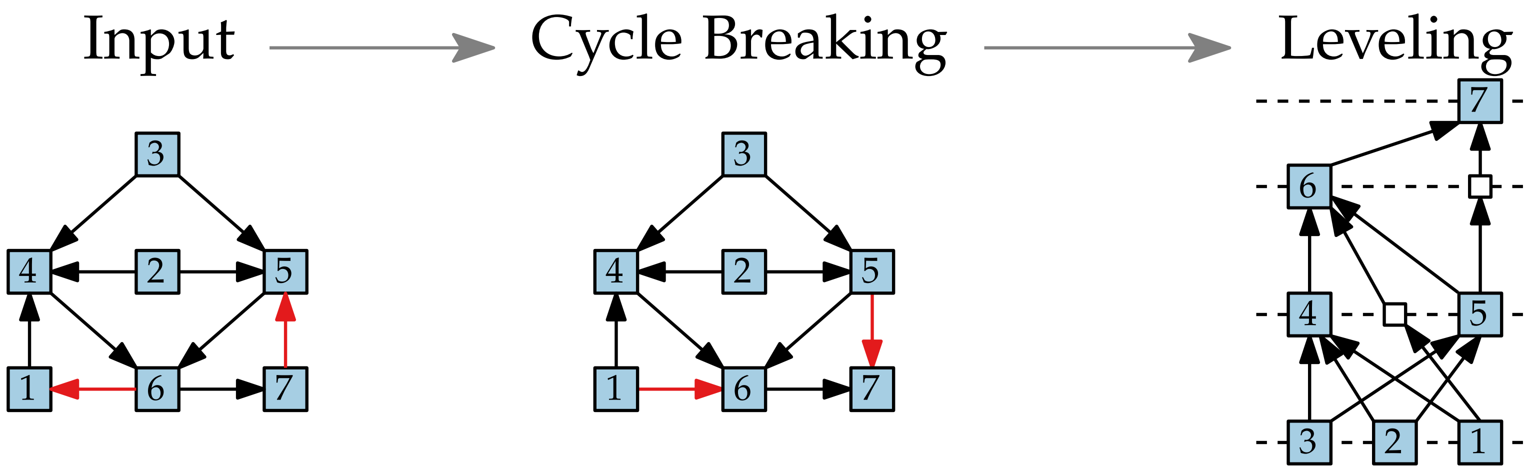

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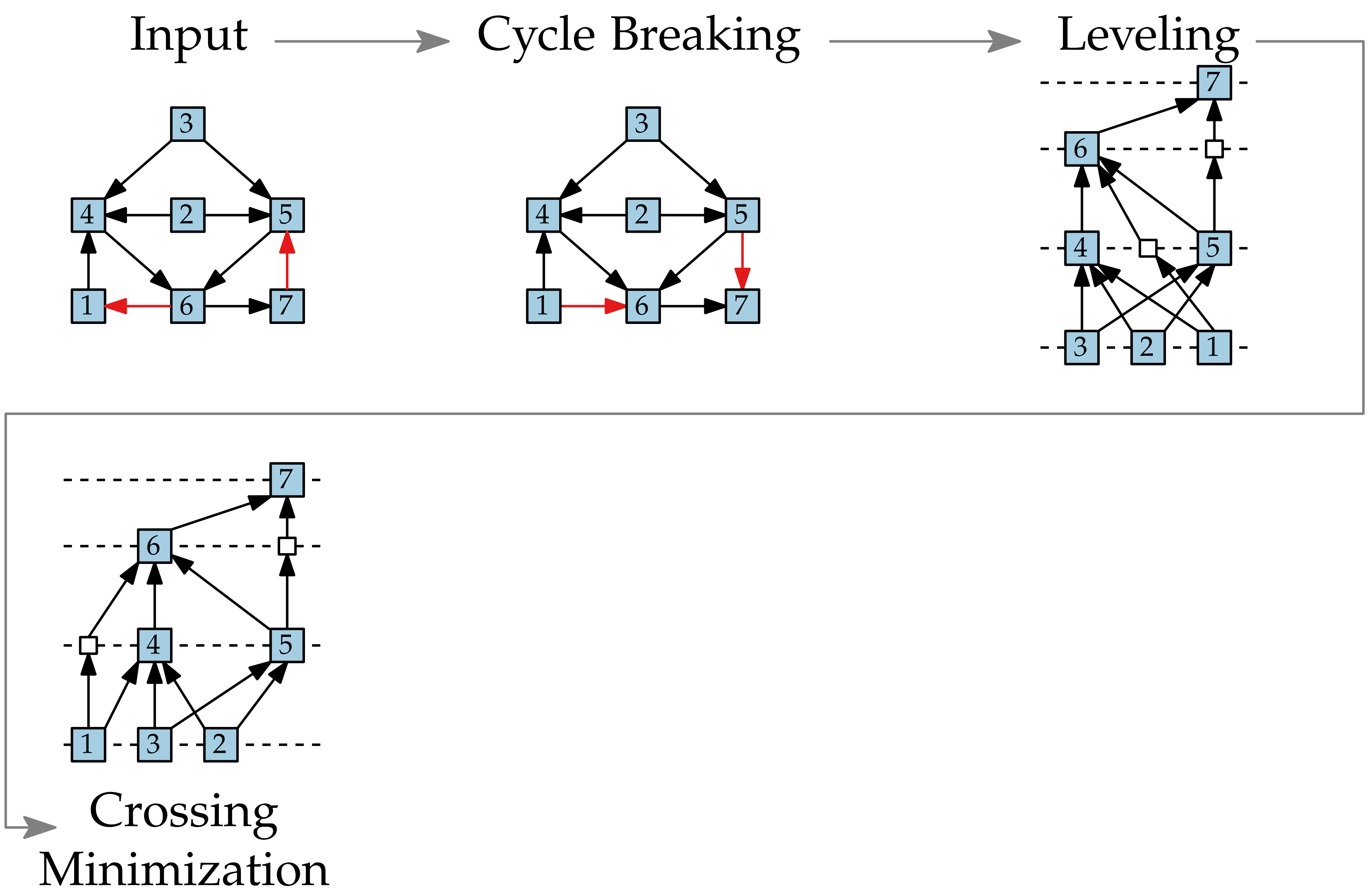

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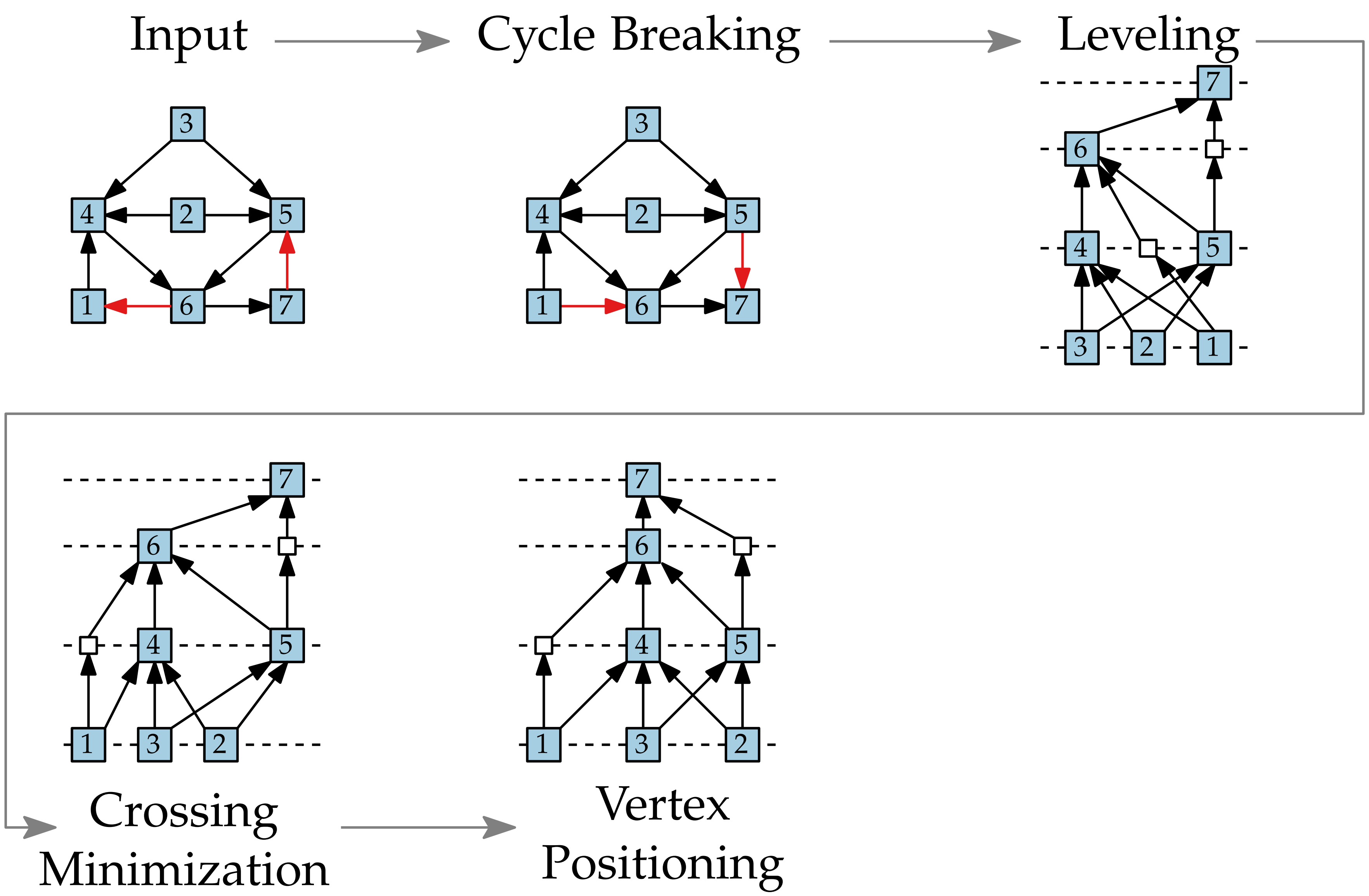

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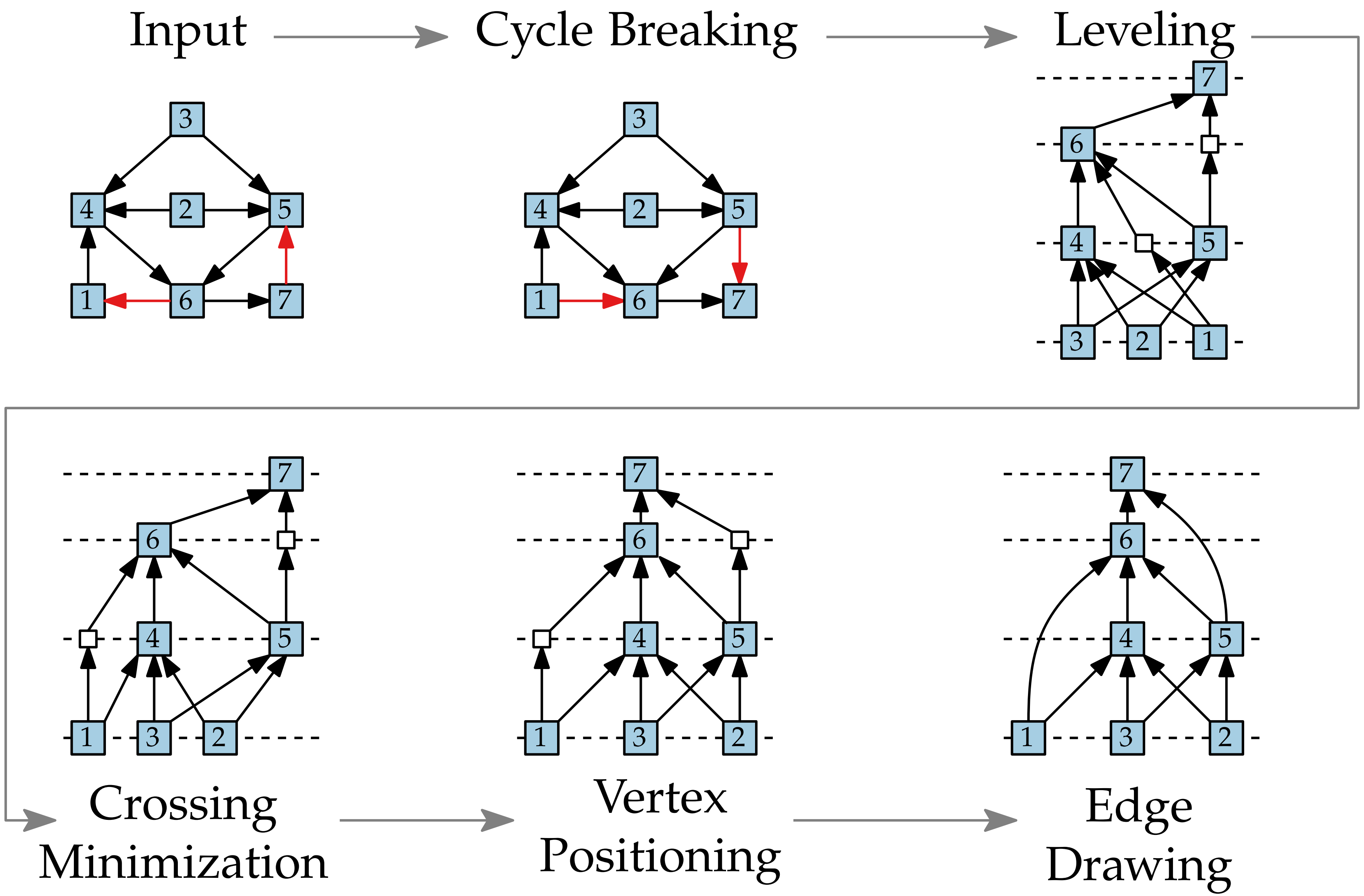
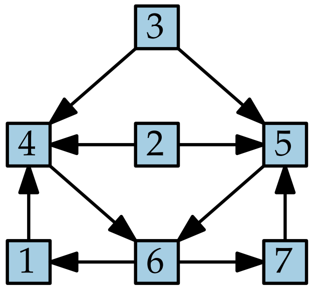
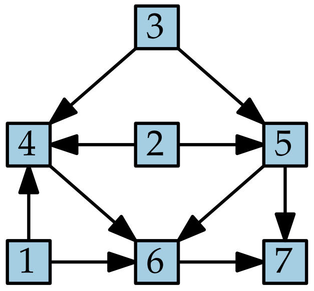


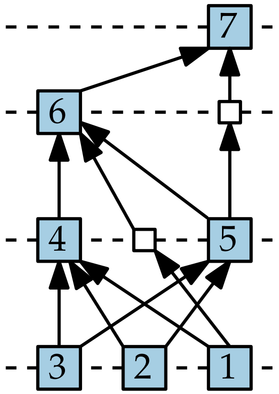
Universität Trier

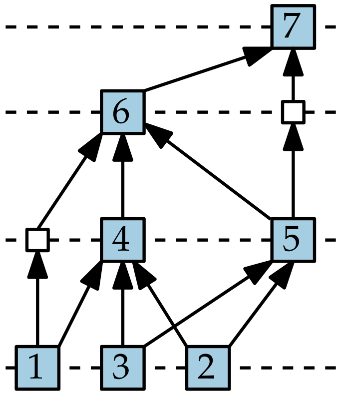
# Visualization of Graphs

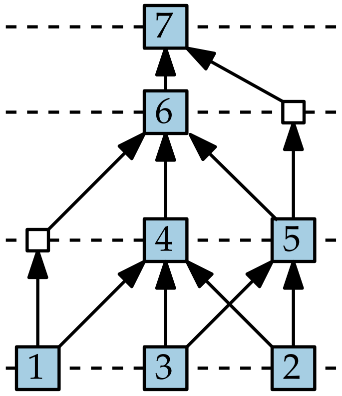


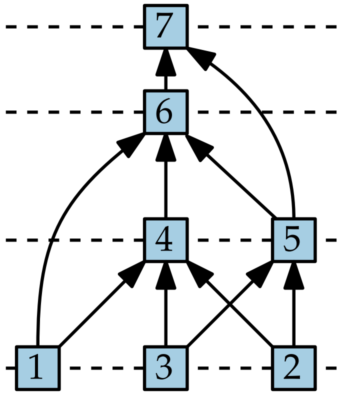




## Lecture 8:
## Hierarchical Layouts:
## Sugiyama Framework



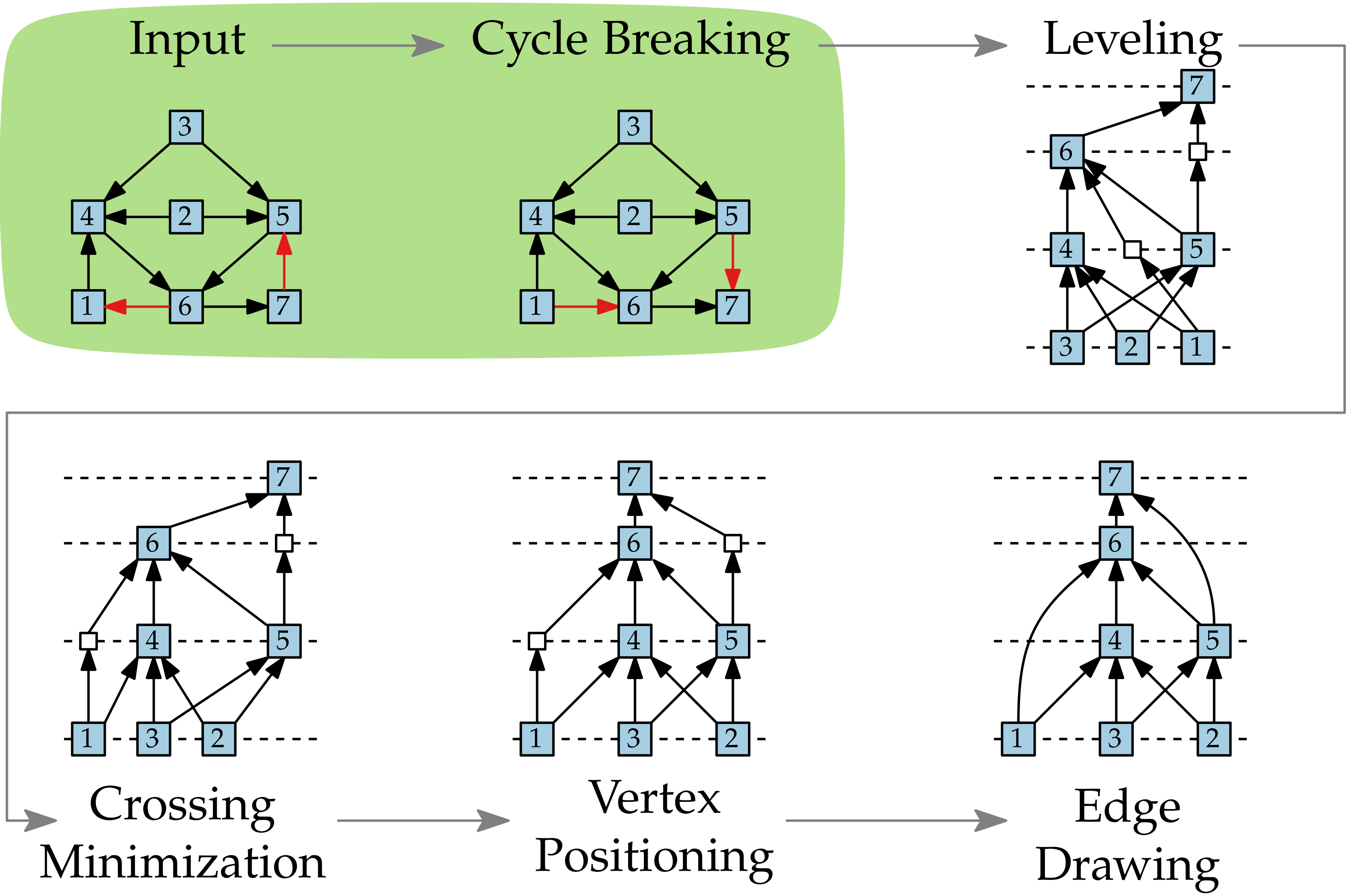




### Part II:
### Cycle Breaking

Philipp Kindermann

# Step 1: Cycle breaking



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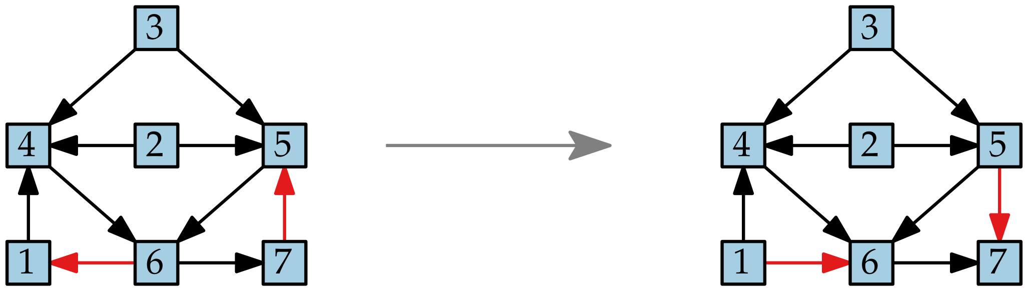
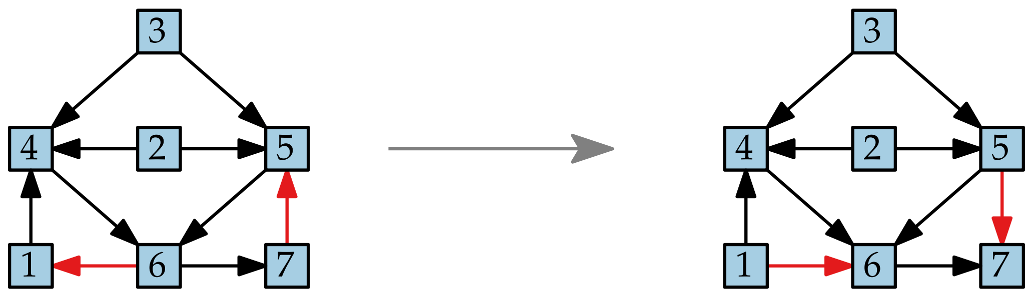



**Approach.**

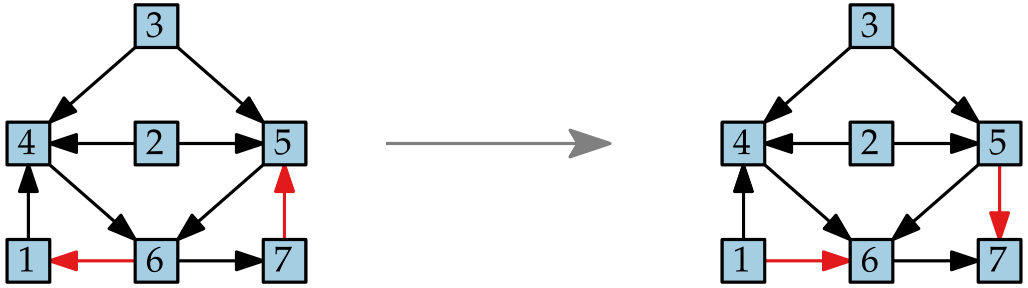
# Step 1: Cycle breaking




**Approach.**

- Find minimum set $E^\star$ of edges which are not upwards.

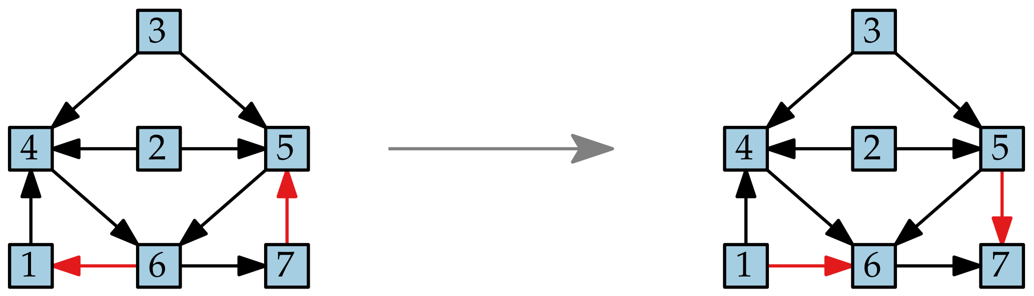
# Step 1: Cycle breaking




**Approach.**

- Find minimum set $E^\star$ of edges which are not upwards.
- Remove $E^\star$ and insert reversed edges.

# Step 1: Cycle breaking




**Approach.**

- Find minimum set $E^\star$ of edges which are not upwards.
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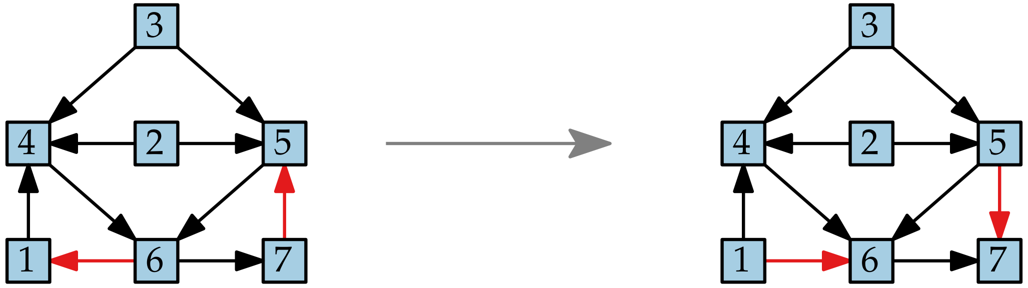
**Problem** MINIMUM FEEDBACK ARC SET **(FAS).**

# Step 1: Cycle breaking




**Approach.**

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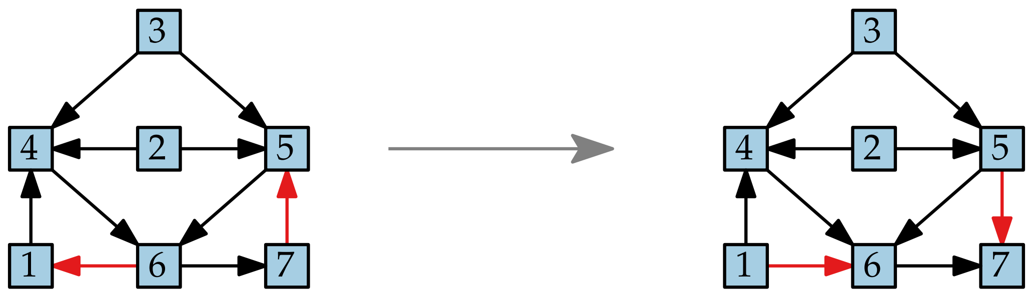
**Problem Minimum Feedback Arc Set (FAS).**

- Input: directed graph $G = (V, E)$
- Output:

# Step 1: Cycle breaking

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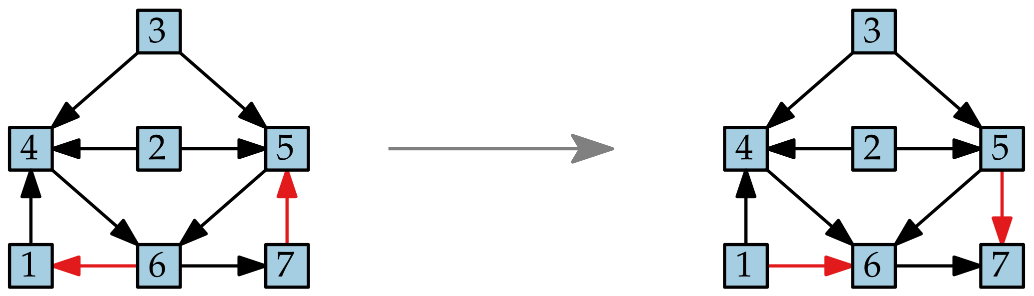
**Problem MINIMUM FEEDBACK ARC SET (FAS).**

- Input: directed graph $G = (V, E)$
- Output: min. set $E^\star \subseteq E$, so that $G - E^\star$ acyclic

# Step 1: Cycle breaking




**Approach.**

- Find minimum set $E^\star$ of edges which are not upwards.
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**Problem** Minimum Feedback ~~Arc~~ Set (F~~A~~S).

- Input: directed graph $G = (V, E)$
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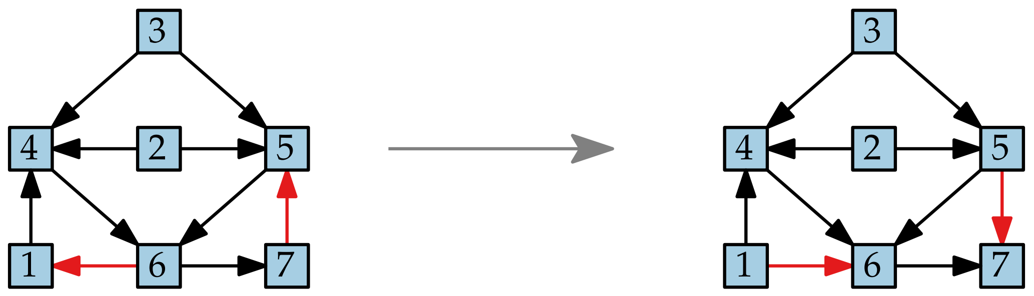
  $G - E^\star + E^\star_r$

# Step 1: Cycle breaking

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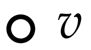
…NP-hard ☹

# Heuristic 1

[Berger, Shor ’90]



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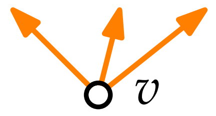


$$N^{\rightarrow}(v) \quad := \quad \{(v,u) | (v,u) \in E\}$$

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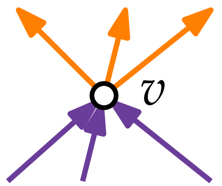
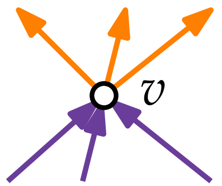
[Berger, Shor ’90]




$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$
$$N^{\leftarrow}(v) := \{(u,v) | (u,v) \in E\}$$

# Heuristic 1

[Berger, Shor '90]




$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$

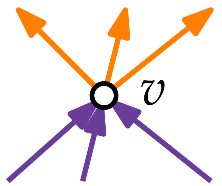
$$N^{\leftarrow}(v) := \{(u,v) | (u,v) \in E\}$$

$$N(v) := N^{\rightarrow}(v) \cup N^{\leftarrow}(v)$$

# Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

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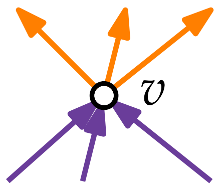
[Berger, Shor '90]




GreedyMakeAcyclic(Digraph $G = (V, E)$)

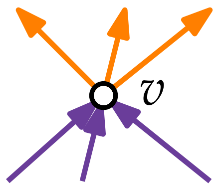
$E' \leftarrow \emptyset$

$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$
$$N^{\leftarrow}(v) := \{(u,v) | (u,v) \in E\}$$
$$N(v) := N^{\rightarrow}(v) \cup N^{\leftarrow}(v)$$

**return** $(V, E')$

# Heuristic 1

[Berger, Shor '90]




GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

**foreach** $v \in V$ **do**

**return** $(V, E')$

$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$

$$N^{\leftarrow}(v) := \{(u,v) | (u,v) \in E\}$$

$$N(v) := N^{\rightarrow}(v) \cup N^{\leftarrow}(v)$$

# Heuristic 1

[Berger, Shor '90]

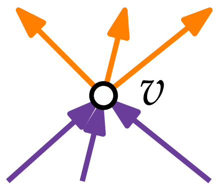
GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

**foreach** $v \in V$ **do**

**if** $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

**return** $(V, E')$

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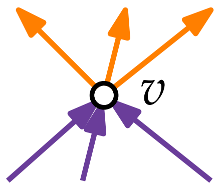


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    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

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$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$
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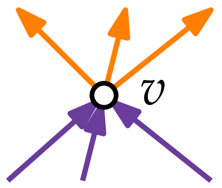
# Heuristic 1

[Berger, Shor ’90]

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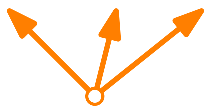

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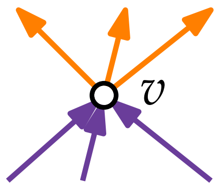


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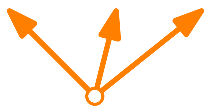

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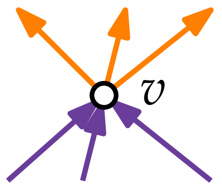


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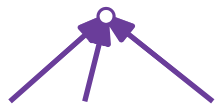

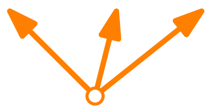

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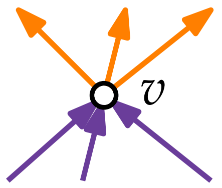


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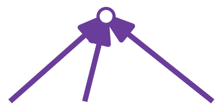

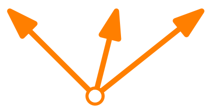

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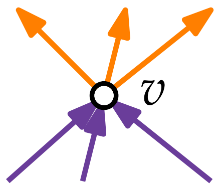


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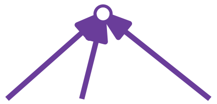

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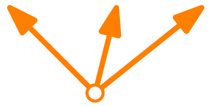

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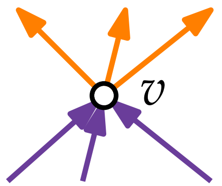


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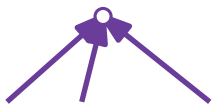

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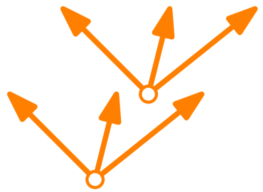

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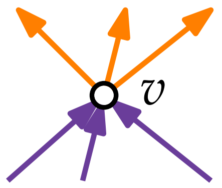


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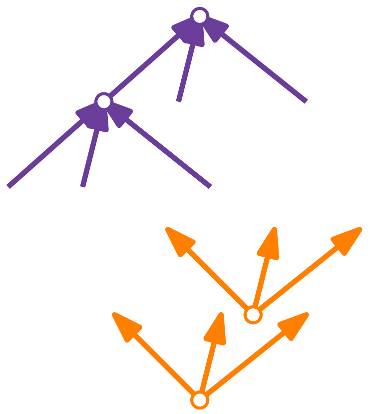

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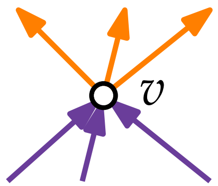


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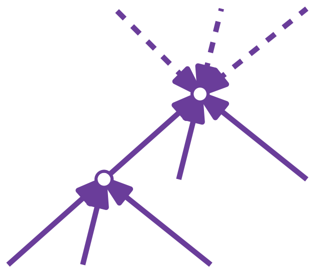

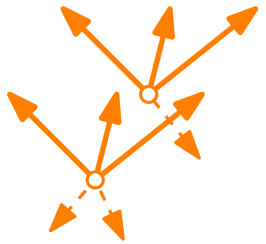

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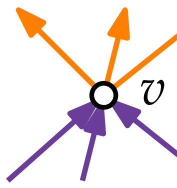


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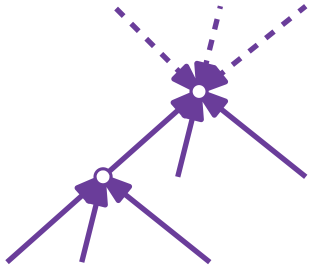

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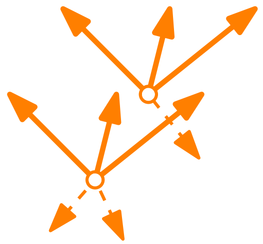

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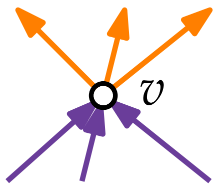


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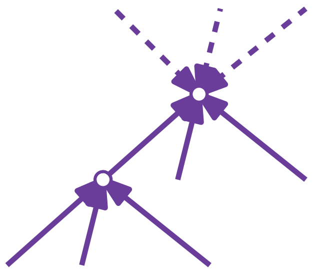

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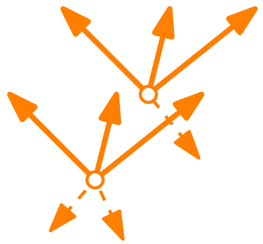

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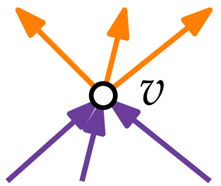


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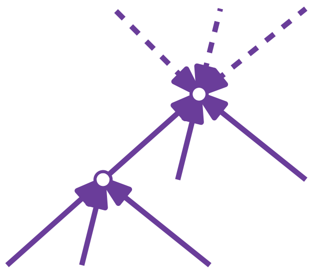

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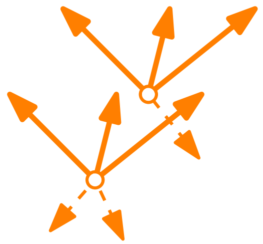

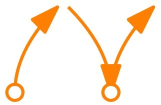

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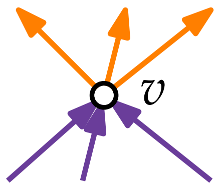


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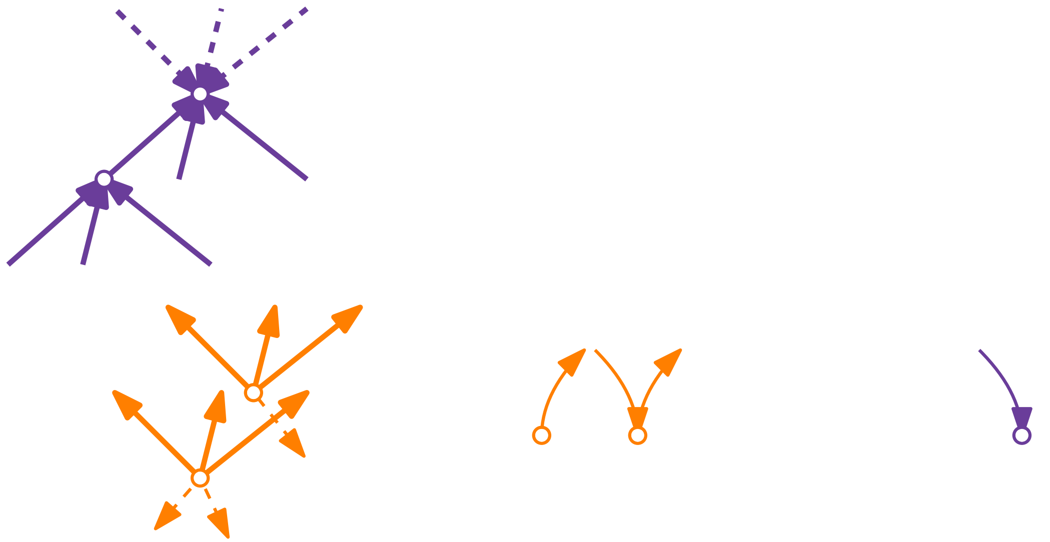

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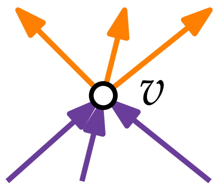


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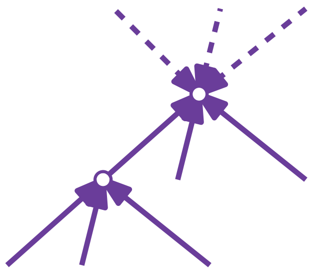

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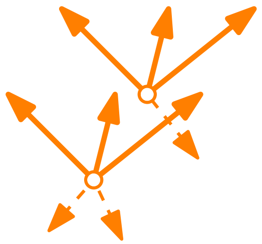

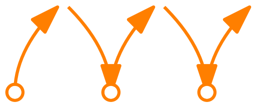

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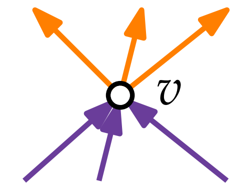


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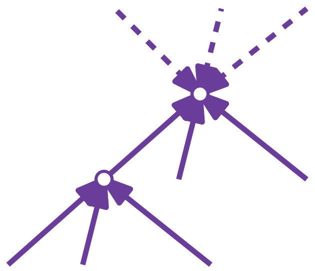

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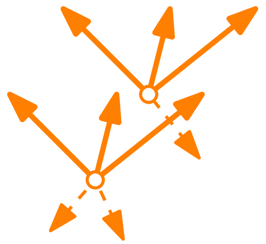

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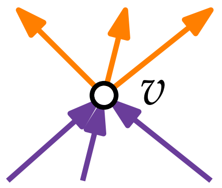


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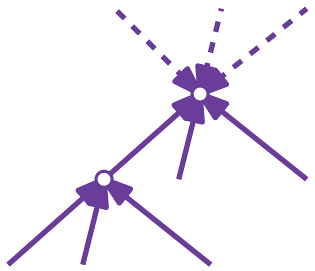

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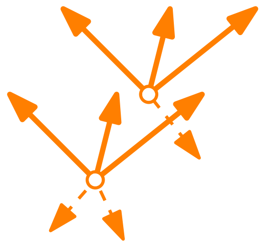

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  remove $v$ and $N(v)$ from $G$.

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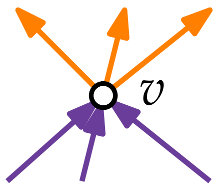


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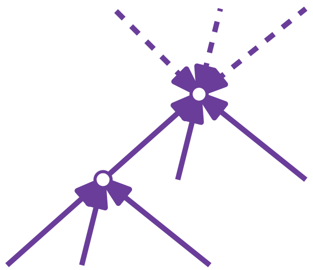

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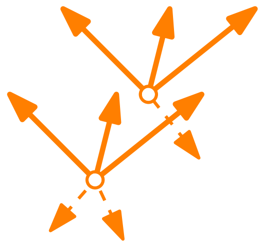

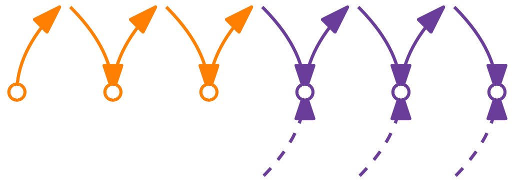

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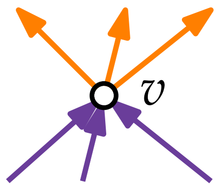


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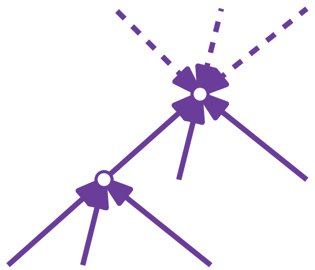

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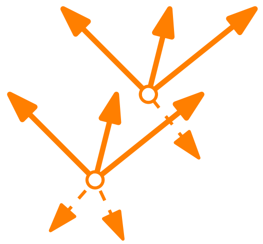

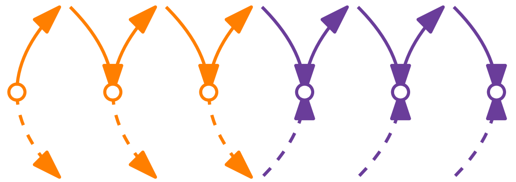

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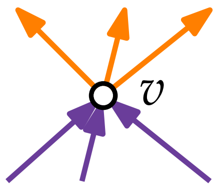


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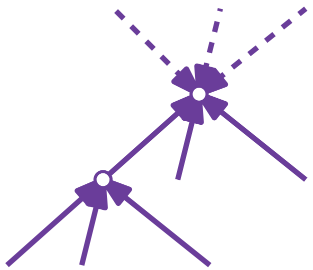
- Time:



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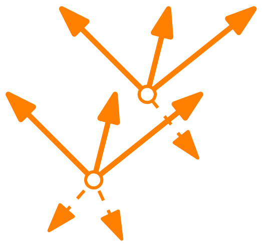

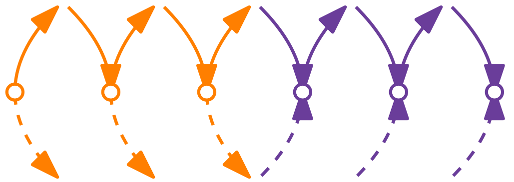

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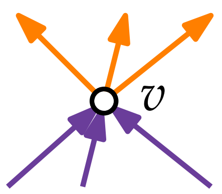


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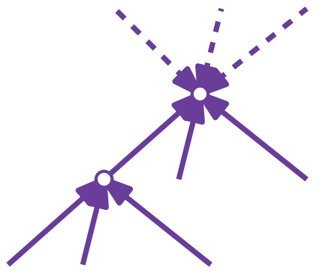

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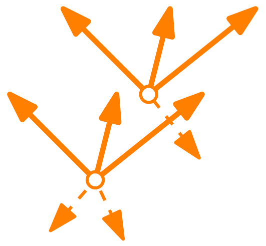

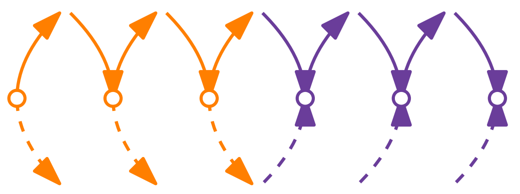

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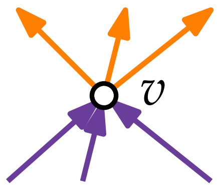


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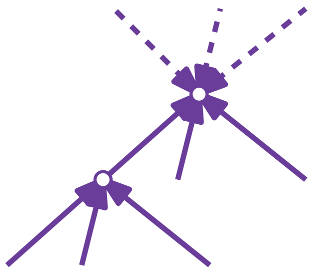

- Time: $\mathcal{O}(n + m)$
- Quality guarantee: $|E'| \geq$

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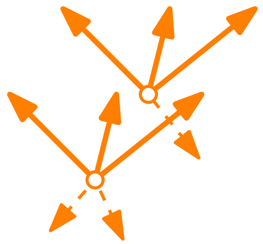

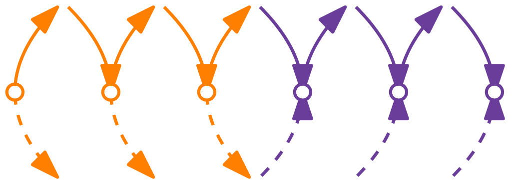

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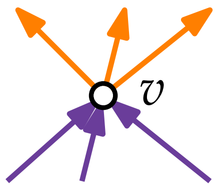


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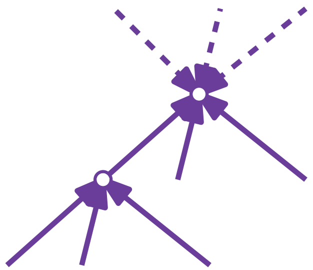
- Time: $\mathcal{O}(n+m)$
- Quality guarantee: $|E'| \geq |E|/2$



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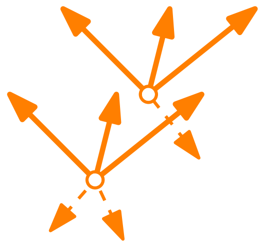

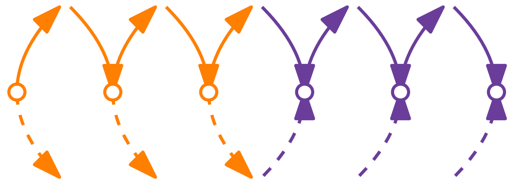

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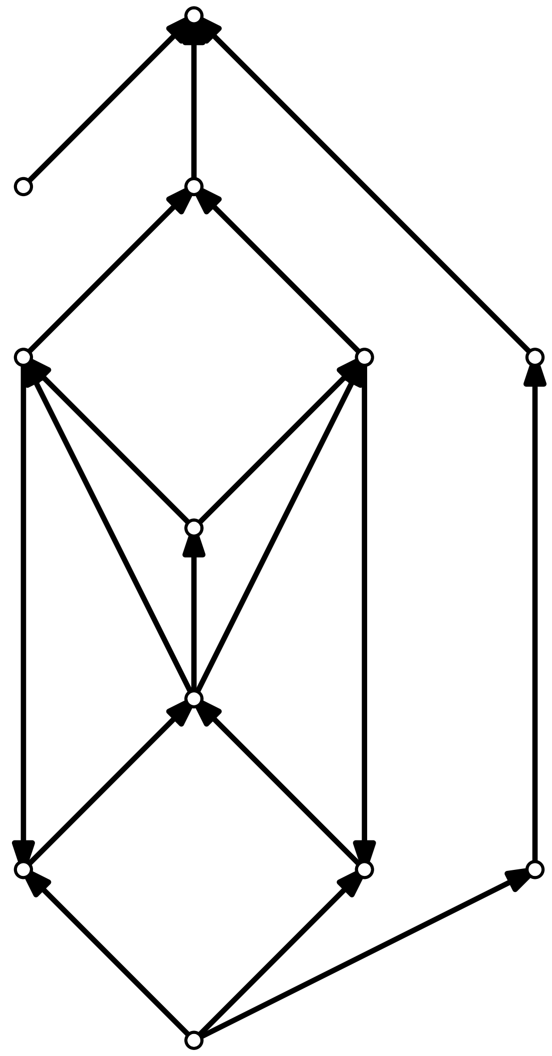
[Eades, Lin, Smyth ’93]



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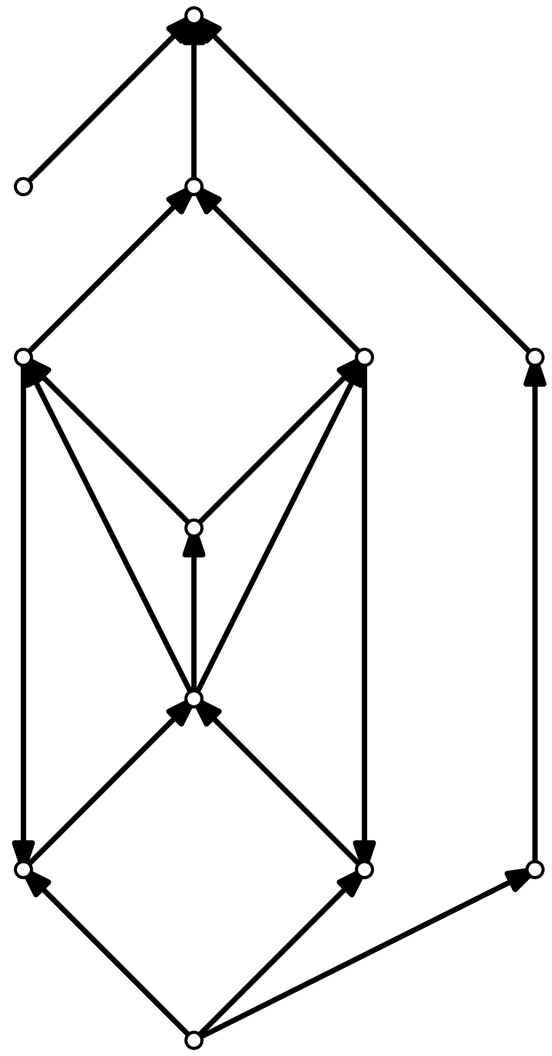

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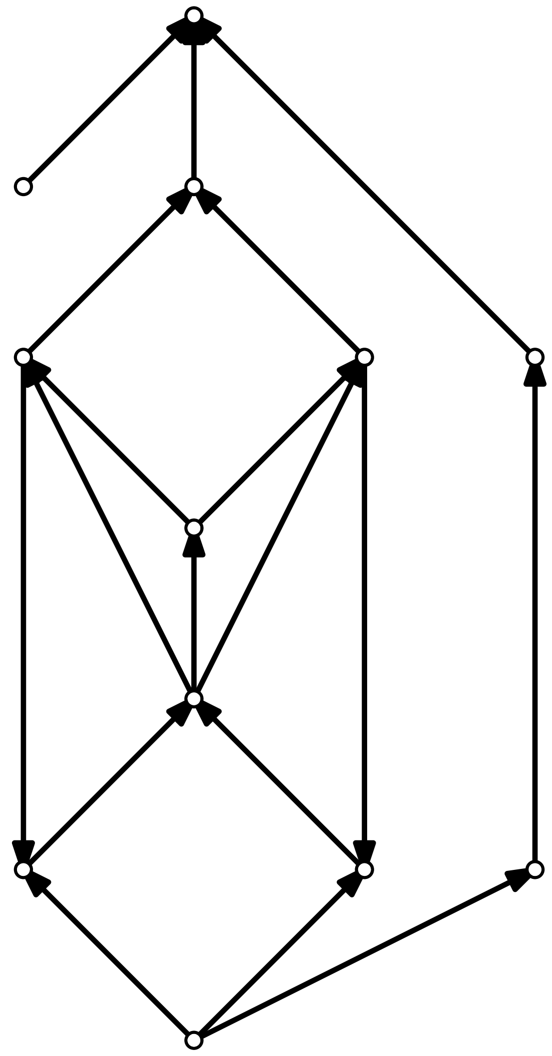

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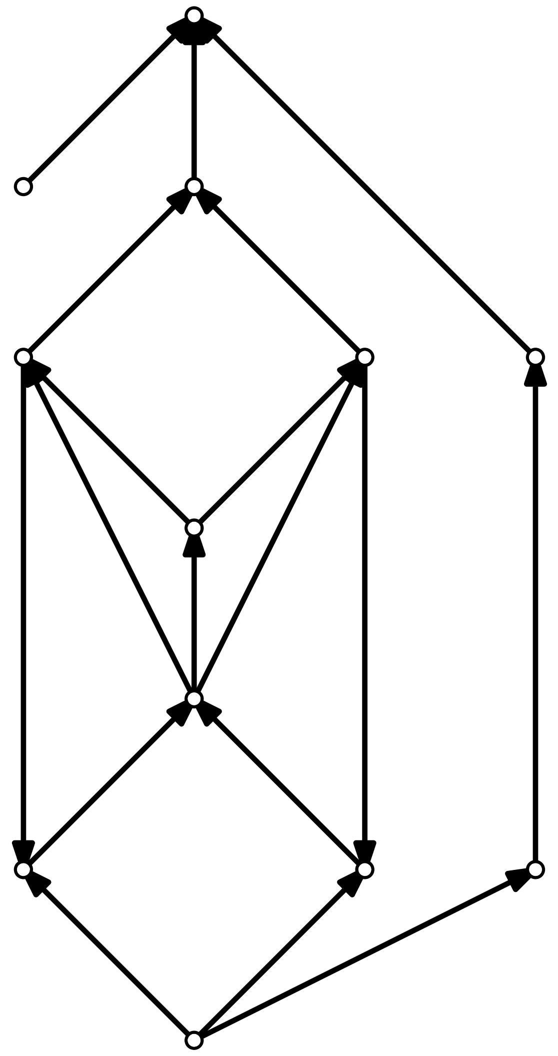

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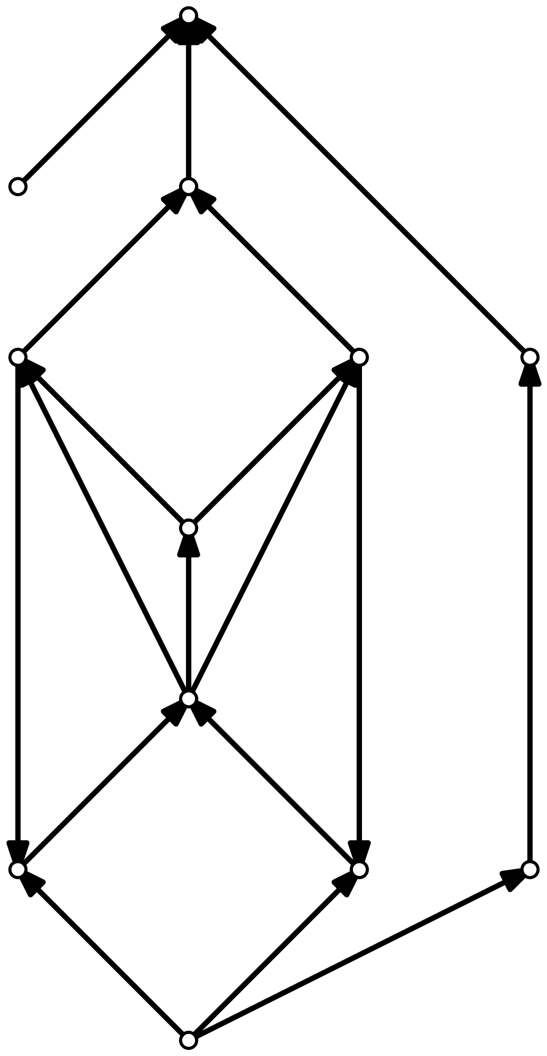

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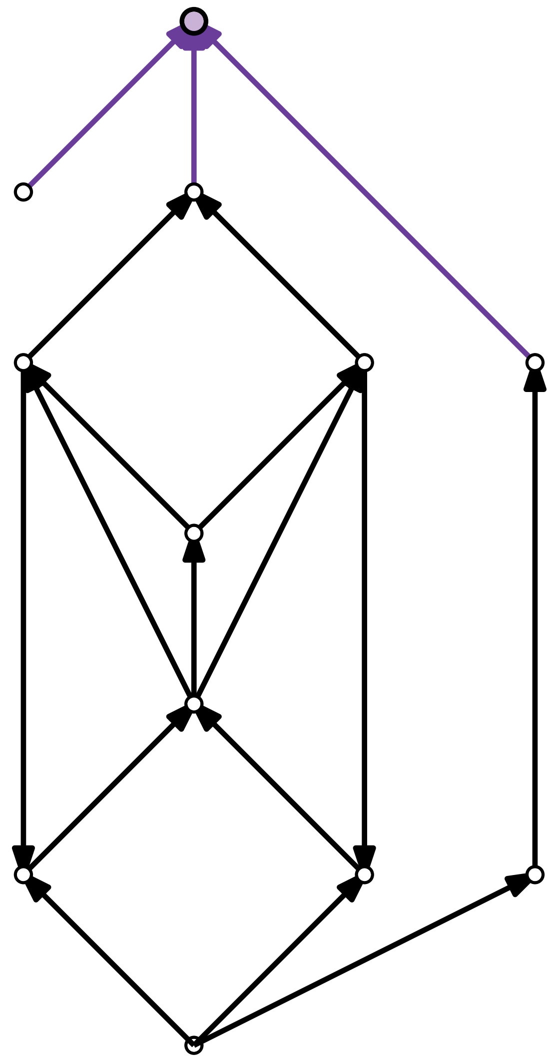

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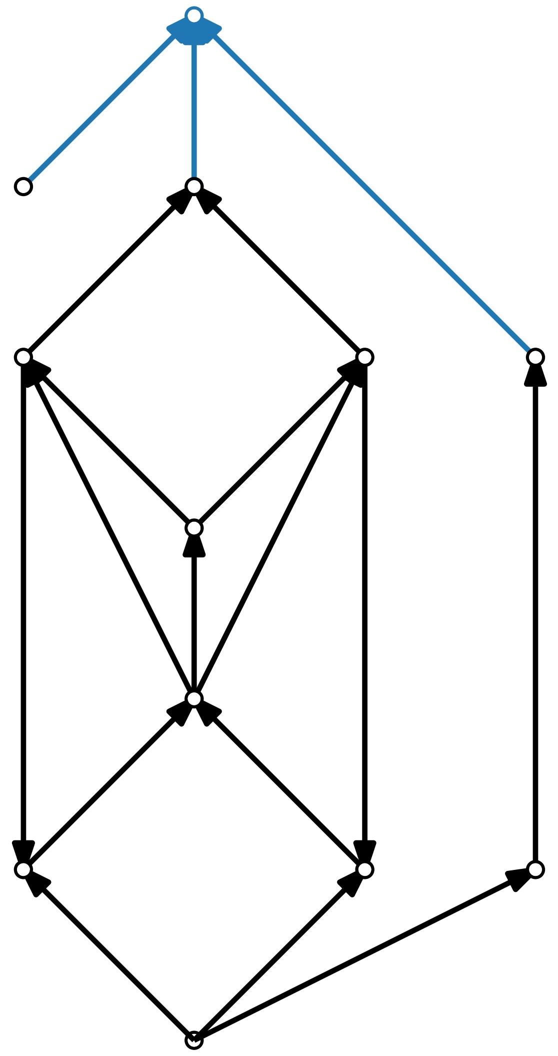

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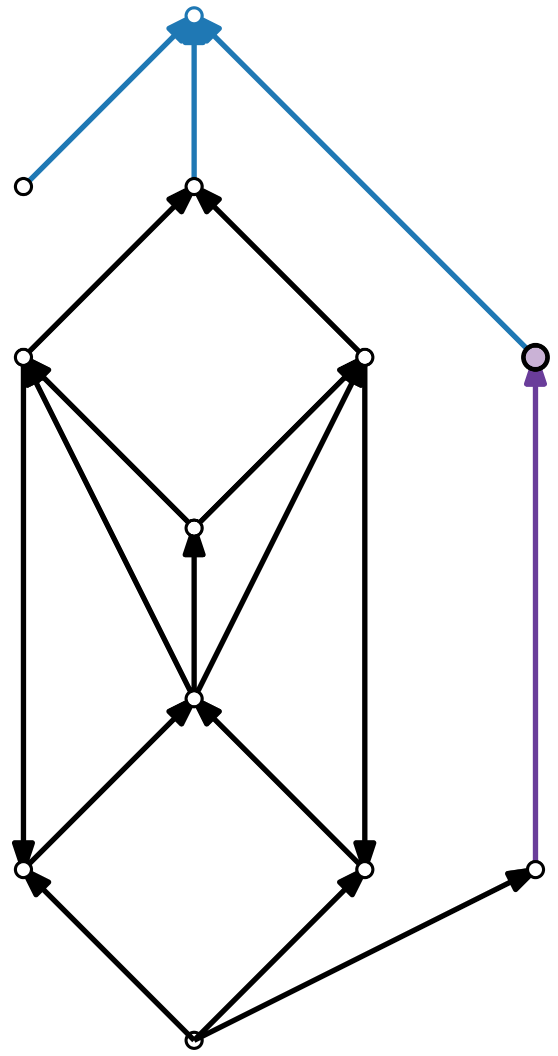

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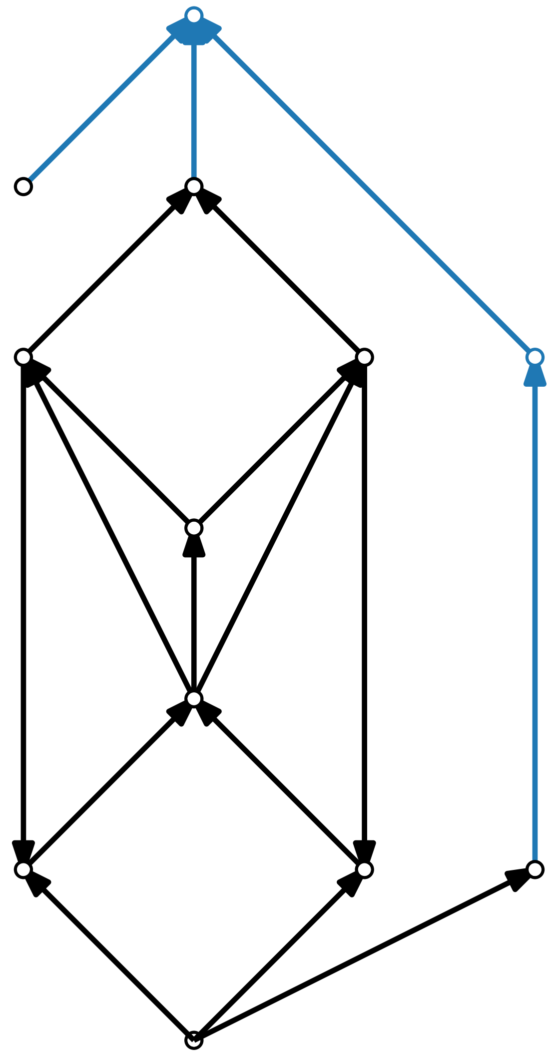

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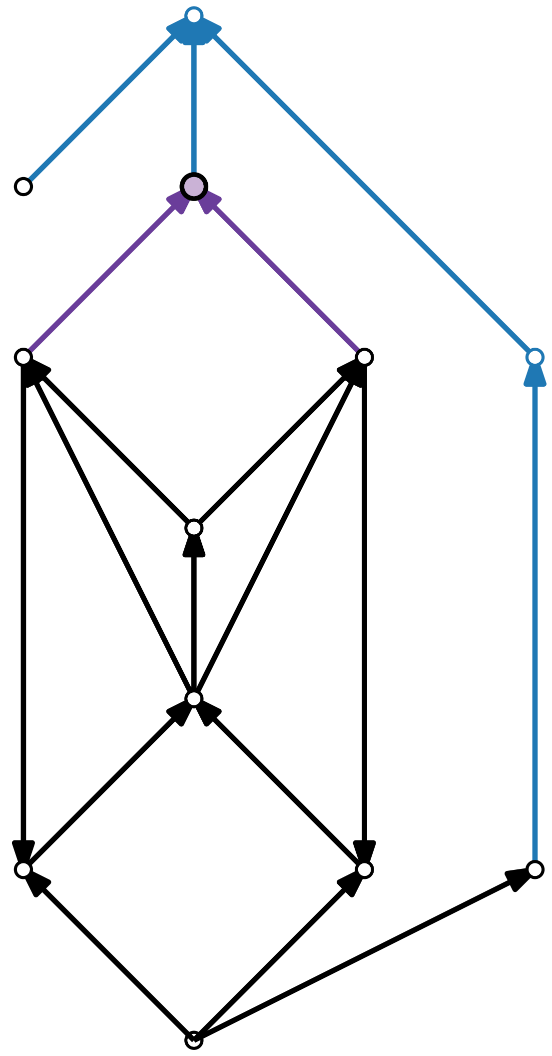

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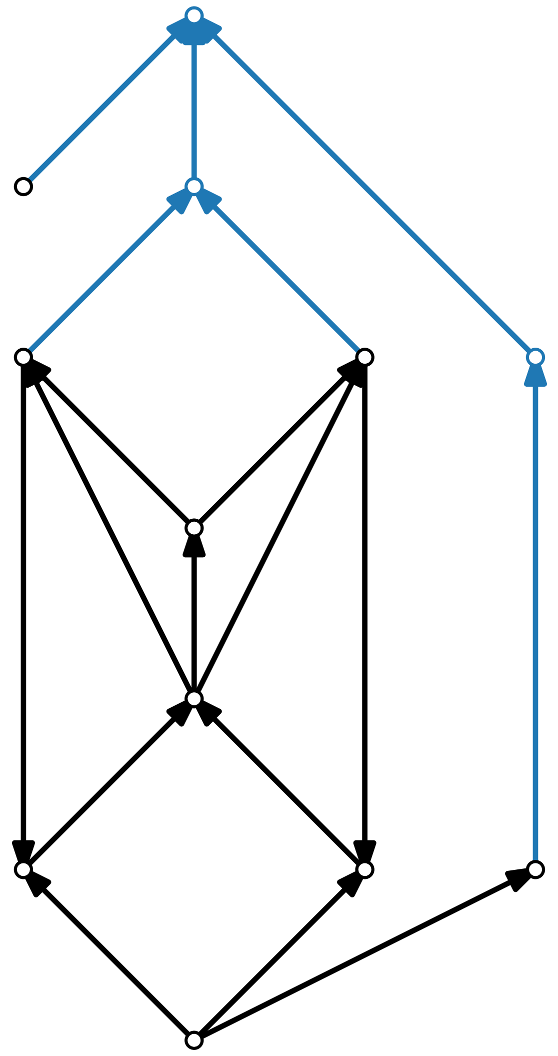

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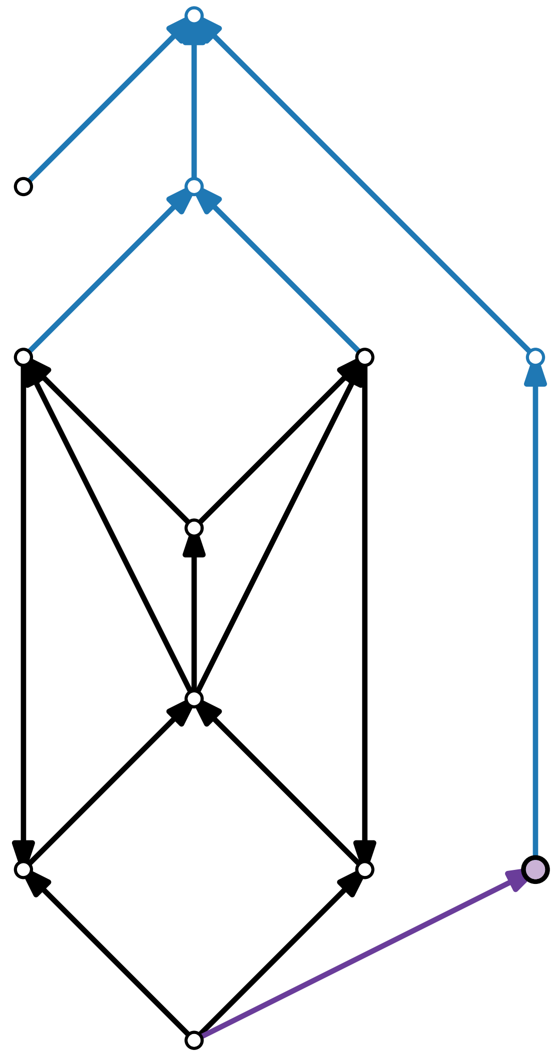

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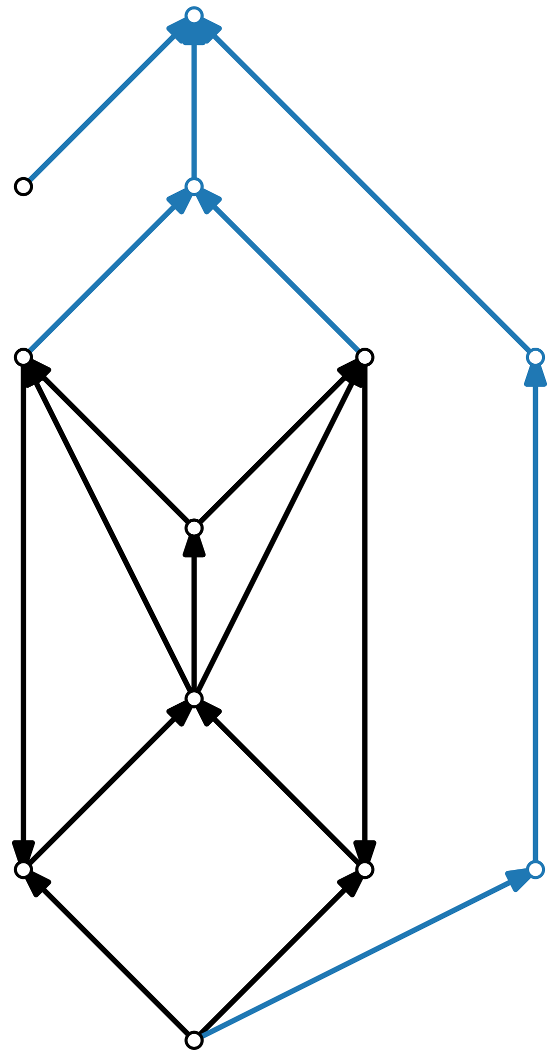

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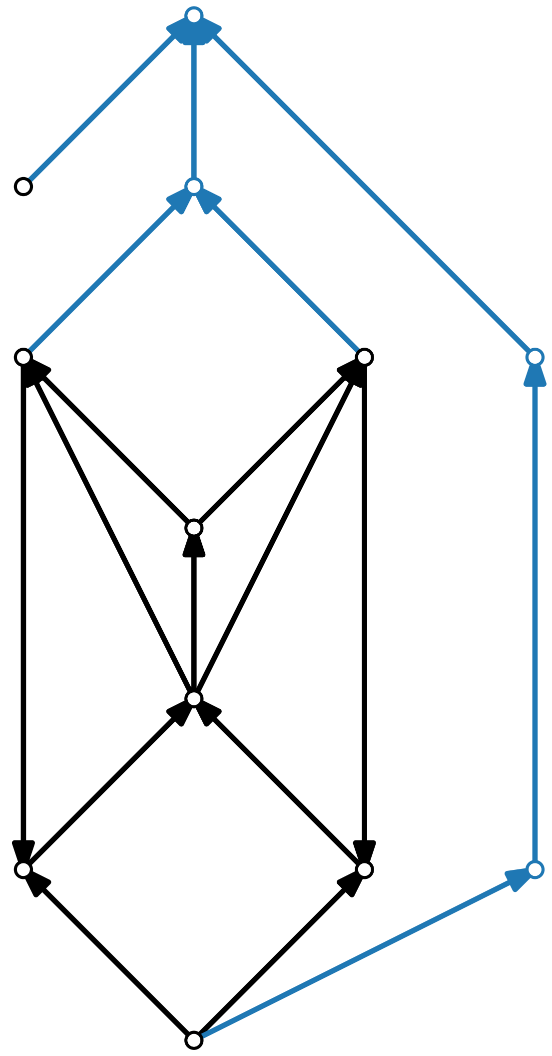

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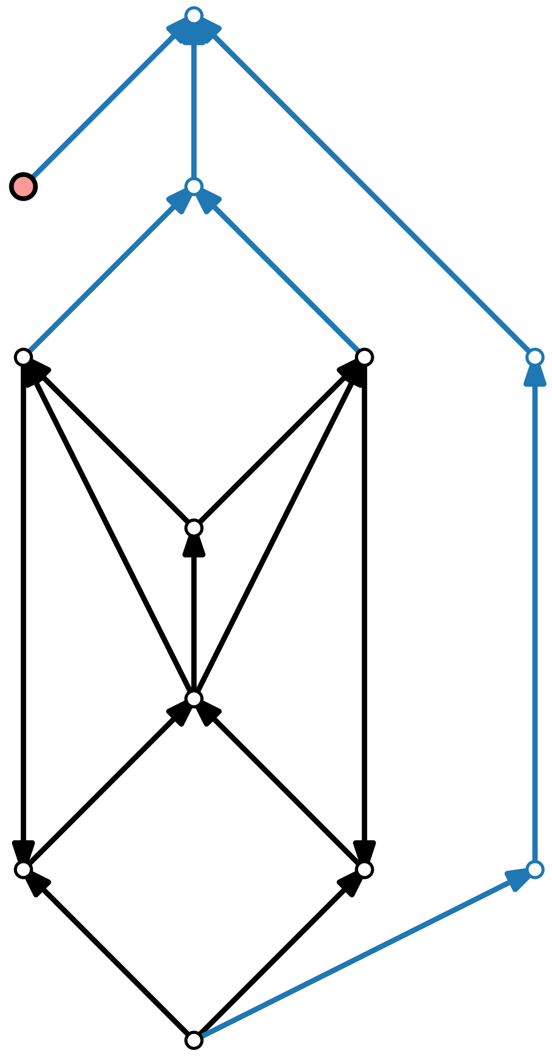

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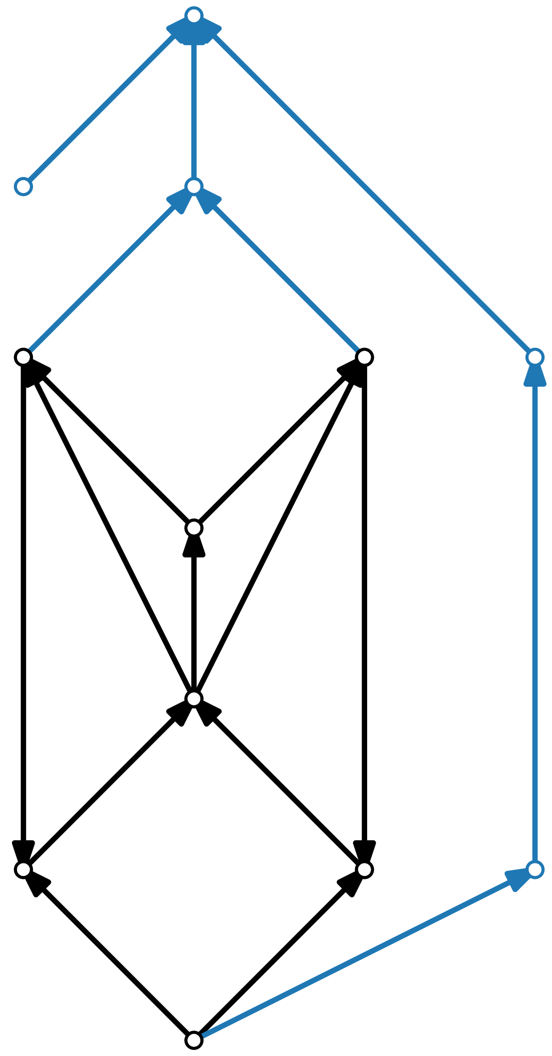

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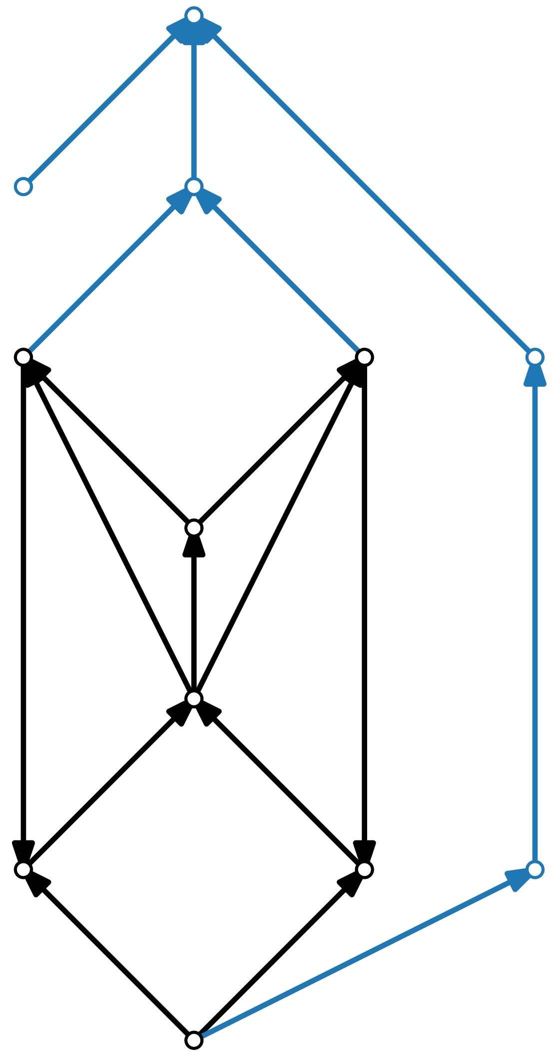

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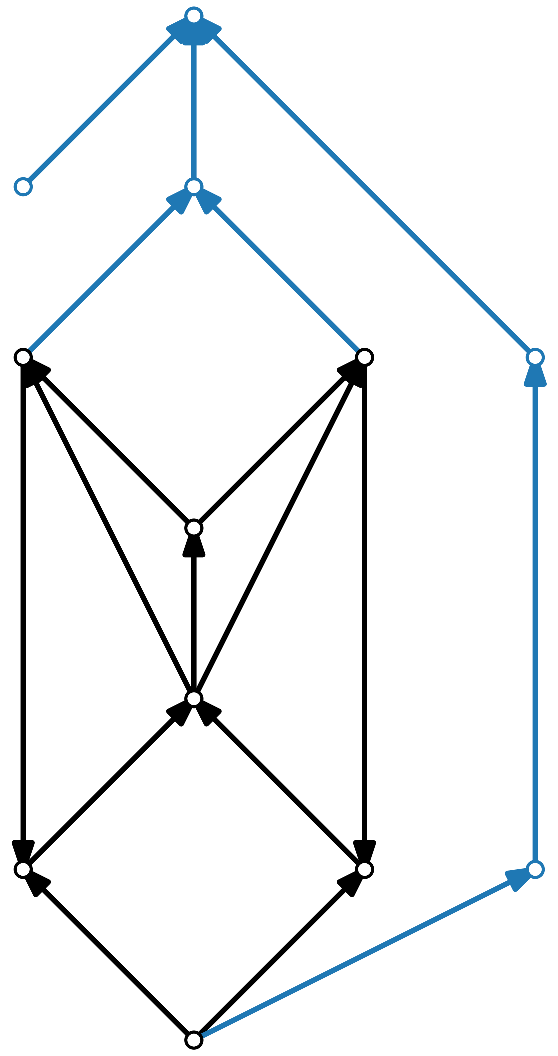

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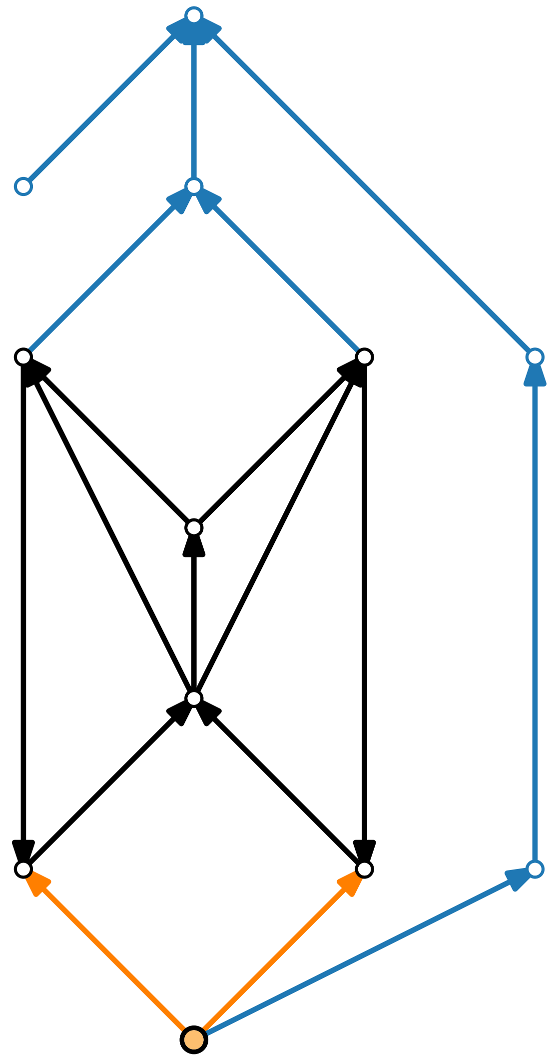

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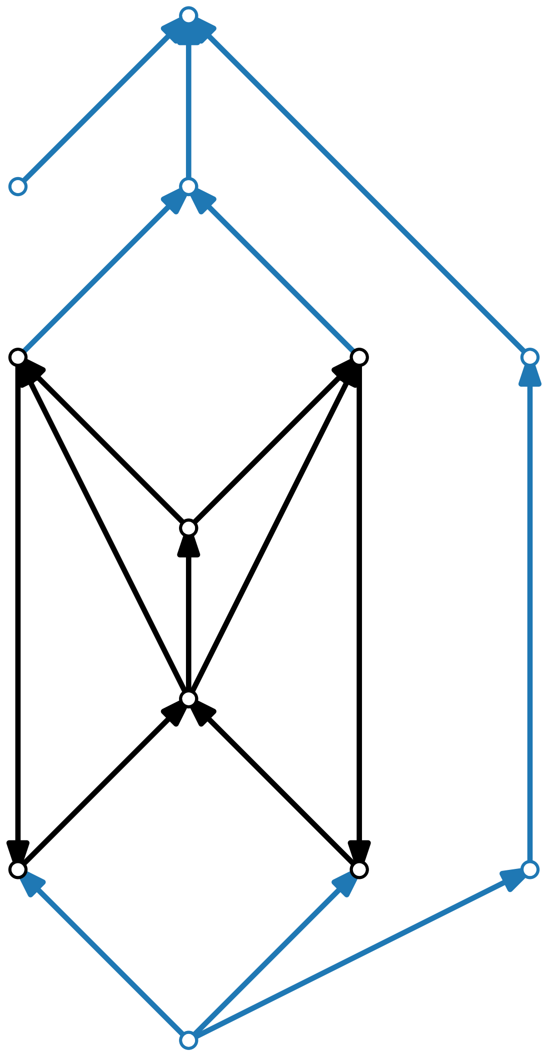

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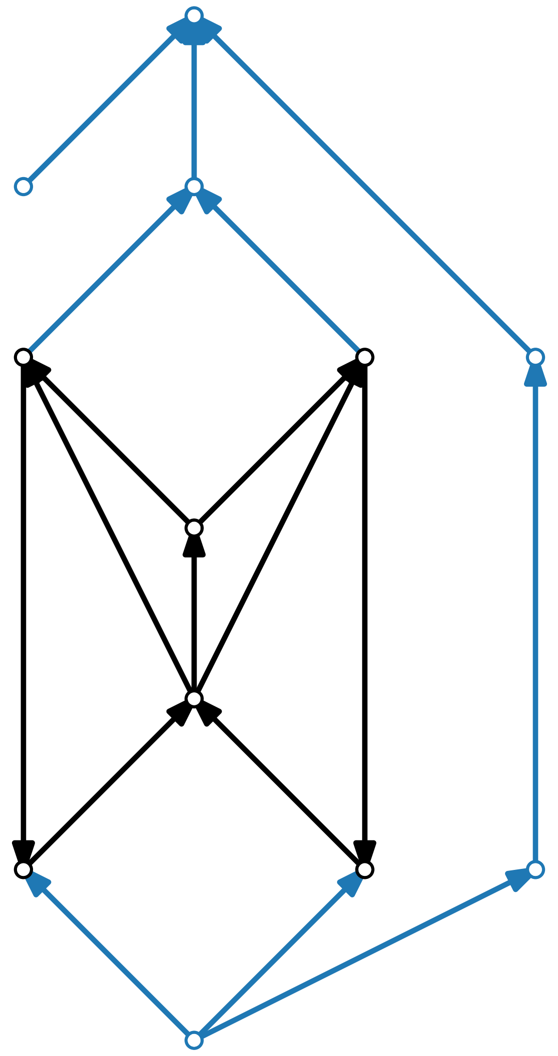

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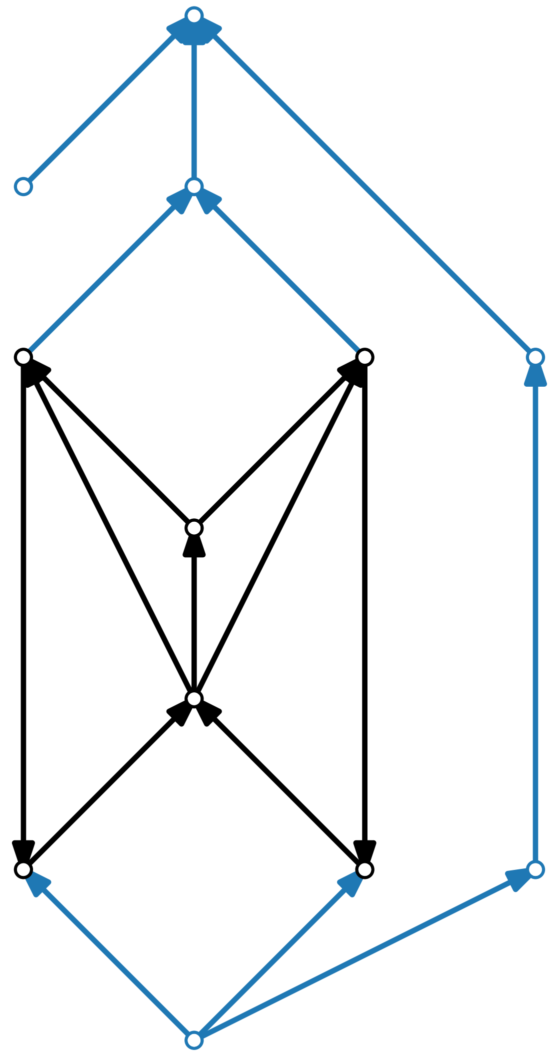

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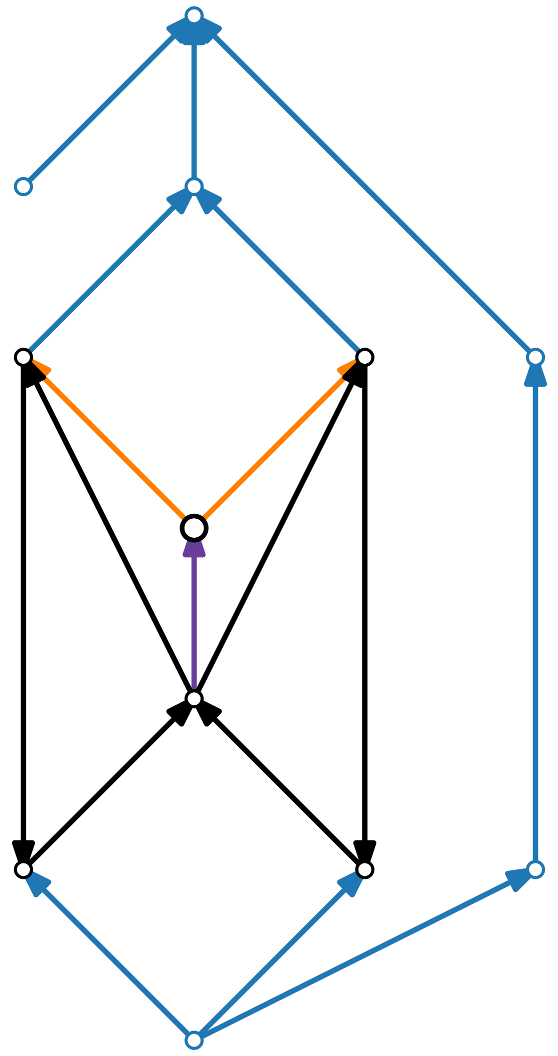

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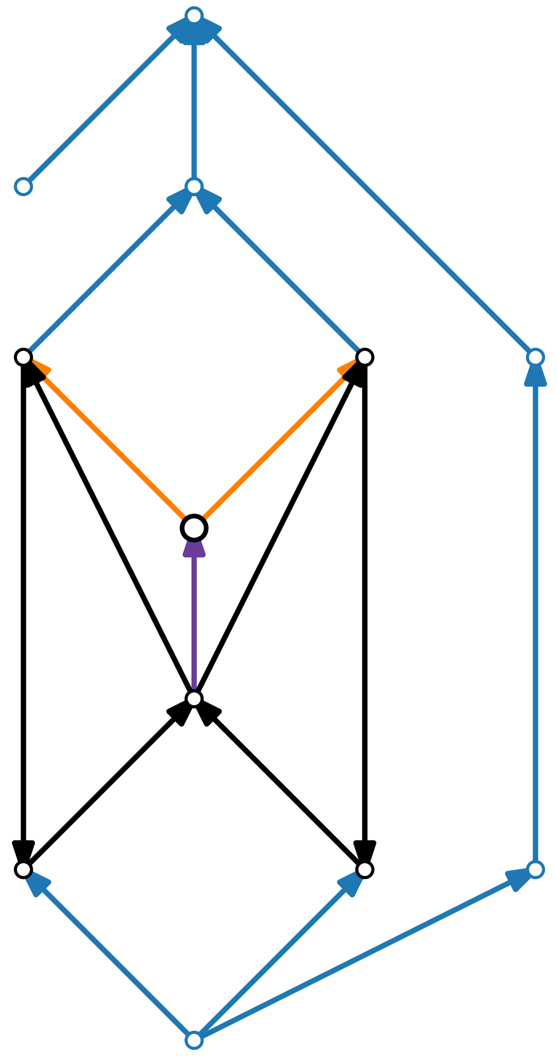

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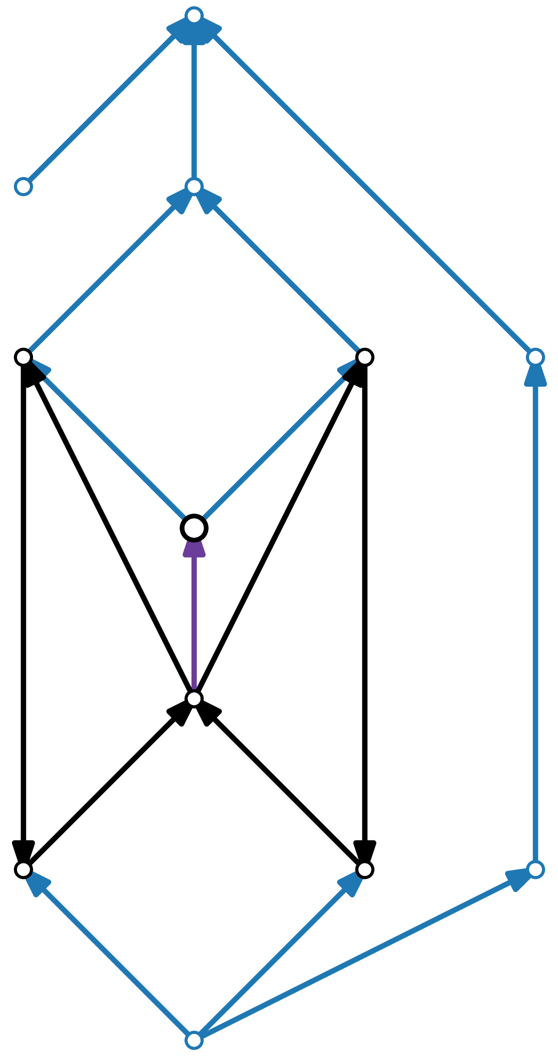

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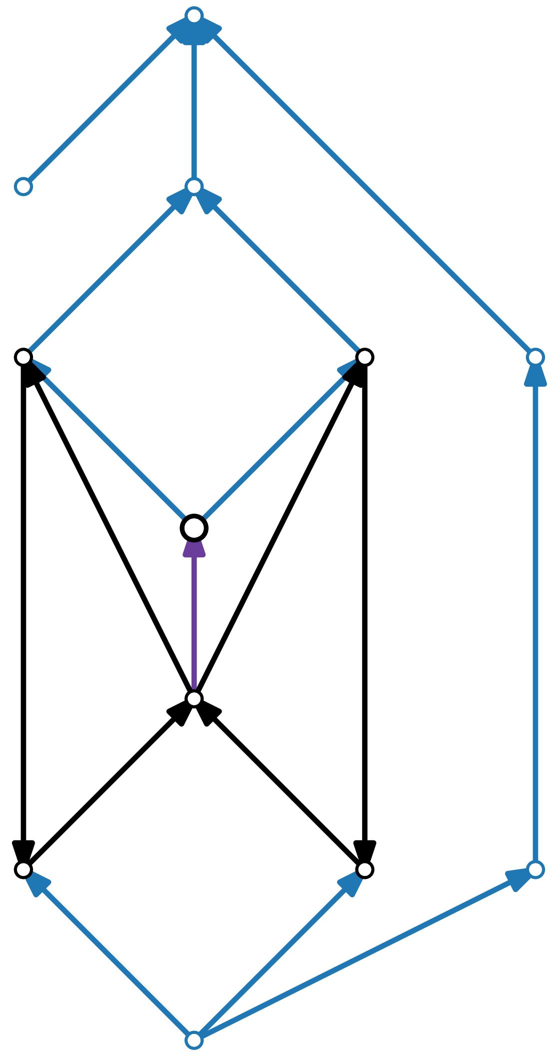

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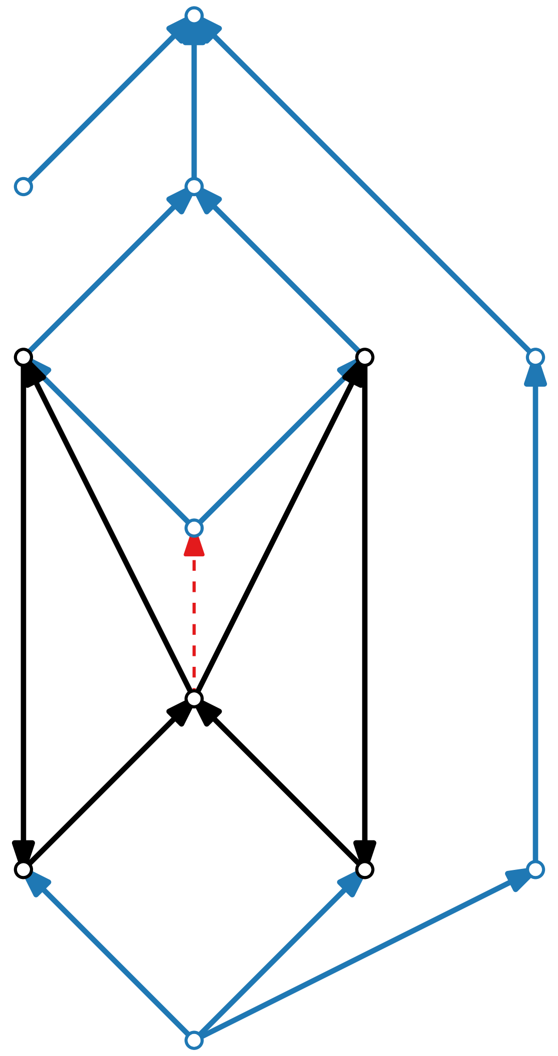

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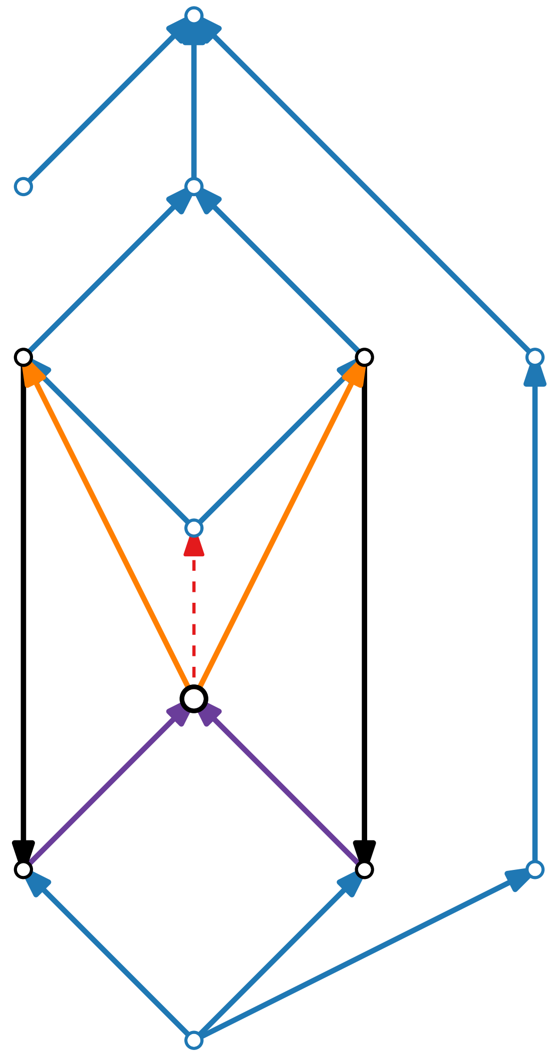

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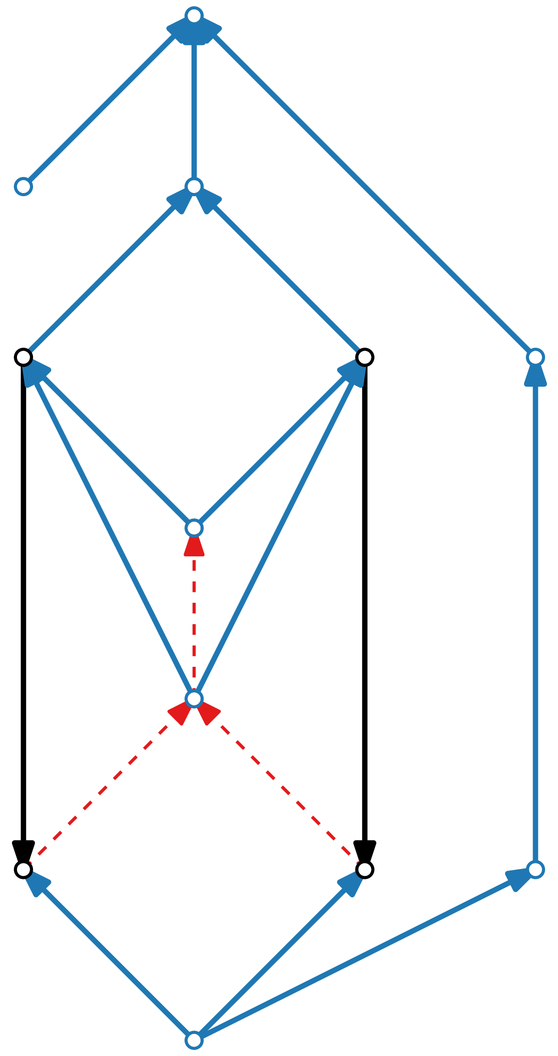

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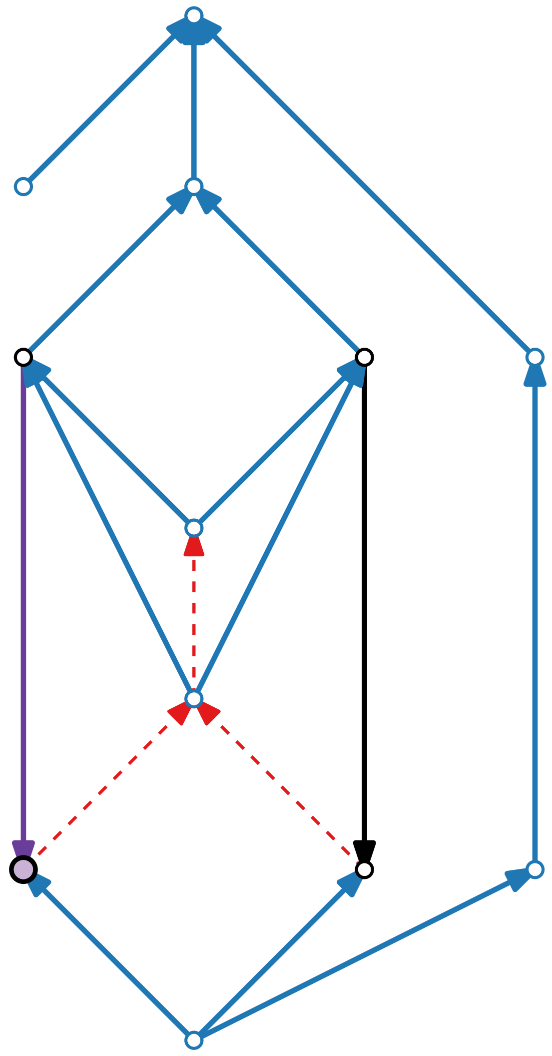

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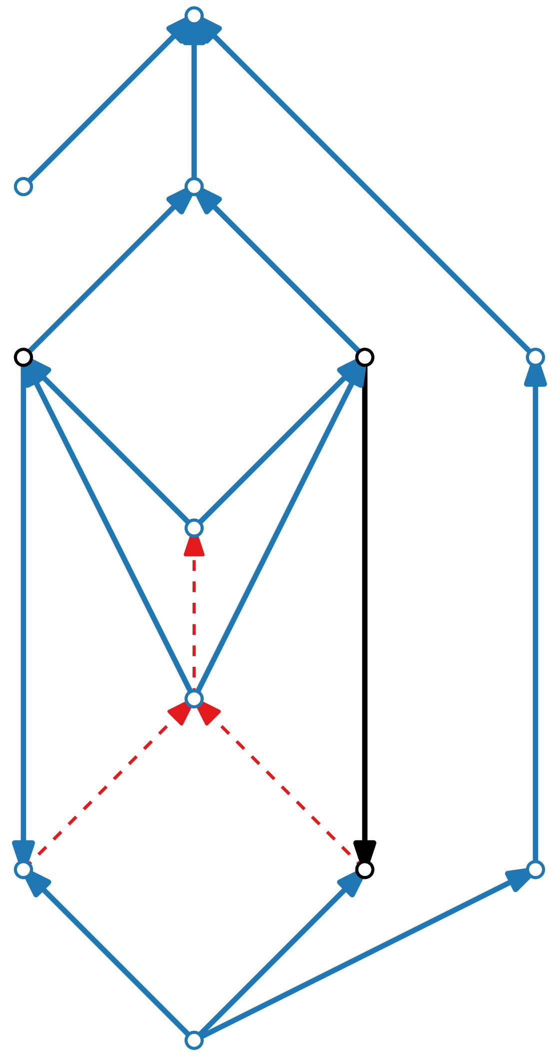

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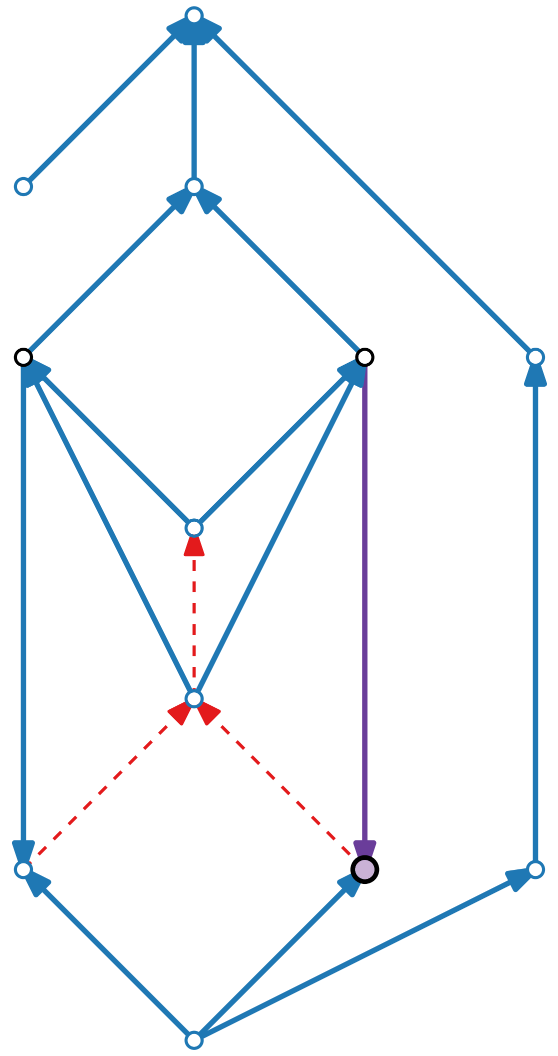

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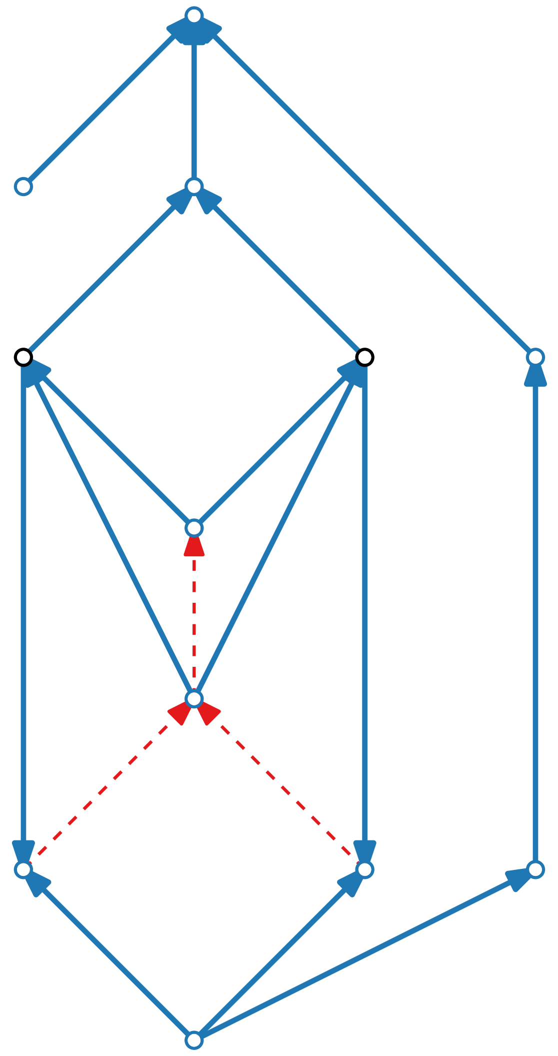

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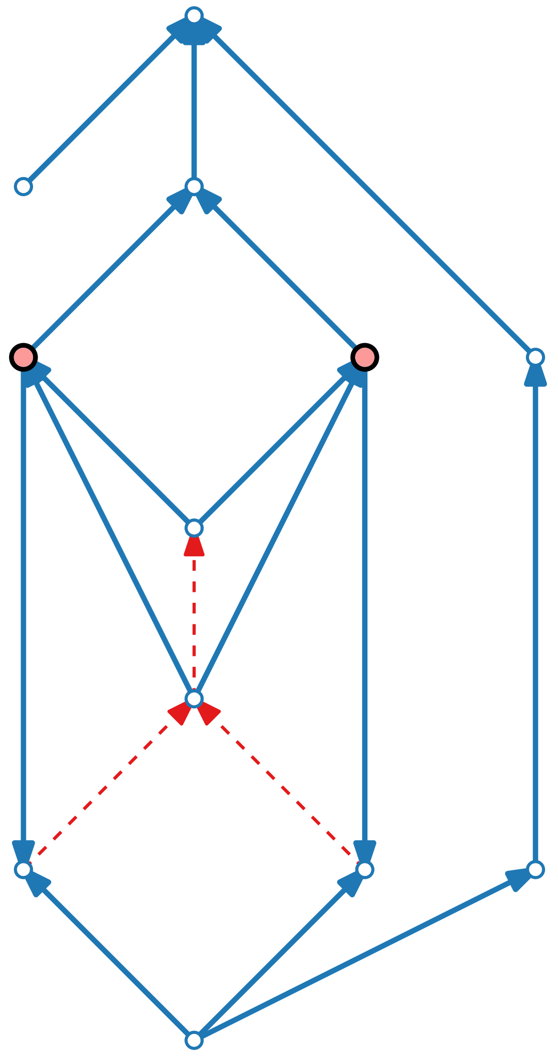

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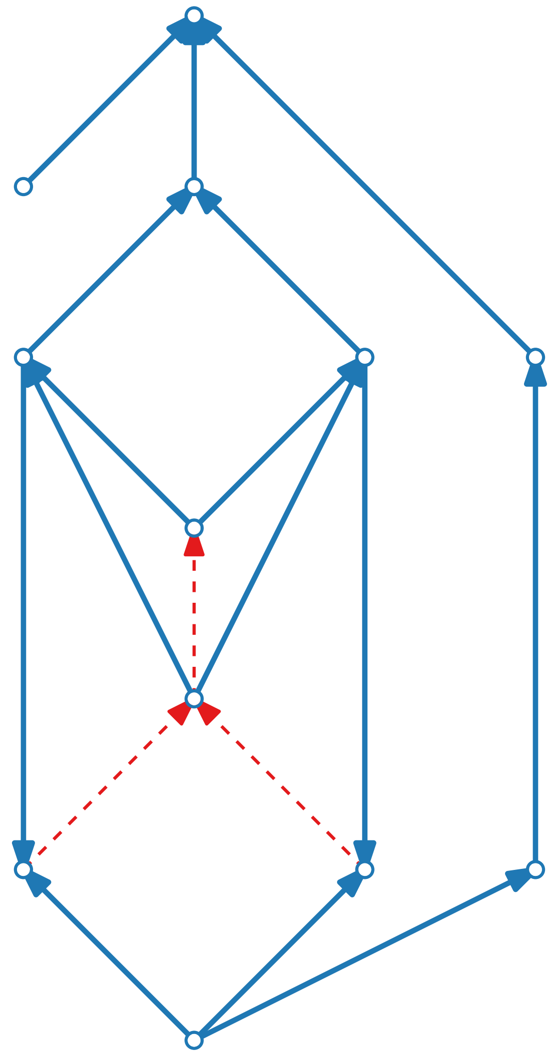

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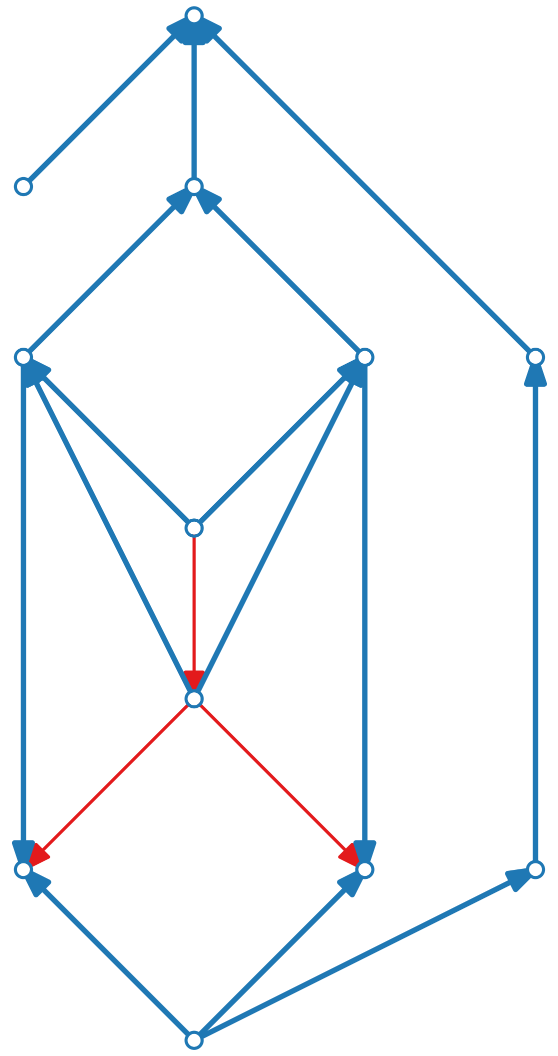

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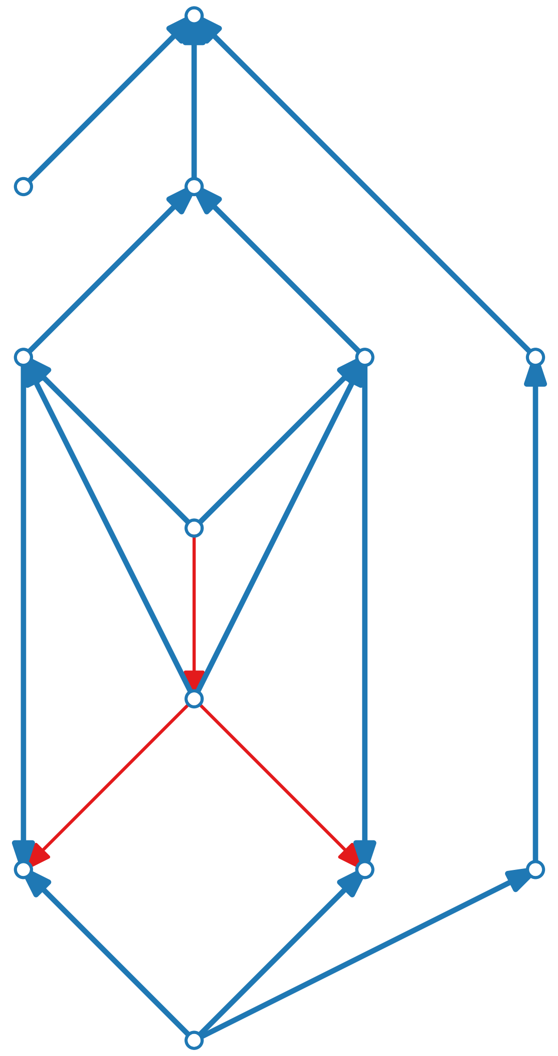

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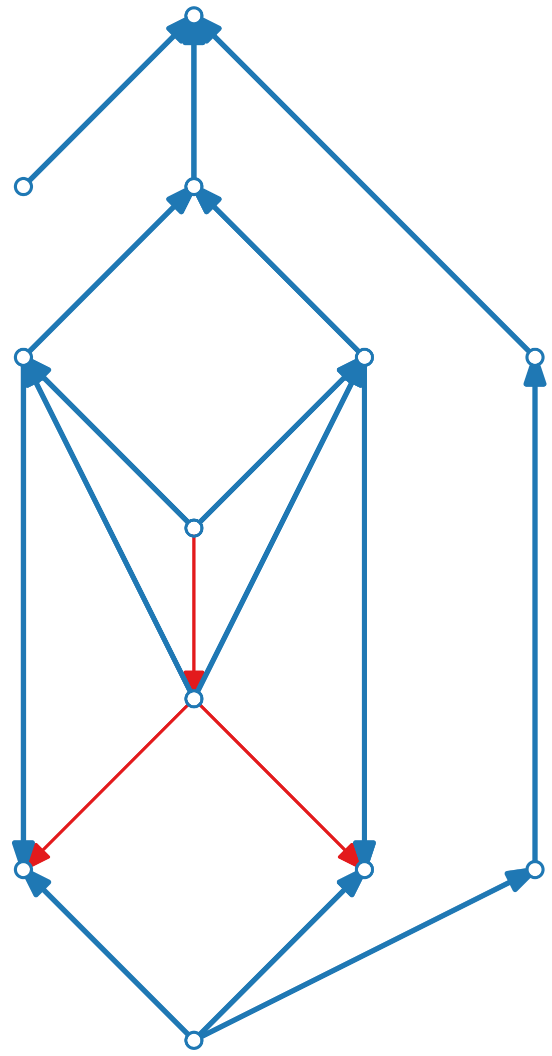
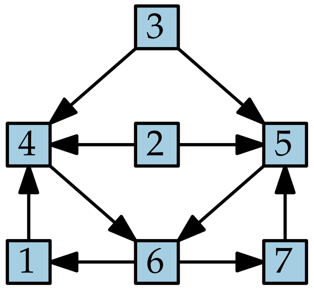
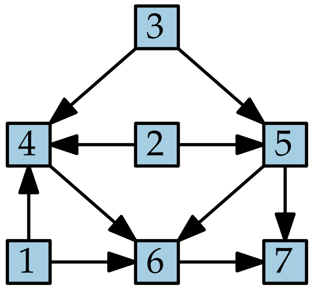


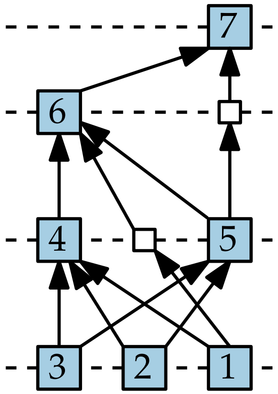
Universität Trier

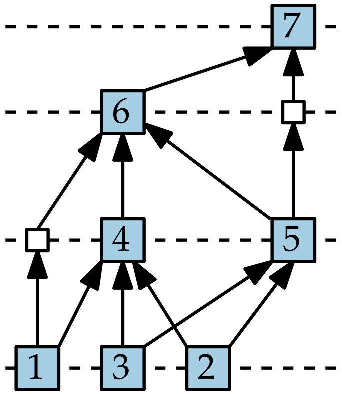
# Visualization of Graphs

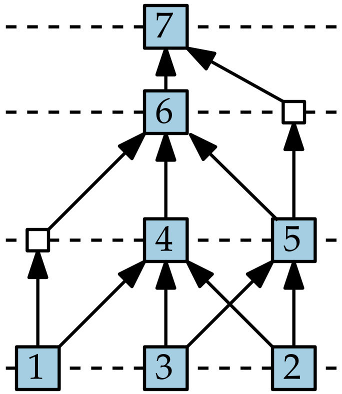


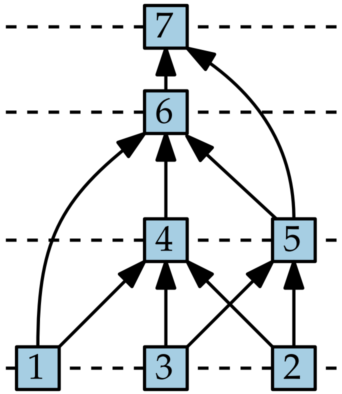




## Lecture 8:
## Hierarchical Layouts:
## Sugiyama Framework



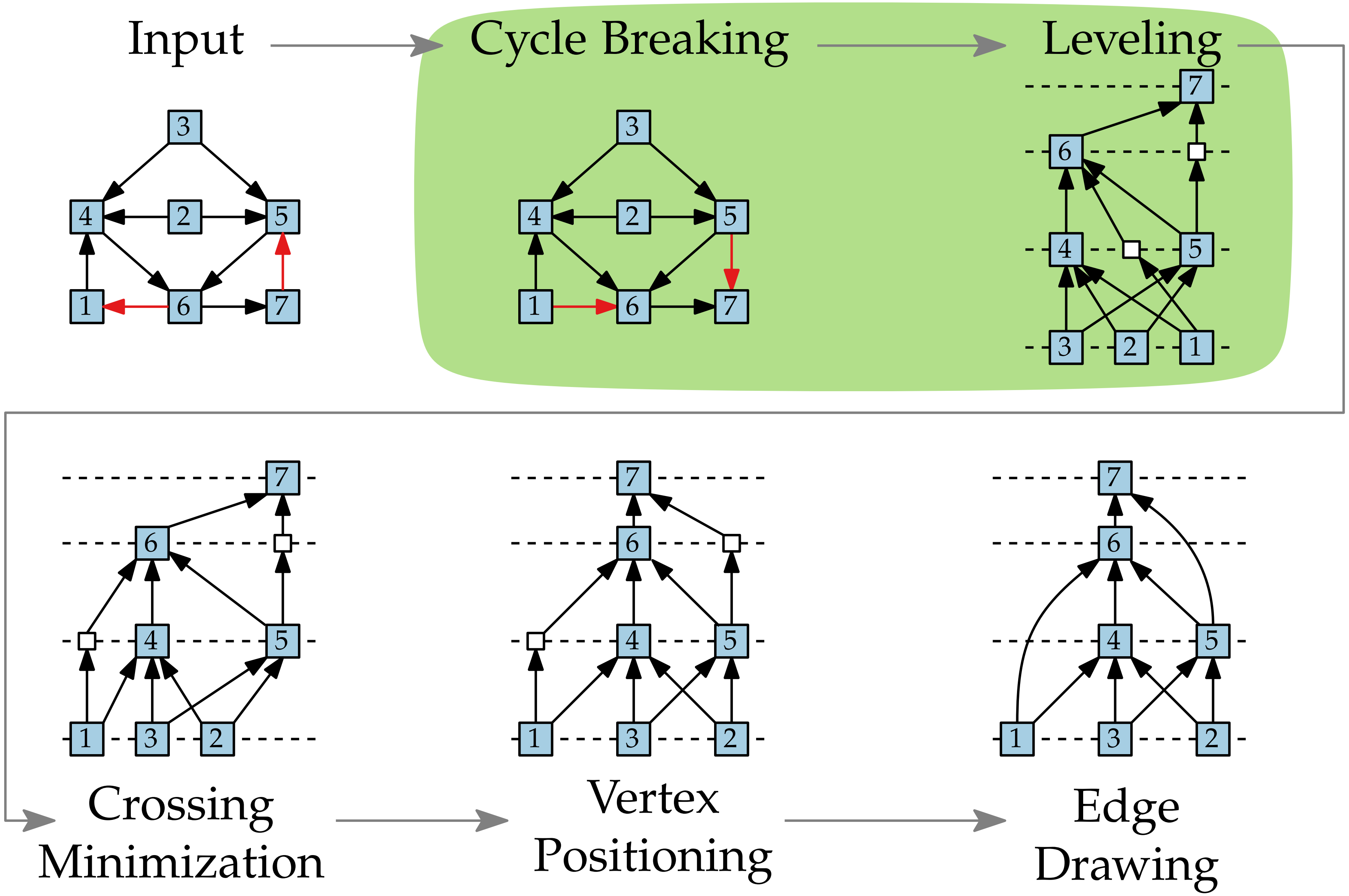




### Part III:
### Leveling

Philipp Kindermann

# Step 2: Leveling



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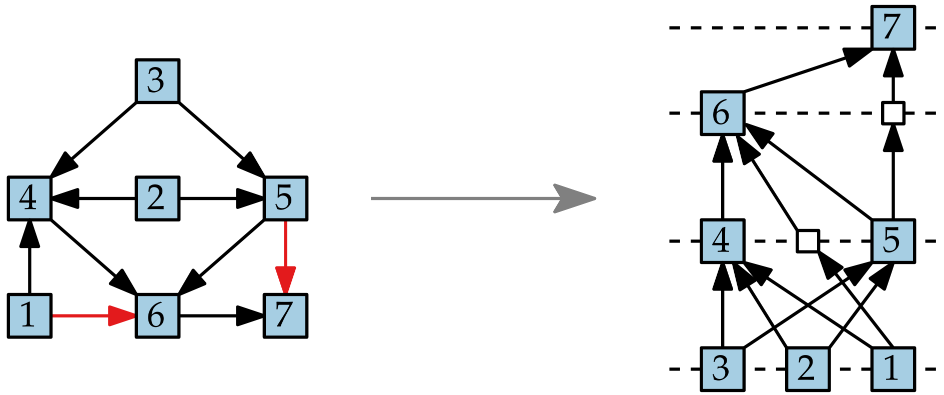
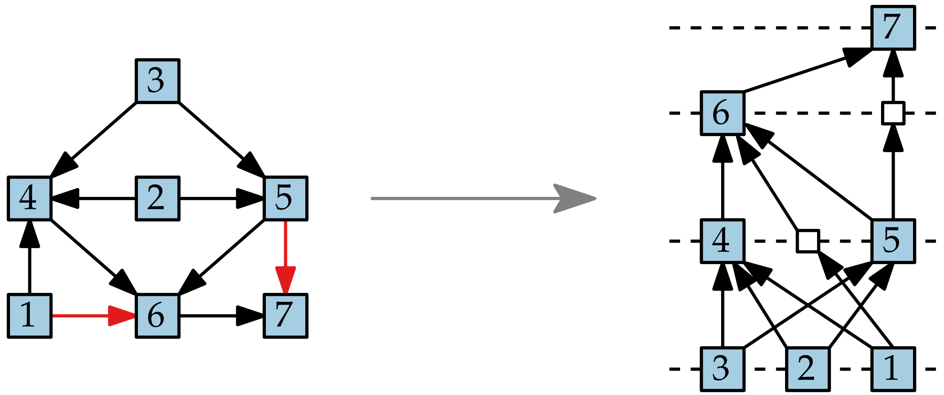



**Problem.**

# Step 2: Leveling

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- Input: acyclic digraph $G = (V, E)$

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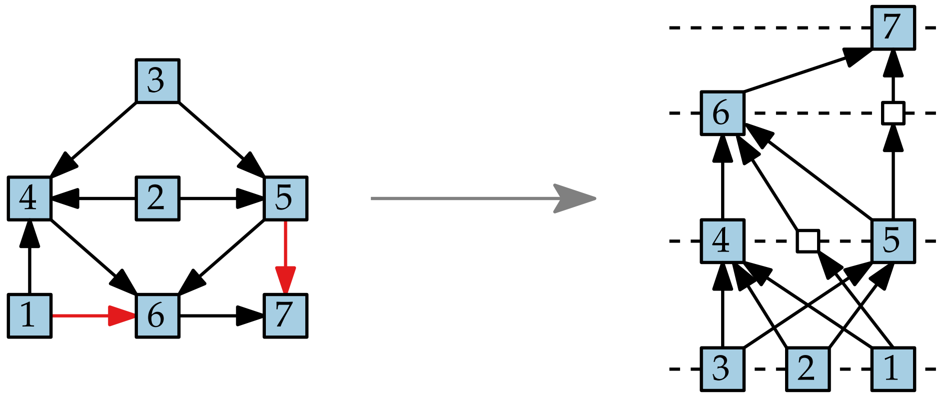


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- Input: acyclic digraph $G = (V, E)$
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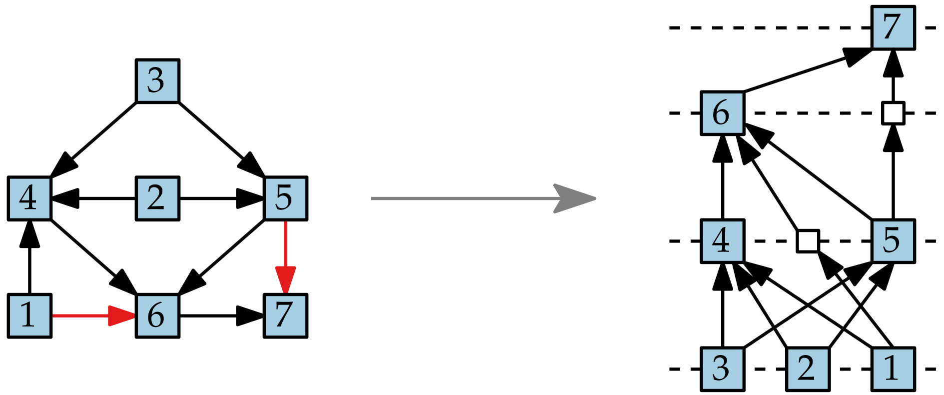


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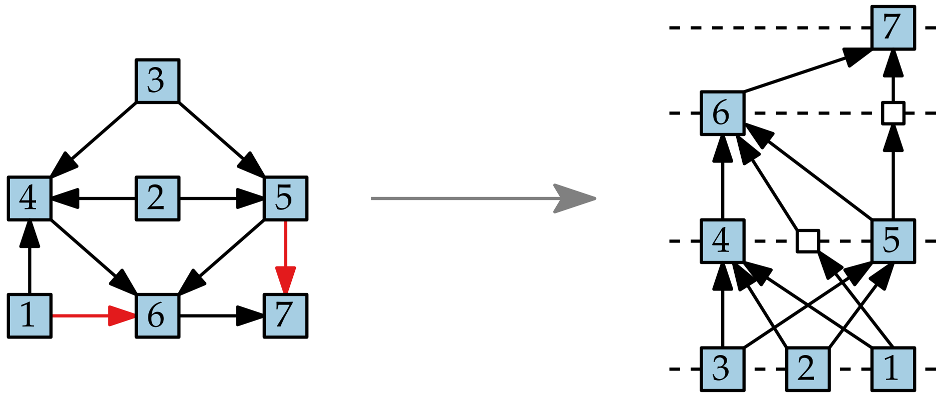


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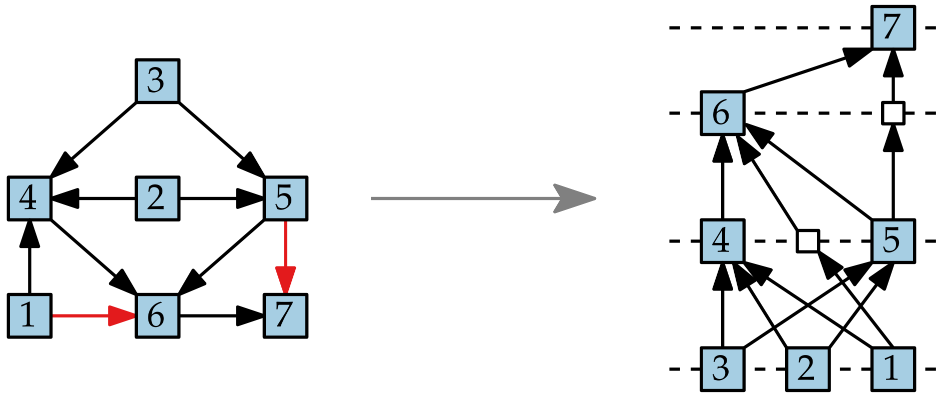


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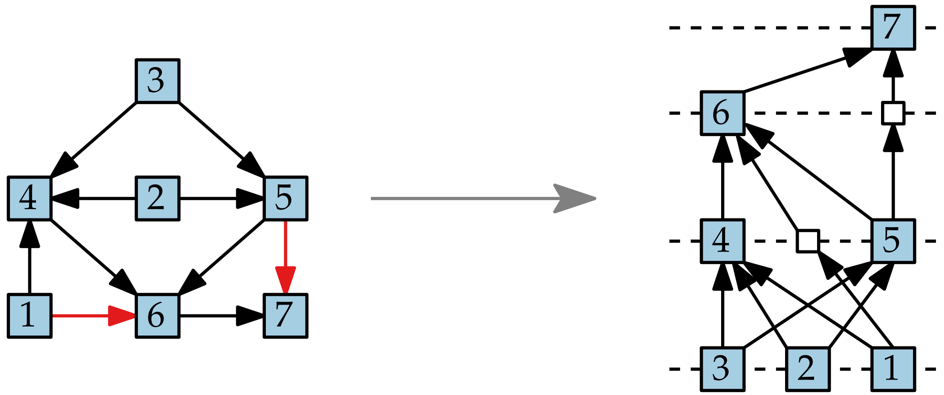


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**Objective** is to *minimize* …

- number of layers, i.e. $|y(V)|$
- length of the longest edge

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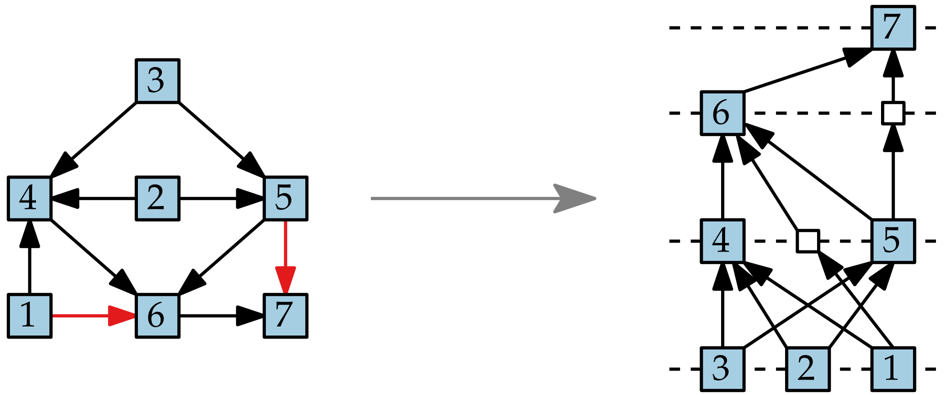


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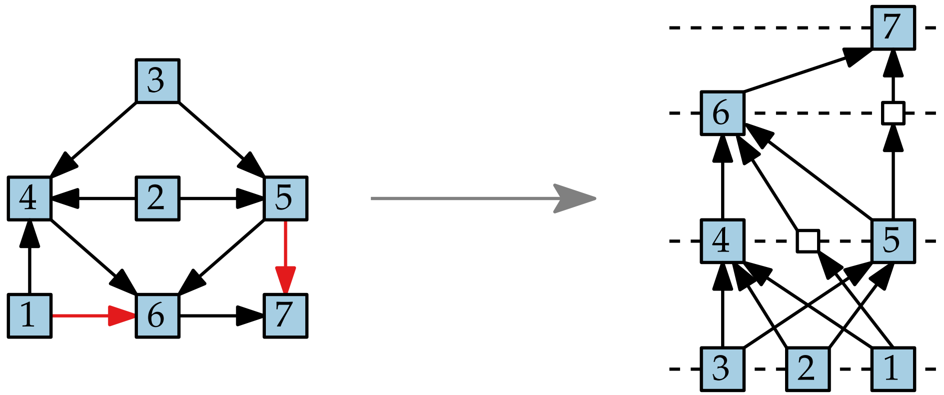


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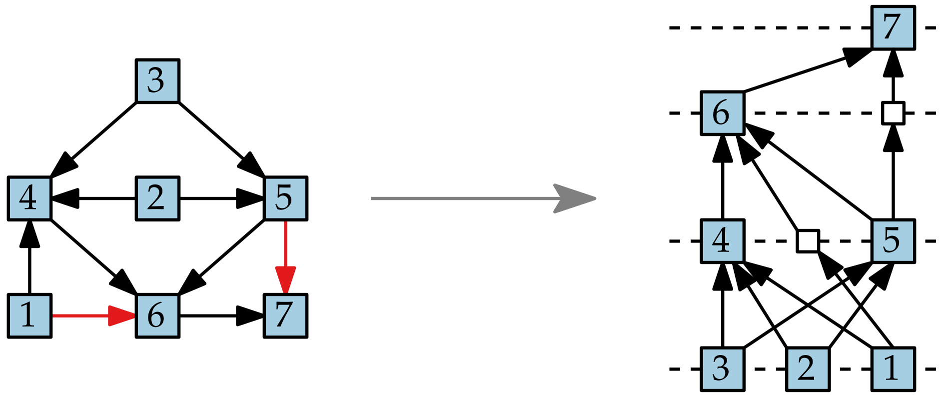


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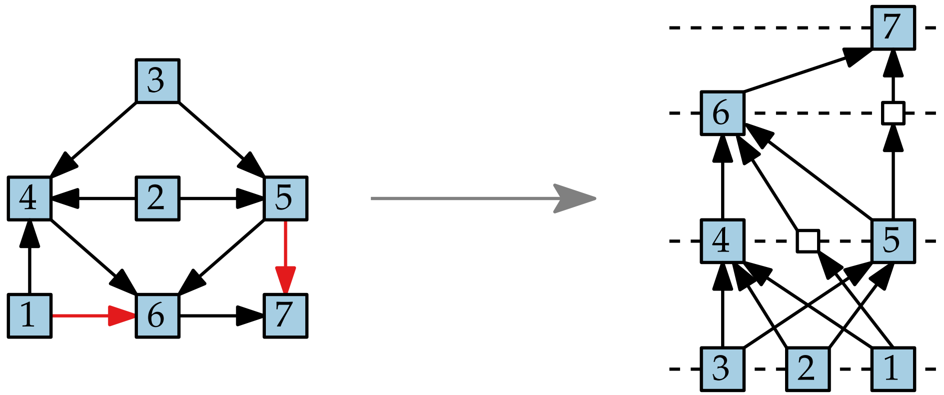


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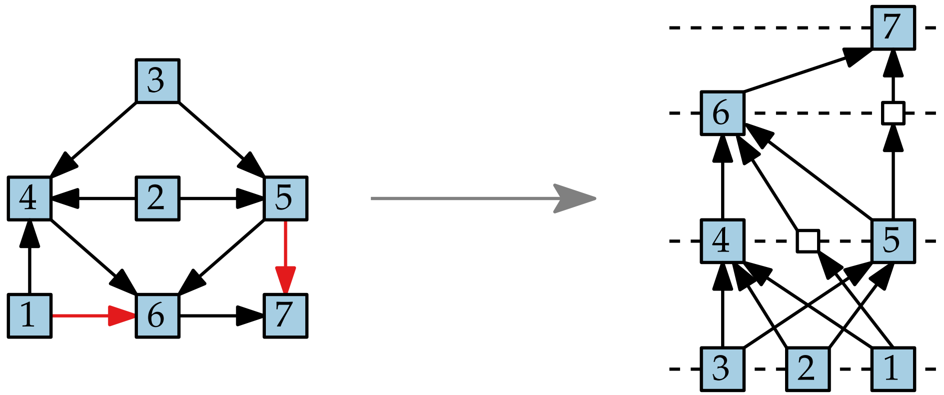


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- total edge length, i.e. number of dummy vertices

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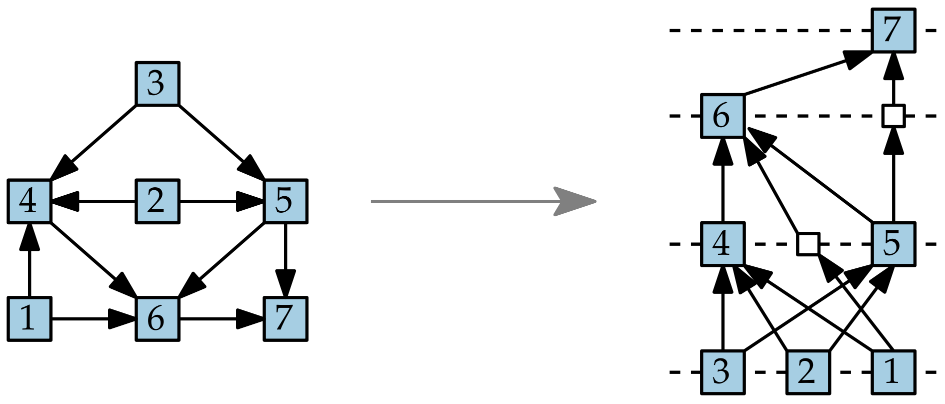

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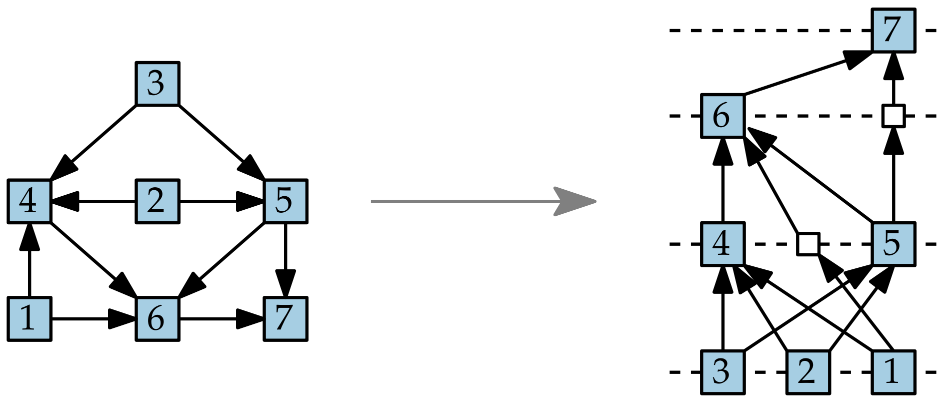

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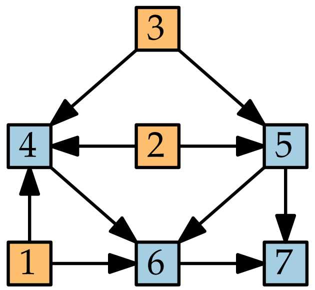


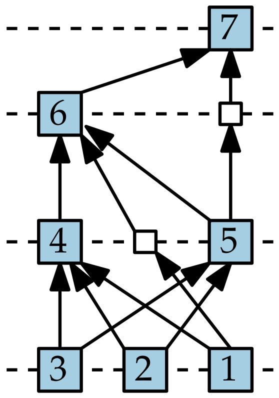

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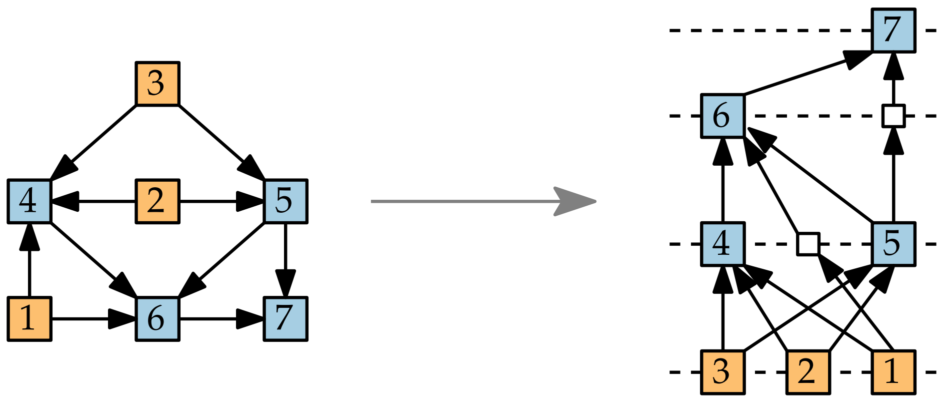

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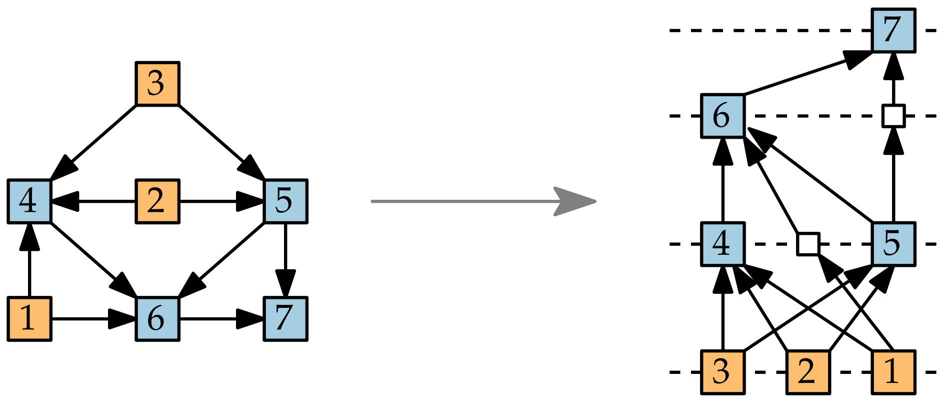

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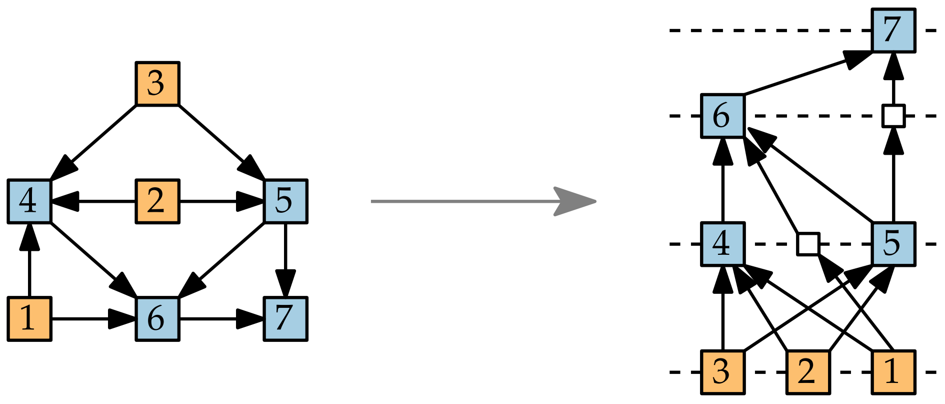


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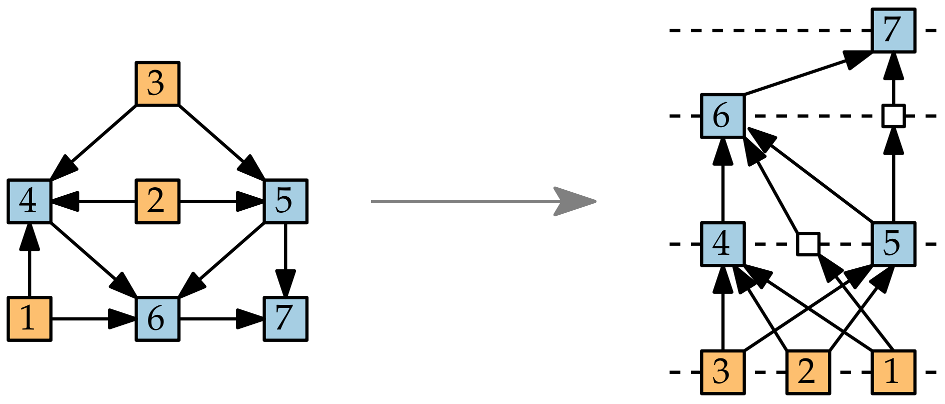


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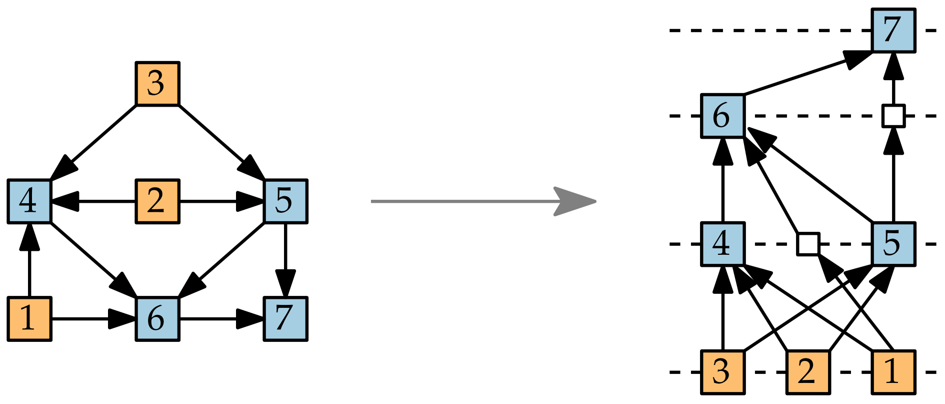


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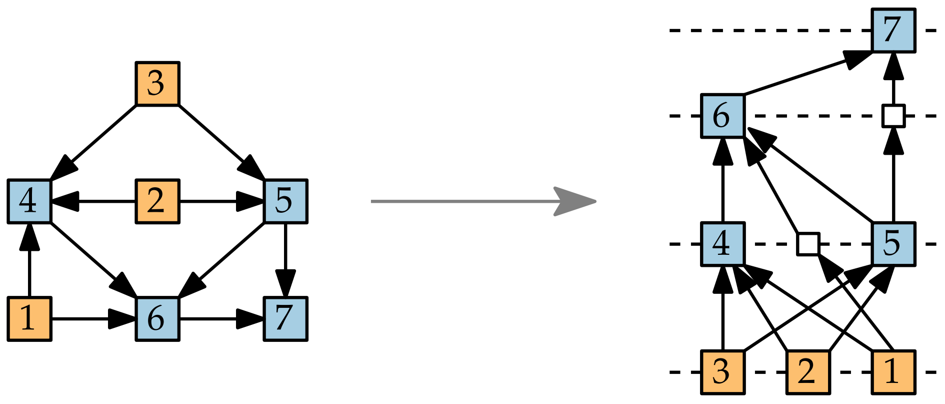


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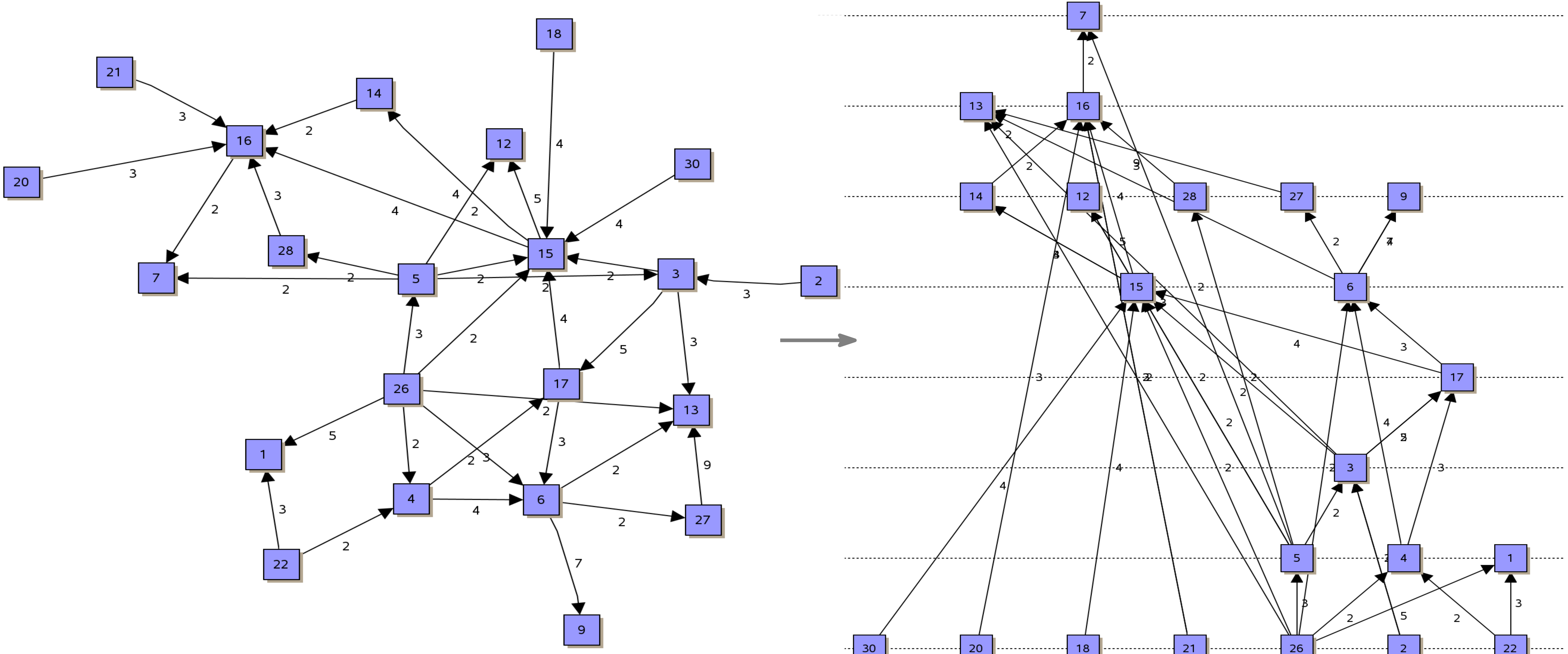
  ... which is optimal!
- Can be implemented in linear time with recursive algorithm.

# Example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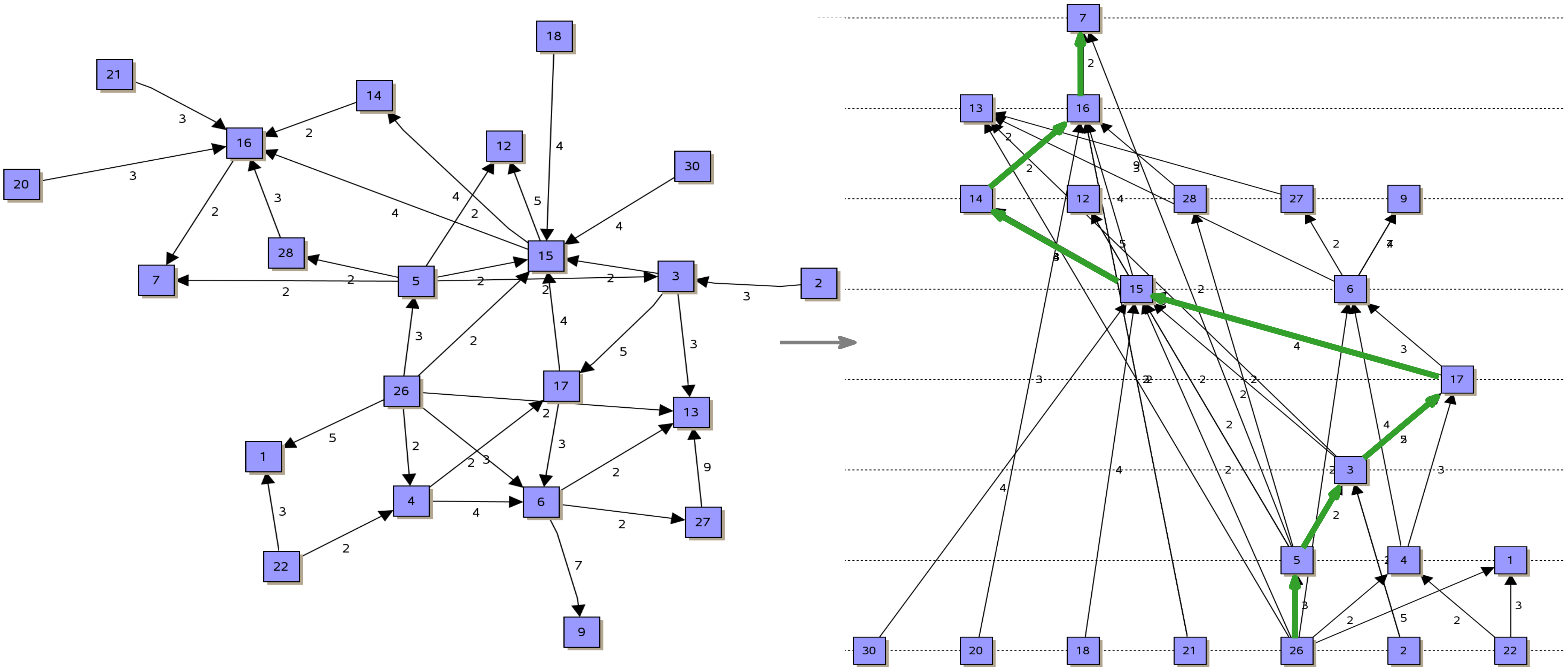
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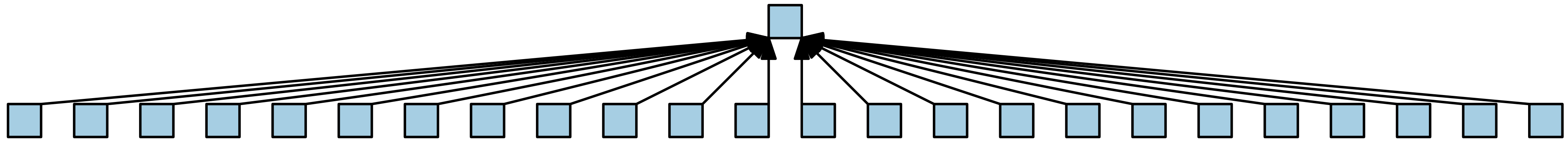
  $\Rightarrow$ Solution of the relaxed linear program is integer
- The total edge length can be minimized in polynomial time

# Width



Drawings can be very wide.

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- as long as there are free machines and available jobs, take the first available job and assign it to a free machine

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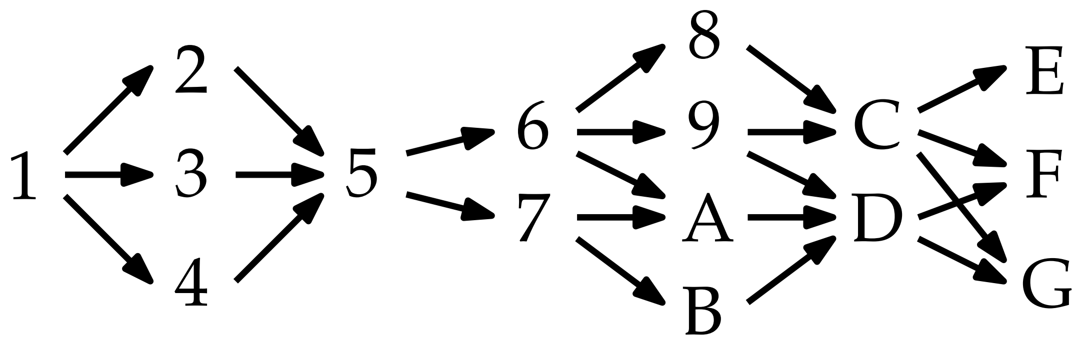
**Input:** Precedence graph (divided into layers of arbitrary width)



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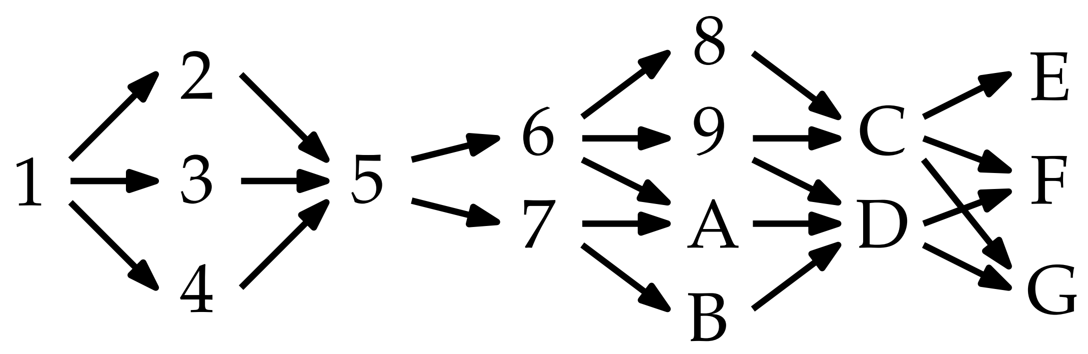


Number of Machines is $W = 2$.

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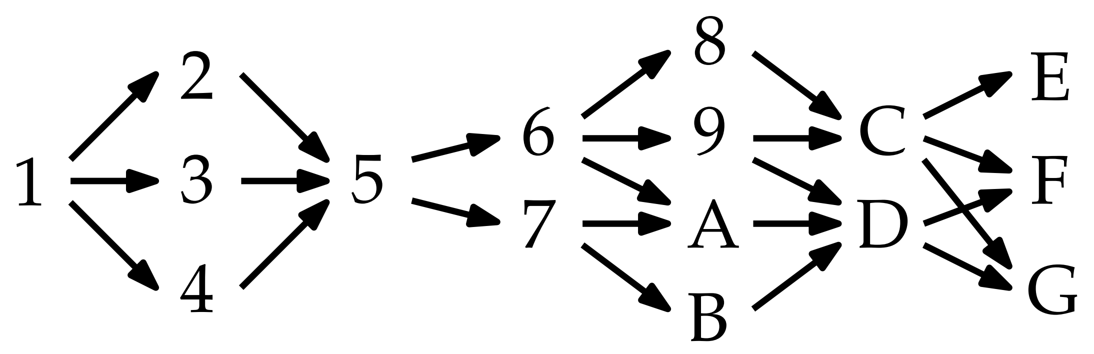


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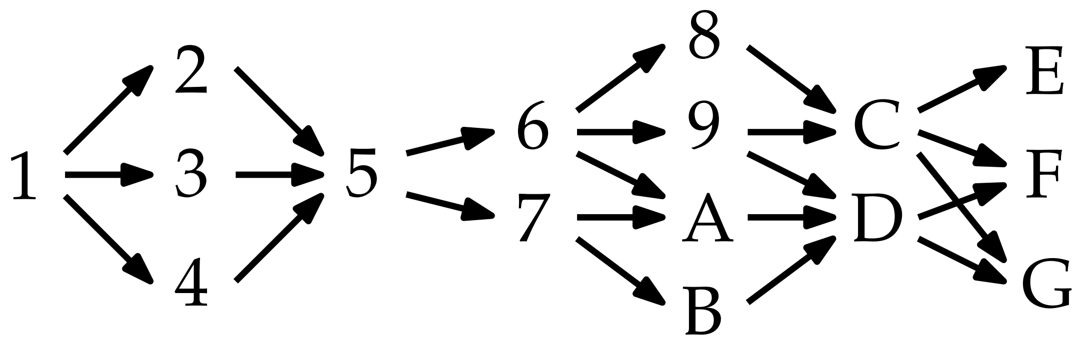


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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $M_1$ | | | | | | | | | | |
| $M_2$ | | | | | | | | | | |
| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

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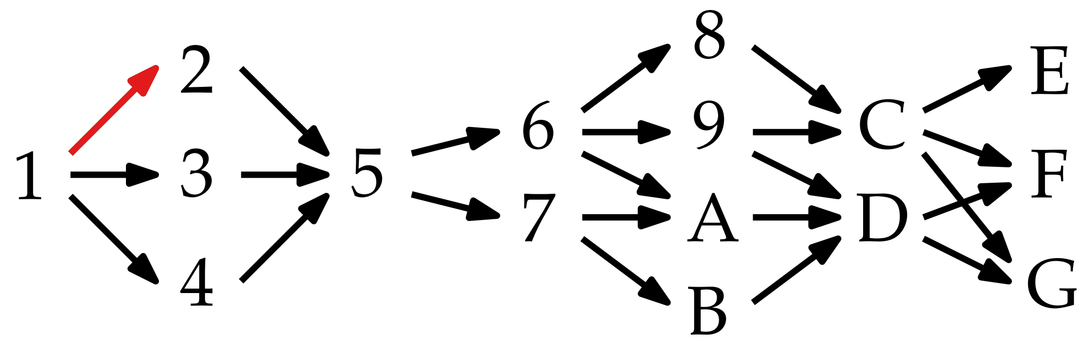


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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $M_1$ | 1 | | | | | | | | | |
| $M_2$ | – | | | | | | | | | |
| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

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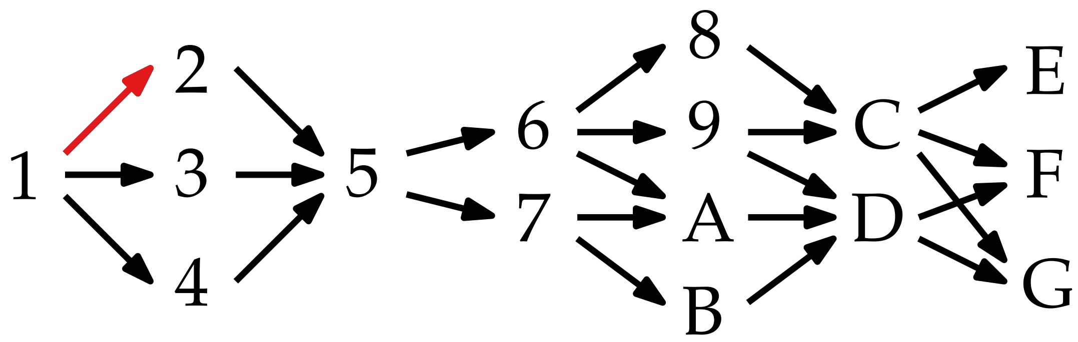


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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $M_1$ | 1 | 2 | | | | | | | | |
| $M_2$ | – | 3 | | | | | | | | |
| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

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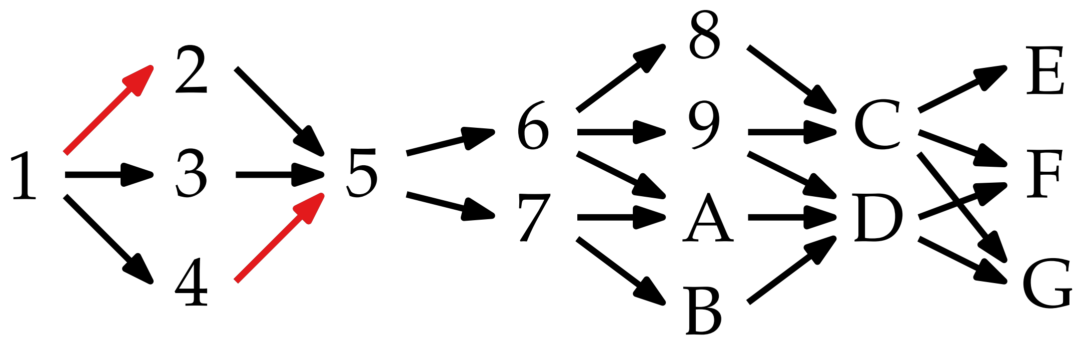


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**Output:** Schedule

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $M_1$ | 1 | 2 | 4 | | | | | | | |
| $M_2$ | – | 3 | – | | | | | | | |
| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

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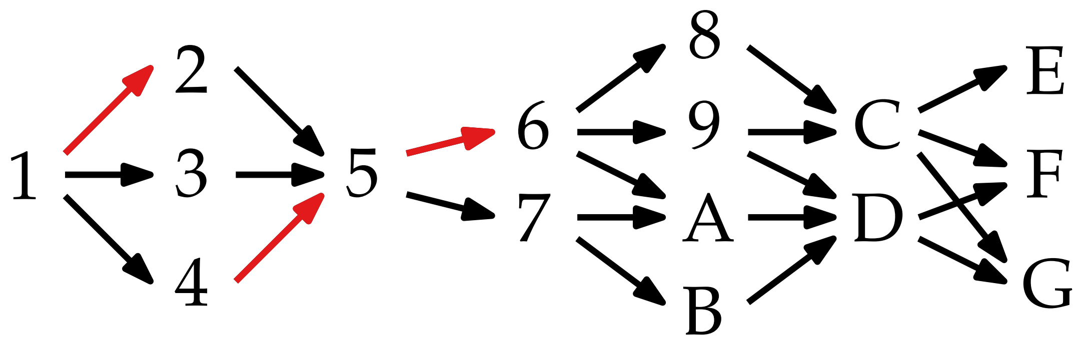


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**Output:** Schedule

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $M_1$ | 1 | 2 | 4 | 5 | | | | | | |
| $M_2$ | – | 3 | – | – | | | | | | |
| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

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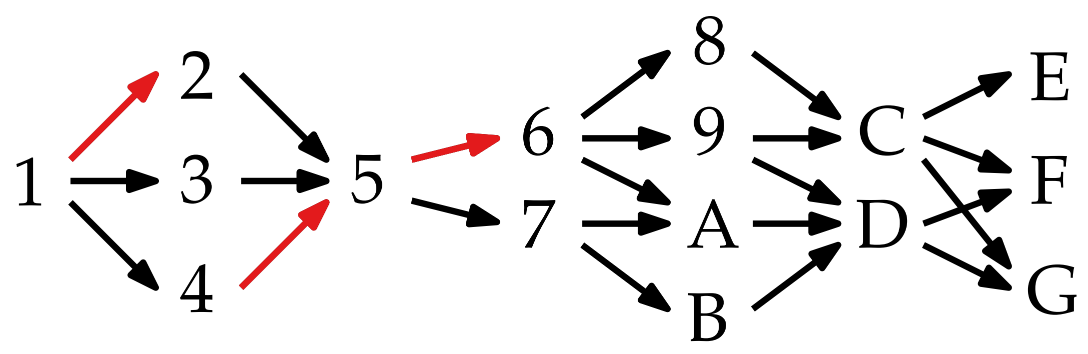


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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
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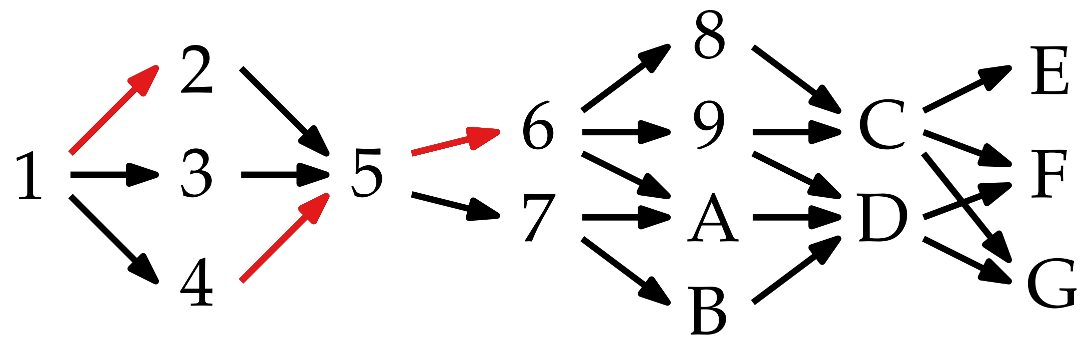


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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $M_1$ | 1 | 2 | 4 | 5 | 6 | 8 | | | | |
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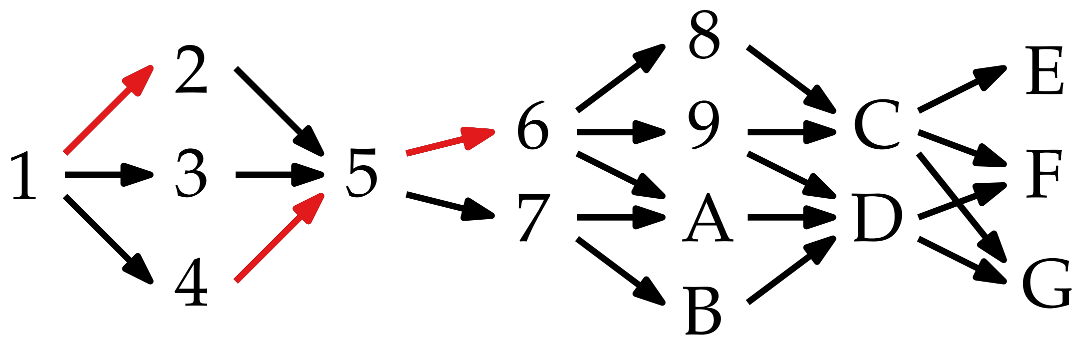


Number of Machines is $W = 2$.

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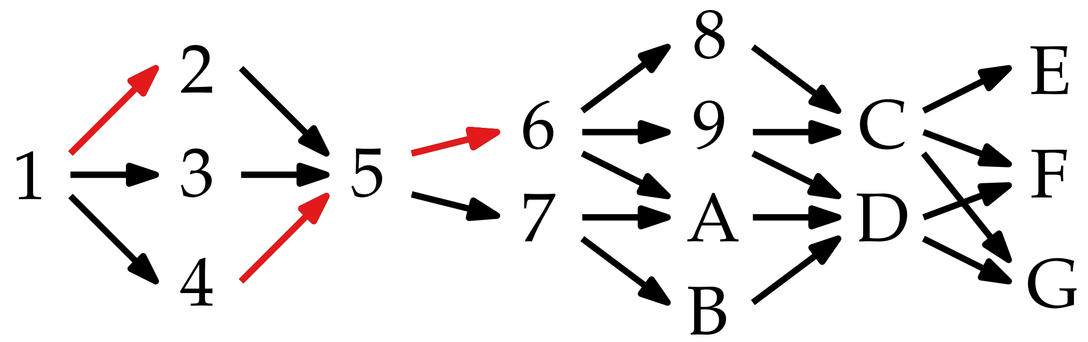
| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $M_1$ | 1 | 2 | 4 | 5 | 6 | 8 | A | | | |
| $M_2$ | – | 3 | – | – | 7 | 9 | B | | | |
| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

# Approximating PCMPS

**Input:** Precedence graph (divided into layers of arbitrary width)




Number of Machines is $W = 2$.

**Output:** Schedule

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $M_1$ | 1 | 2 | 4 | 5 | 6 | 8 | A | C | | |
| $M_2$ | – | 3 | – | – | 7 | 9 | B | D | | |
| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

# Approximating PCMPS

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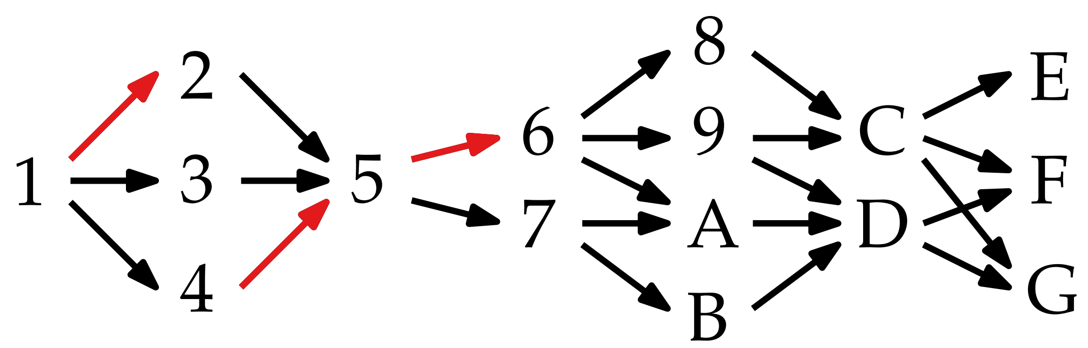


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|---|---|---|---|---|---|---|---|---|---|---|
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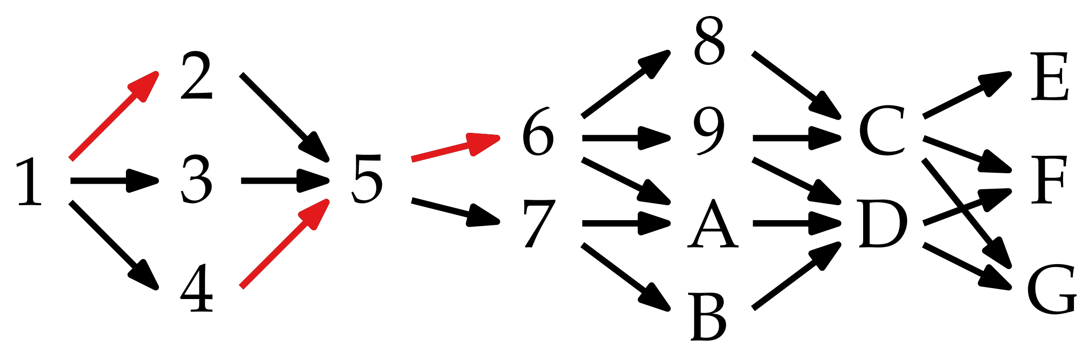


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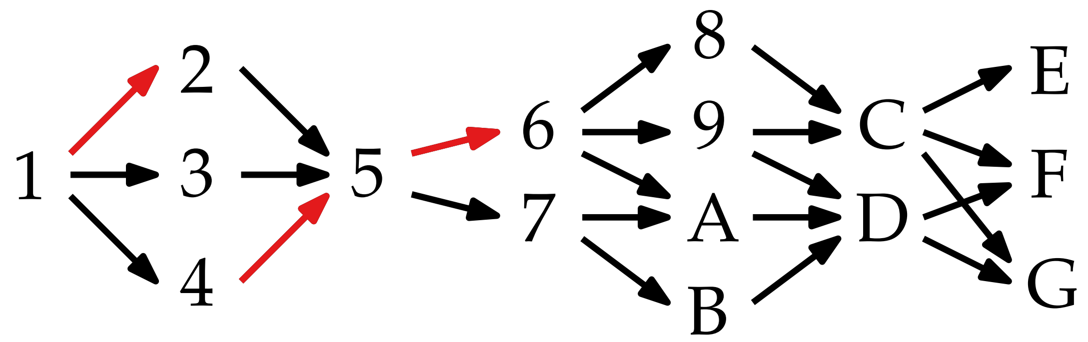


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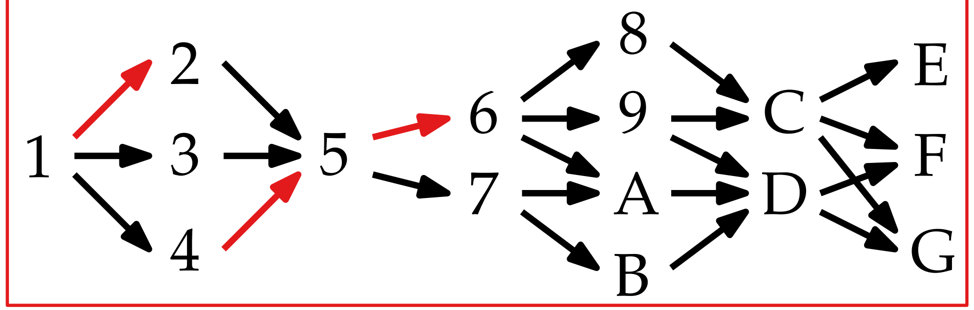
| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

**Question:** Good approximation factor?

# Approximating PCMPS - Analysis for $W = 2$

Precedence graph $G_<$




Schedule

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*„The art of the lower bound“*

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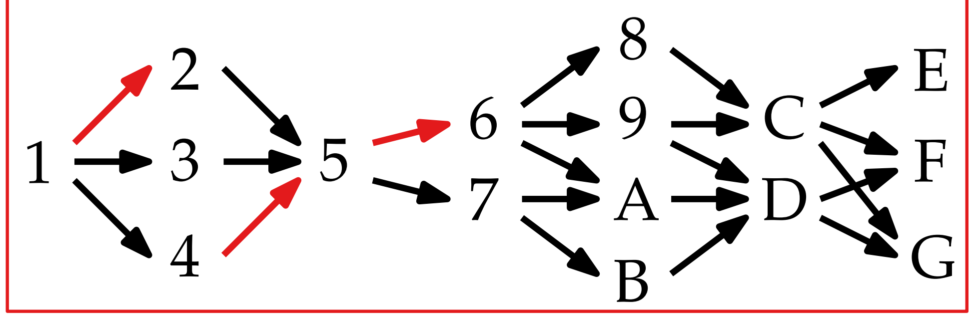


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*„The art of the lower bound"*

OPT $\geq$

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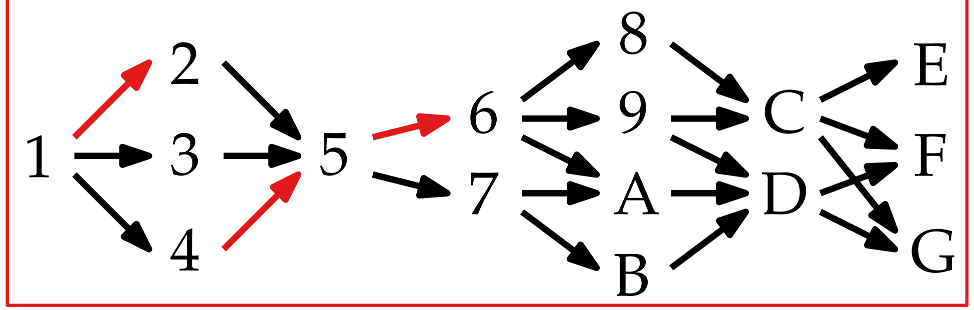
Precedence graph $G_<$

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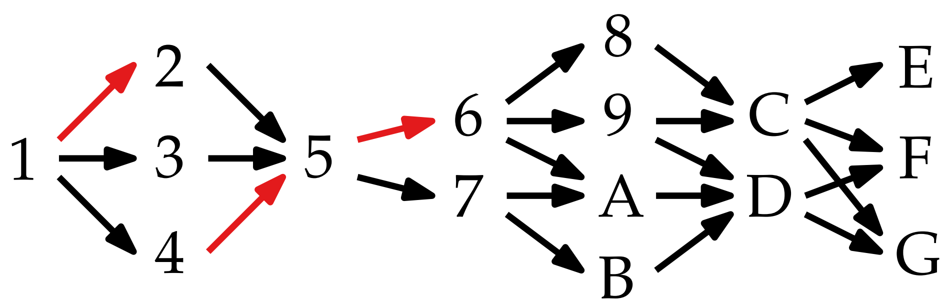
*„The art of the lower bound"*

OPT $\geq \lceil n/2 \rceil$

# Approximating PCMPS - Analysis for $W = 2$

Precedence graph $G_<$

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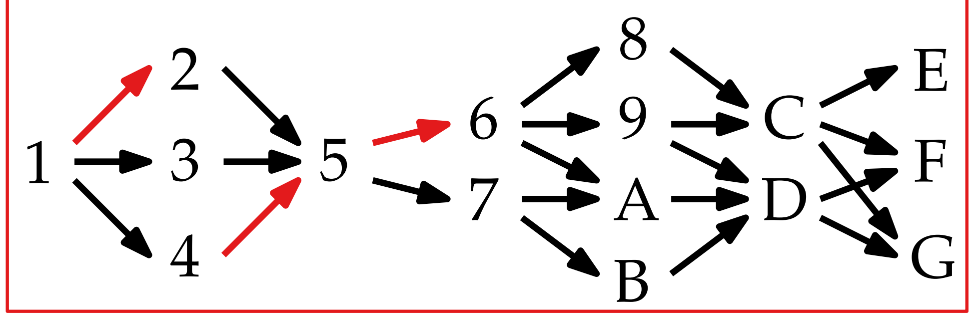
*„The art of the lower bound"*

OPT $\geq$ $\lceil n/2 \rceil$ and OPT $\geq$

# Approximating PCMPS - Analysis for $W = 2$

Precedence graph $G_<$

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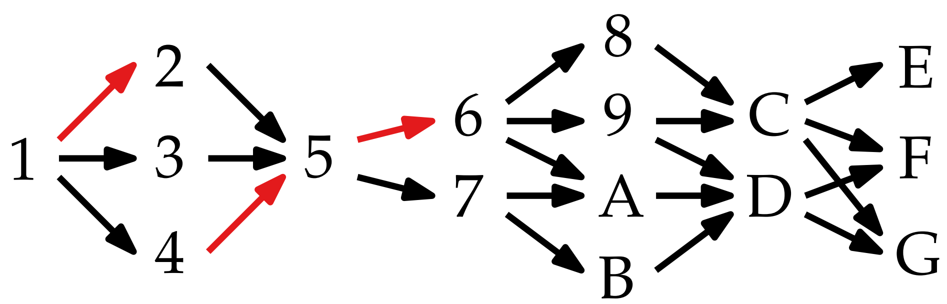
*„The art of the lower bound“*

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

# Approximating PCMPS - Analysis for $W = 2$

Precedence graph $G_<$

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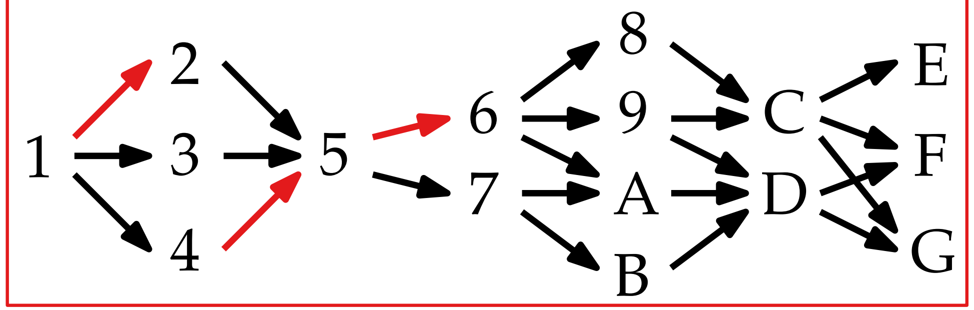
*„The art of the lower bound"*

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

**Goal:** measure the quality of our algorithm using the lower bounds

# Approximating PCMPS - Analysis for $W = 2$

Precedence graph $G_<$

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*„The art of the lower bound"*

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

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**Bound.** ALG $\leq$

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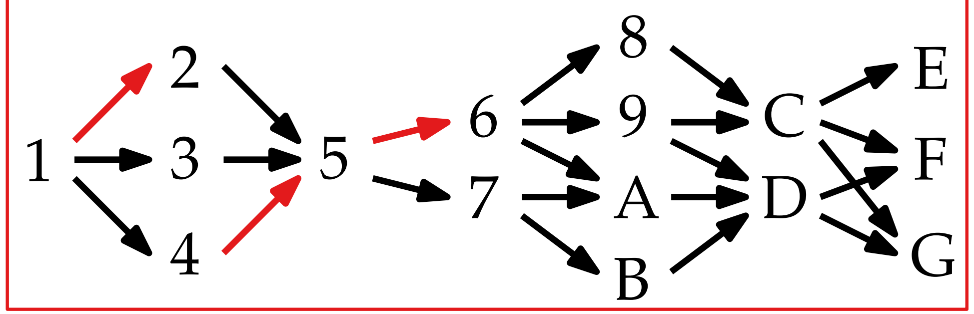
Precedence graph $G_<$

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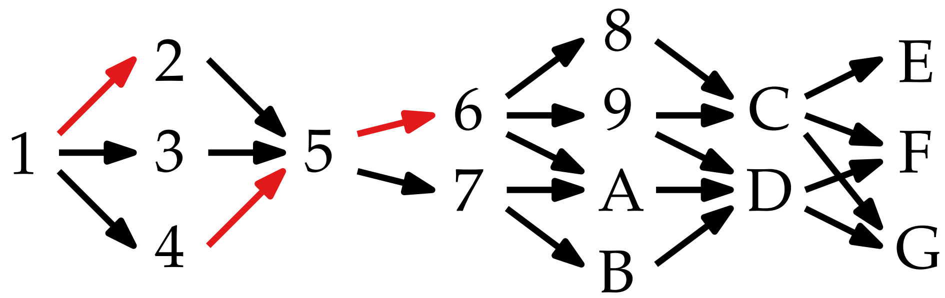
Precedence graph $G_<$




Schedule

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**Bound.** ALG $\leq$

insertion of pauses (–) in the schedule
(except the last) maps to layers of $G_<$

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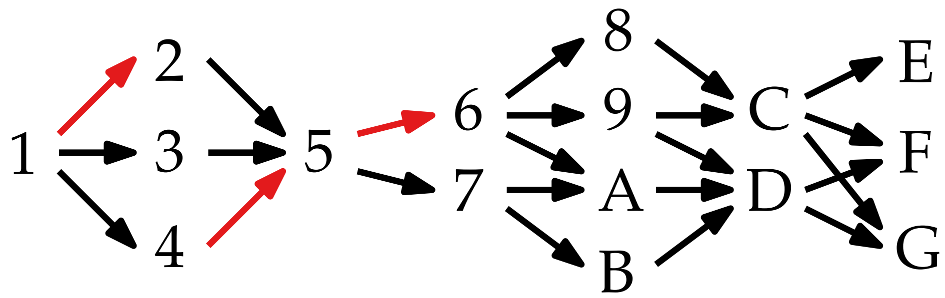
Precedence graph $G_<$

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*„The art of the lower bound“*

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq$ $\ell :=$ Number of layers of $G_<$

**Goal:** measure the quality of our algorithm using the lower bounds

**Bound.** ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil$

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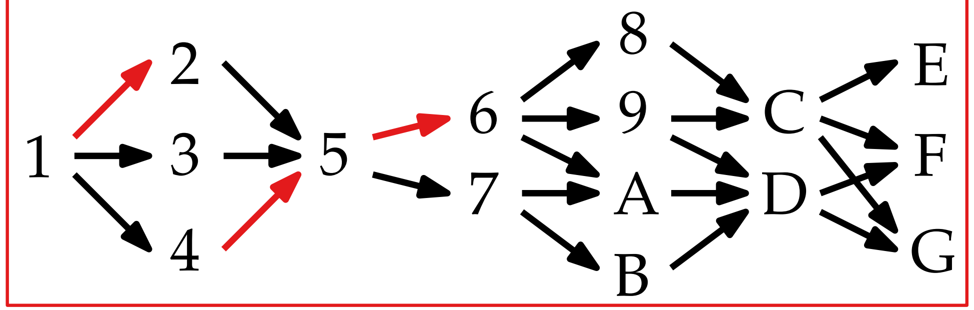
Precedence graph $G_<$

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*„The art of the lower bound“*

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

**Goal:** measure the quality of our algorithm using the lower bounds

**Bound.** ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx$

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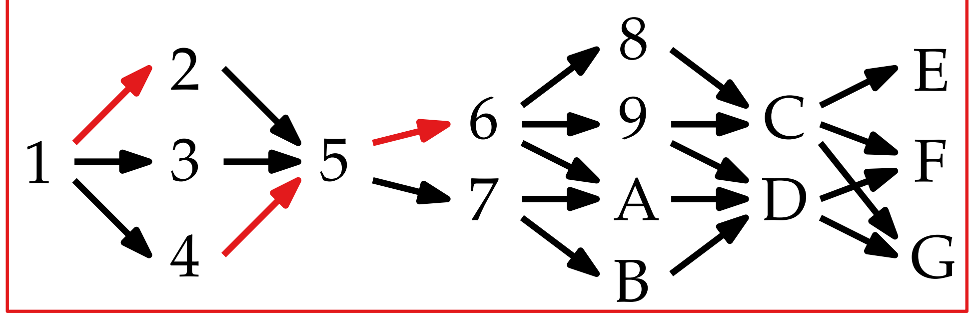
Precedence graph $G_<$

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*„The art of the lower bound"*

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq$ $\ell :=$ Number of layers of $G_<$

**Goal:** measure the quality of our algorithm using the lower bounds

**Bound.** ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx \lceil n/2 \rceil + \ell/2$

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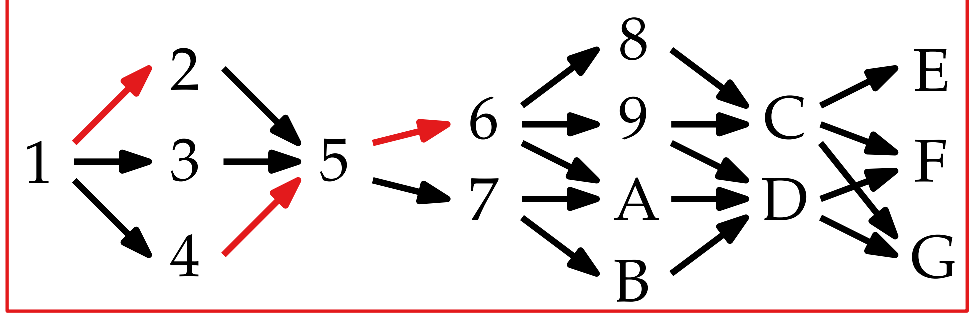
# Approximating PCMPS - Analysis for $W = 2$

Precedence graph $G_<$

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*„The art of the lower bound“*

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

**Goal:** measure the quality of our algorithm using the lower bounds

**Bound.** ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx \lceil n/2 \rceil + \ell/2 \leq$

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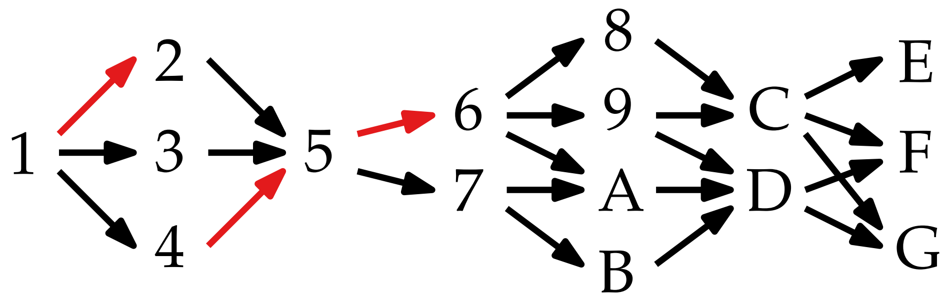


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OPT $\geq \lceil n/2 \rceil$ and OPT $\geq$ $\ell :=$ Number of layers of $G_<$

**Goal:** measure the quality of our algorithm using the lower bounds

**Bound.** $\text{ALG} \leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx \lceil n/2 \rceil + \ell/2 \leq 3/2 \cdot \text{OPT}$

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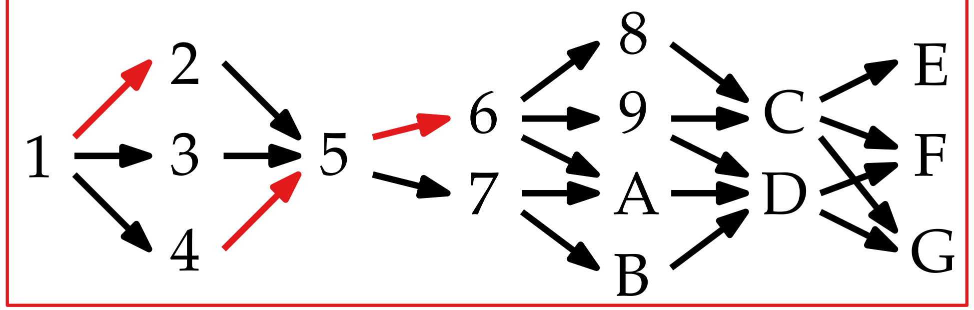


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*„The art of the lower bound"*

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

**Goal:** measure the quality of our algorithm using the lower bounds

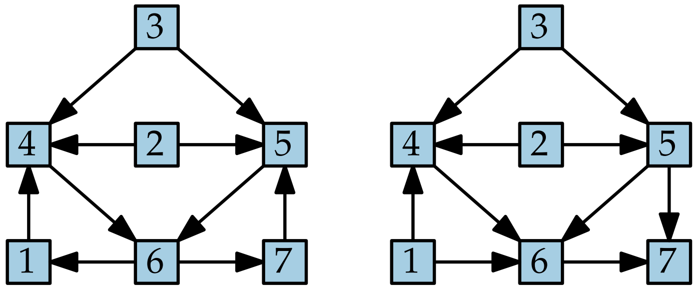
$\leq (2 - 1/W) \cdot$ OPT in general case

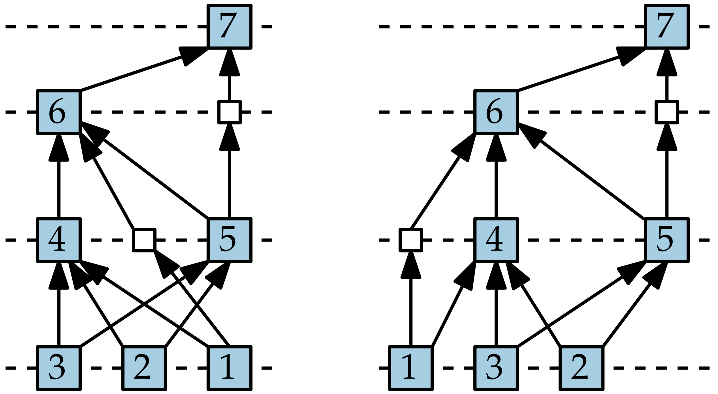
**Bound.** ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx \lceil n/2 \rceil + \ell/2 \leq 3/2 \cdot$ OPT

insertion of pauses (–) in the schedule (except the last) maps to layers of $G_<$

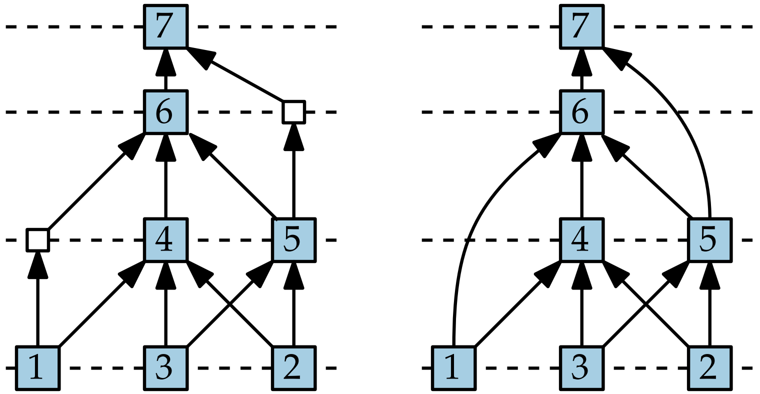
Universität Trier

# Visualization of Graphs





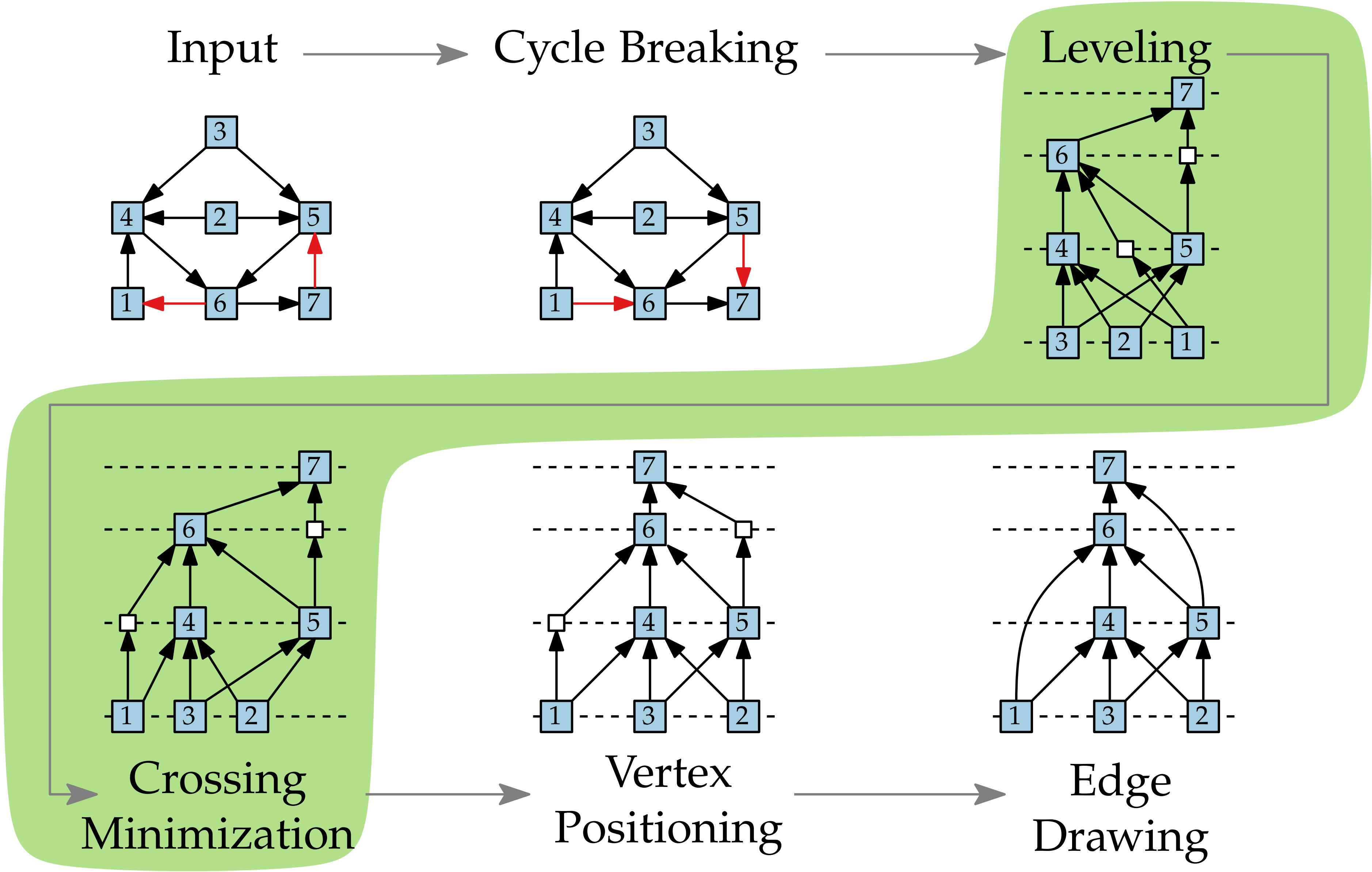
## Lecture 8:
## Hierarchical Layouts:
## Sugiyama Framework



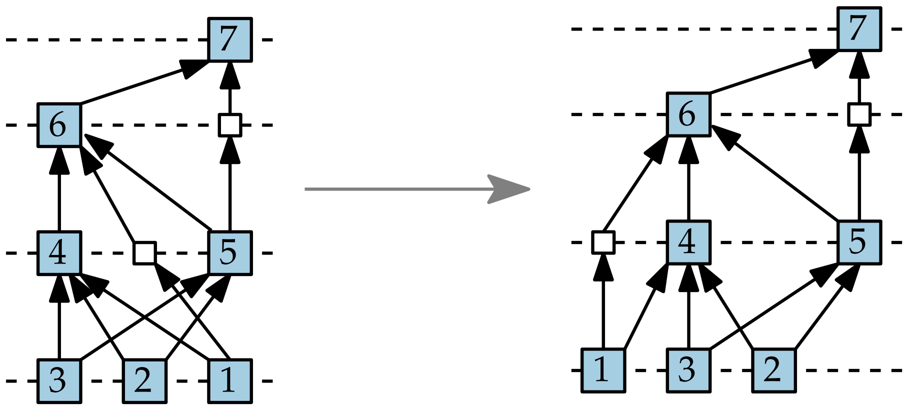
### Part IV:
### Crossing Minimization

Philipp Kindermann

# Step 3: Crossing Minimization

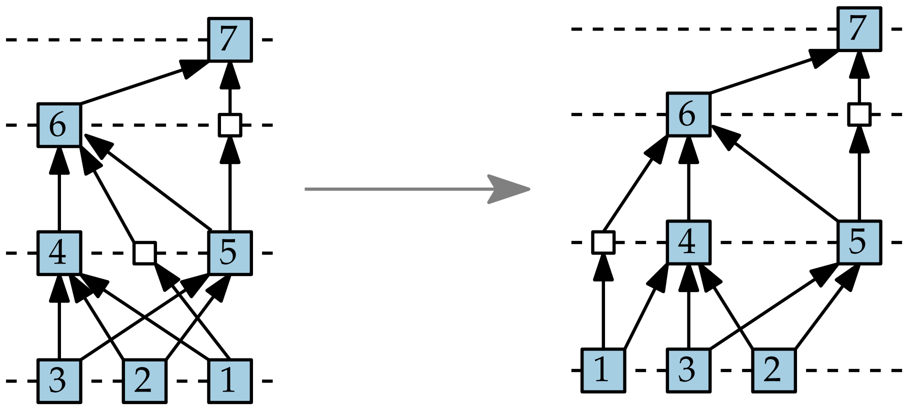


# Step 3: Crossing Minimization




**Problem.**

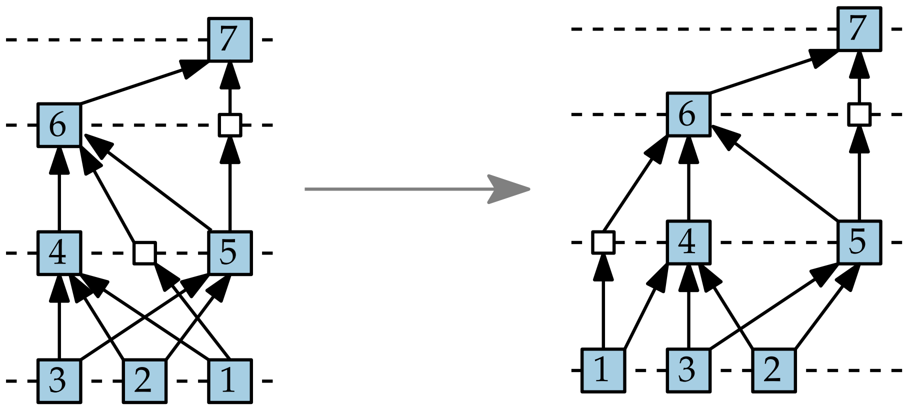
# Step 3: Crossing Minimization




**Problem.**

- Input: Graph $G$, layering $y \colon V \to \{1, \ldots, n\}$

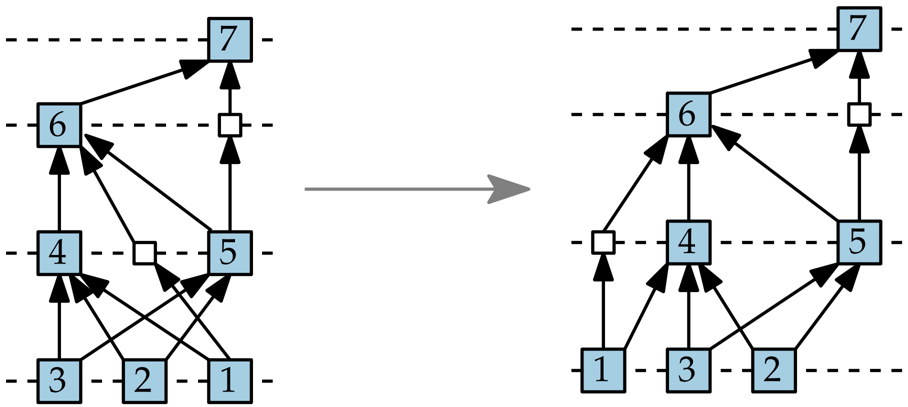
# Step 3: Crossing Minimization




**Problem.**

- Input: Graph $G$, layering $y\colon V \to \{1, \ldots, n\}$
- Output: (Re-)ordering of vertices in each layer so that the number of crossings in minimized.

# Step 3: Crossing Minimization




**Problem.**

- Input: Graph $G$, layering $y \colon V \to \{1, \ldots, n\}$
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- NP-hard, even for 2 layers [Garey & Johnson '83]

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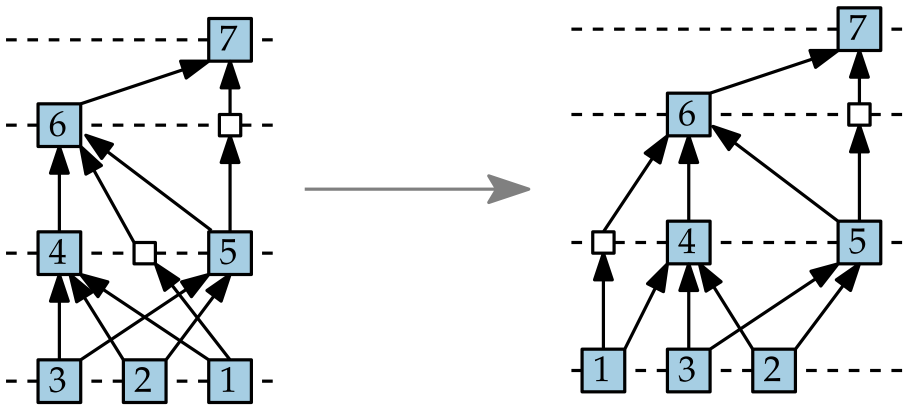


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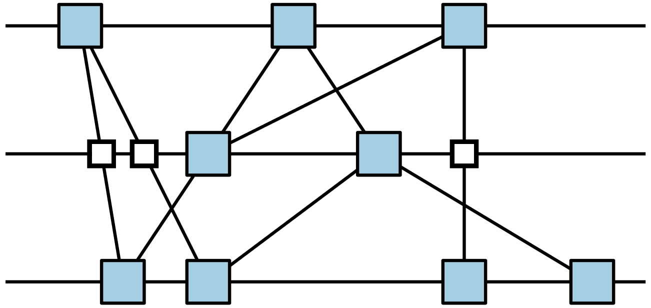
- NP-hard, even for 2 layers [Garey & Johnson '83]
- hardly any approaches optimize over multiple layers :(

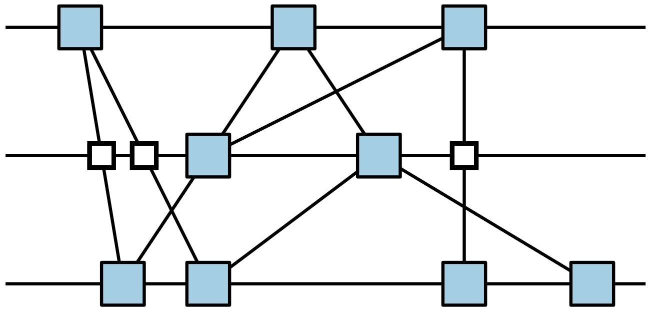
# Iterative Crossing Reduction – Idea



# Iterative Crossing Reduction – Idea

**Observation.**

The number of crossings only depends on permutations of adjacent layers.



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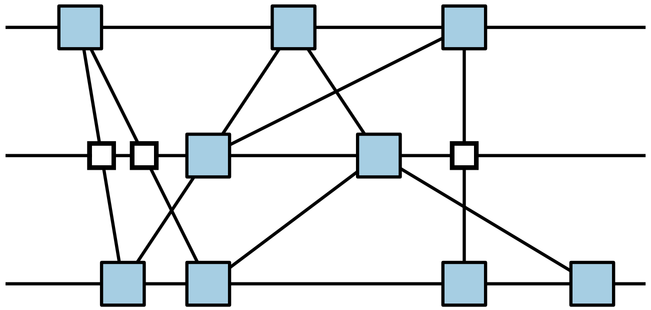

- Add dummy-vertices for edges connecting “far” layers.

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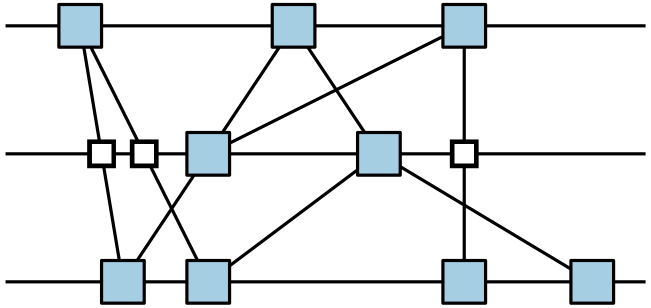

- Add dummy-vertices for edges connecting "far" layers.
- Consider adjacent layers $(L_1, L_2), (L_2, L_3), \ldots$ bottom-to-top.

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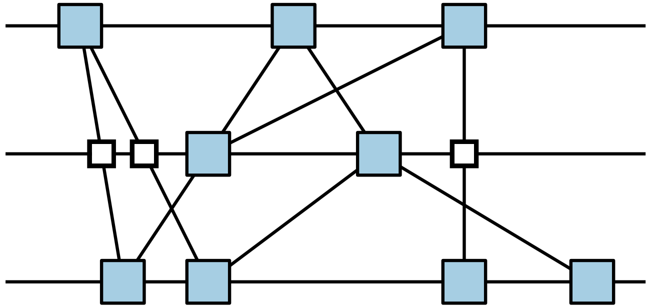

- Add dummy-vertices for edges connecting "far" layers.
- Consider adjacent layers $(L_1, L_2), (L_2, L_3), \ldots$ bottom-to-top.
- Minimize crossings by permuting $L_{i+1}$ while keeping $L_i$ fixed.

# Iterative Crossing Reduction – Algorithm

(1) choose a random permutation of $L_1$

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(2) iteratively consider adjacent layers $L_i$ and $L_{i+1}$

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*one-sided crossing minimization*

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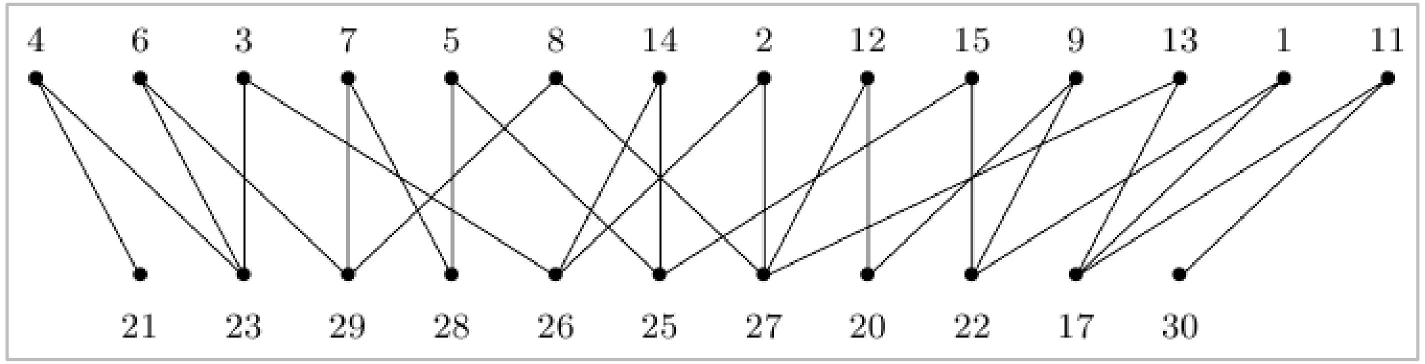


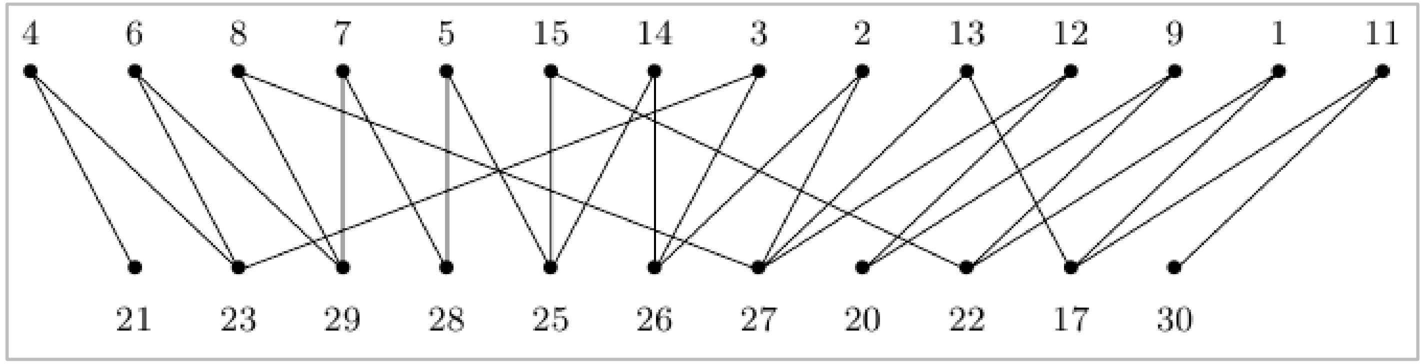


Abb. aus [Kaufmann und Wagner: Drawing Graphs]
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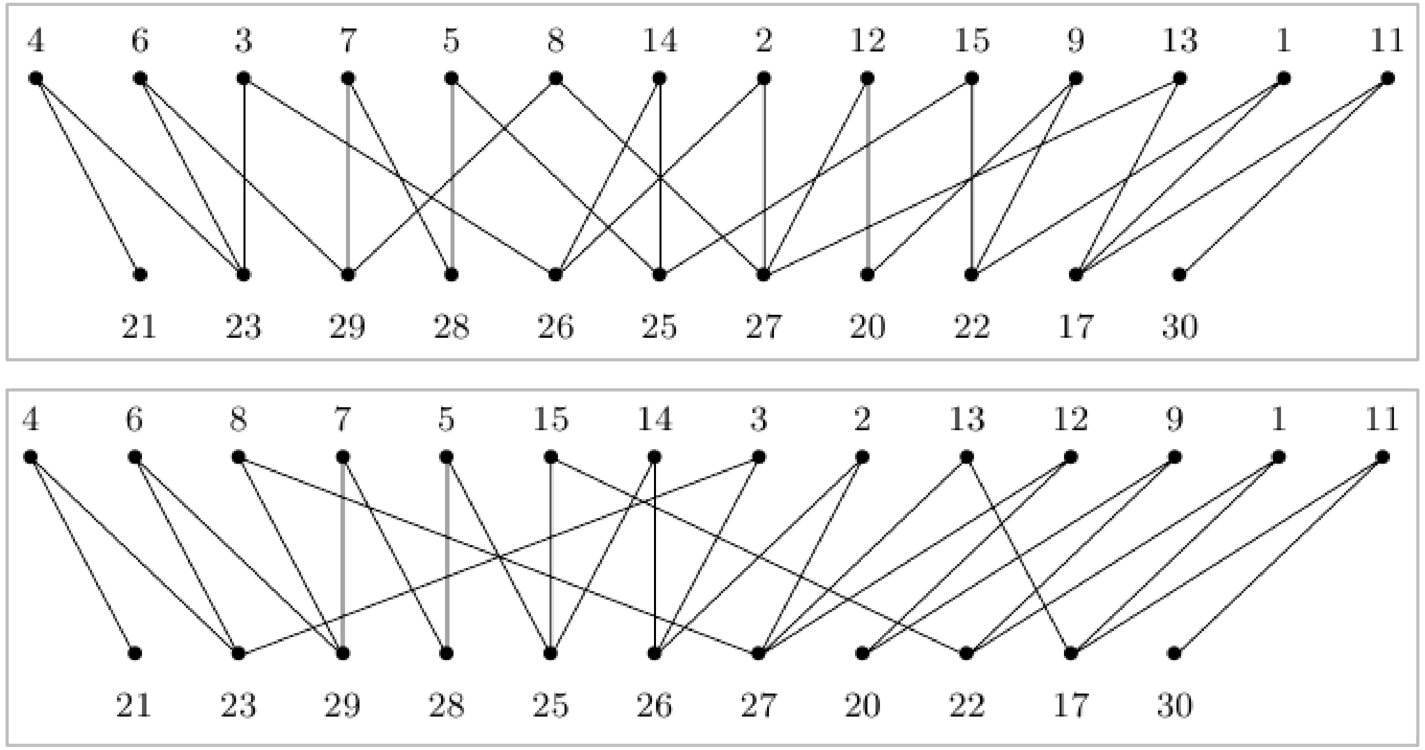


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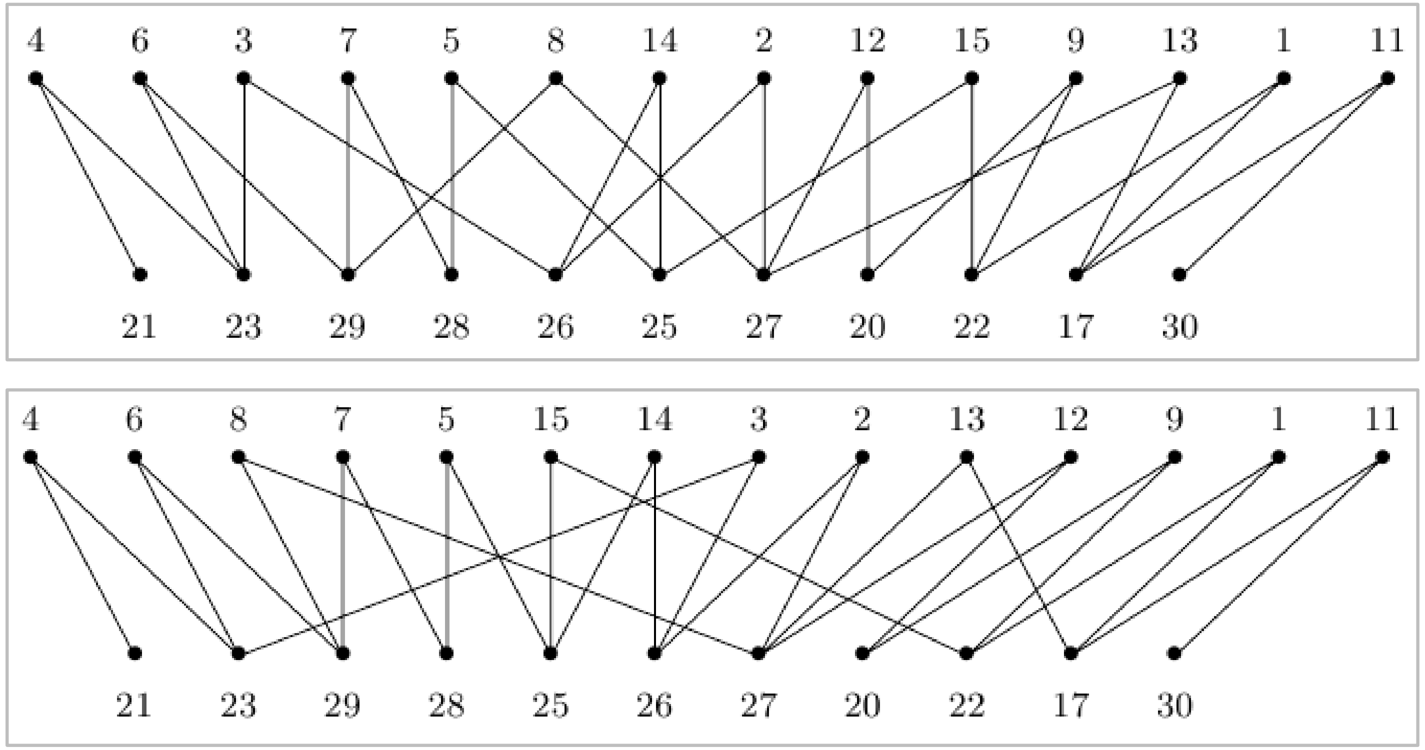


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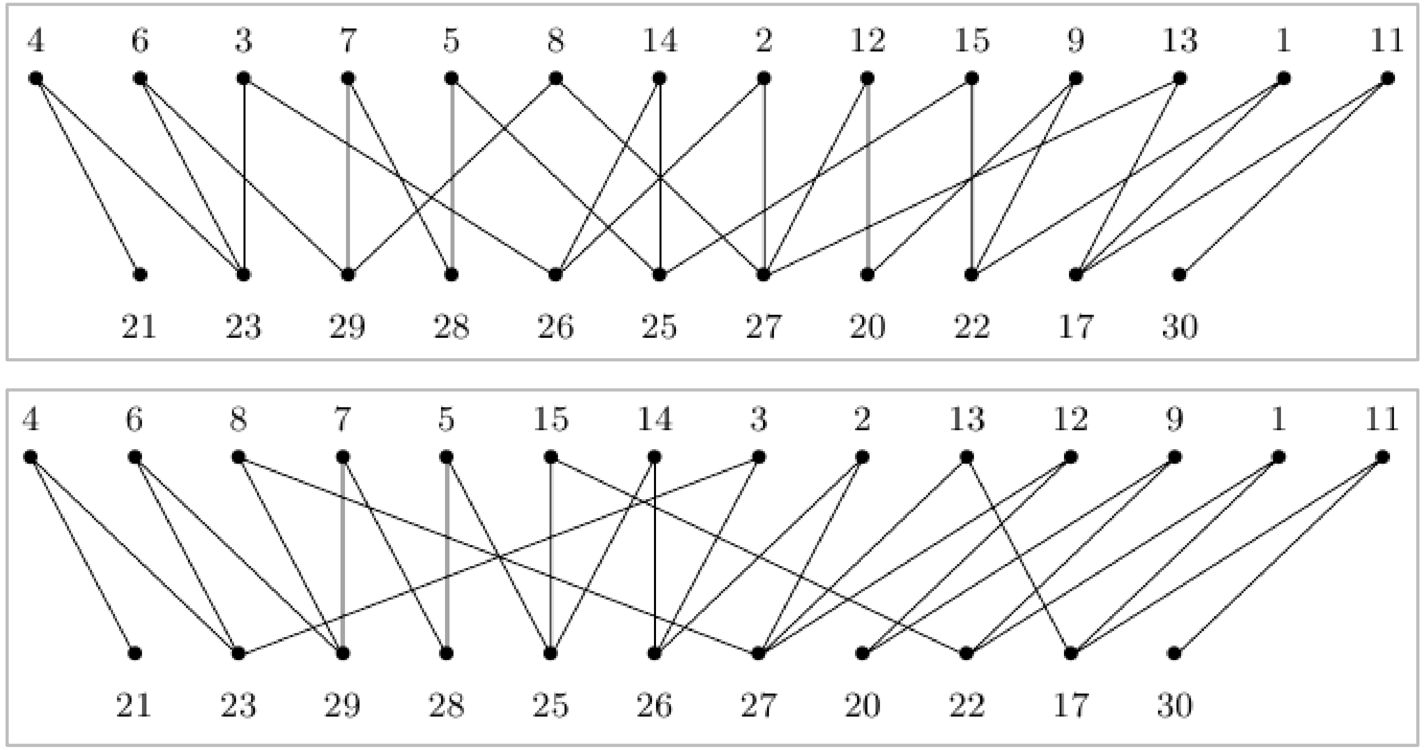


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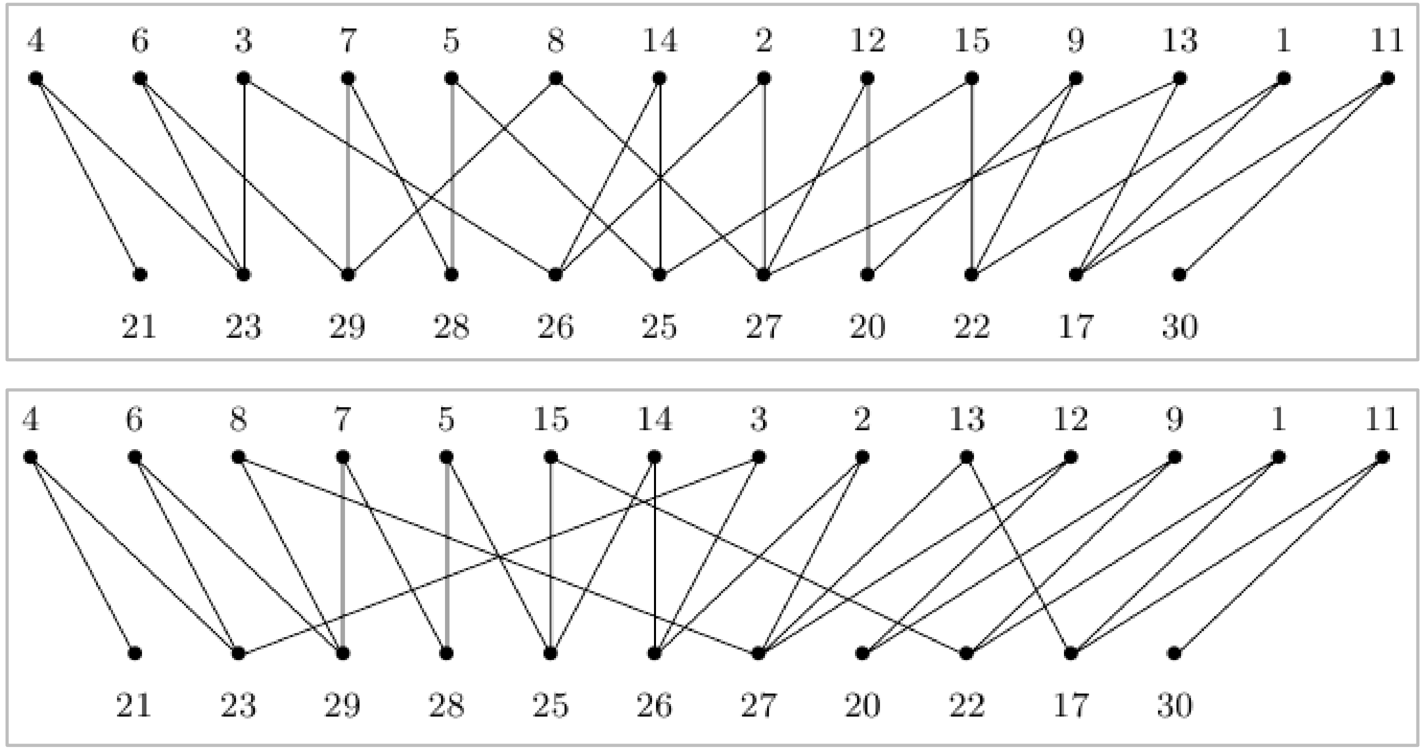


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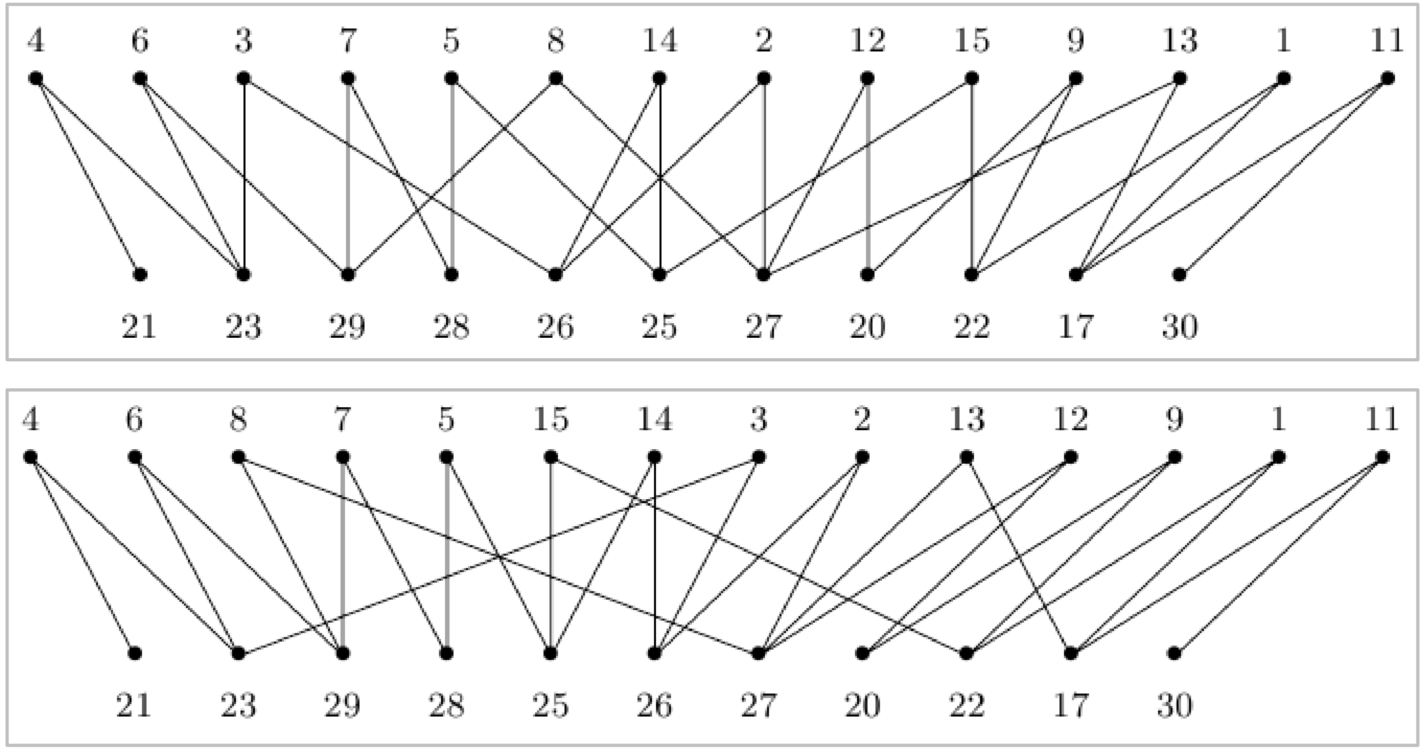


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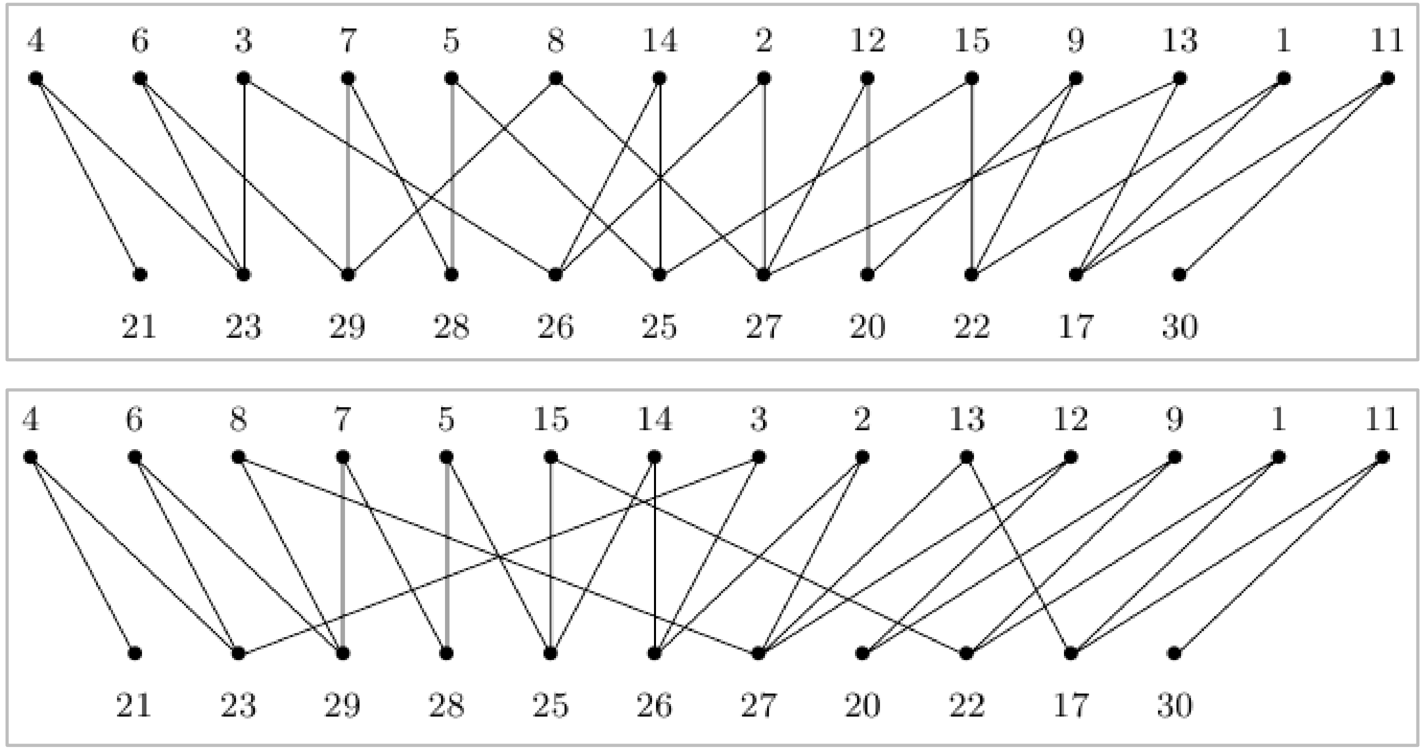




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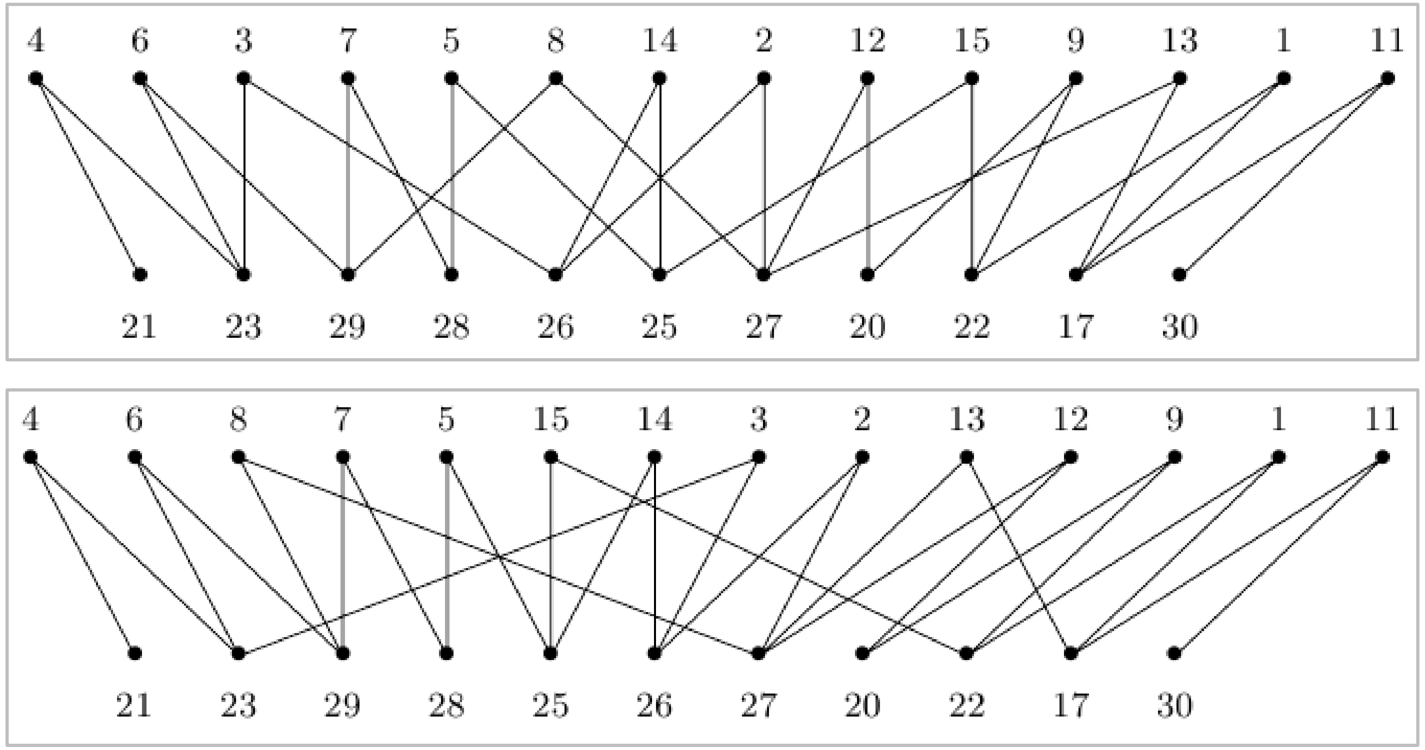


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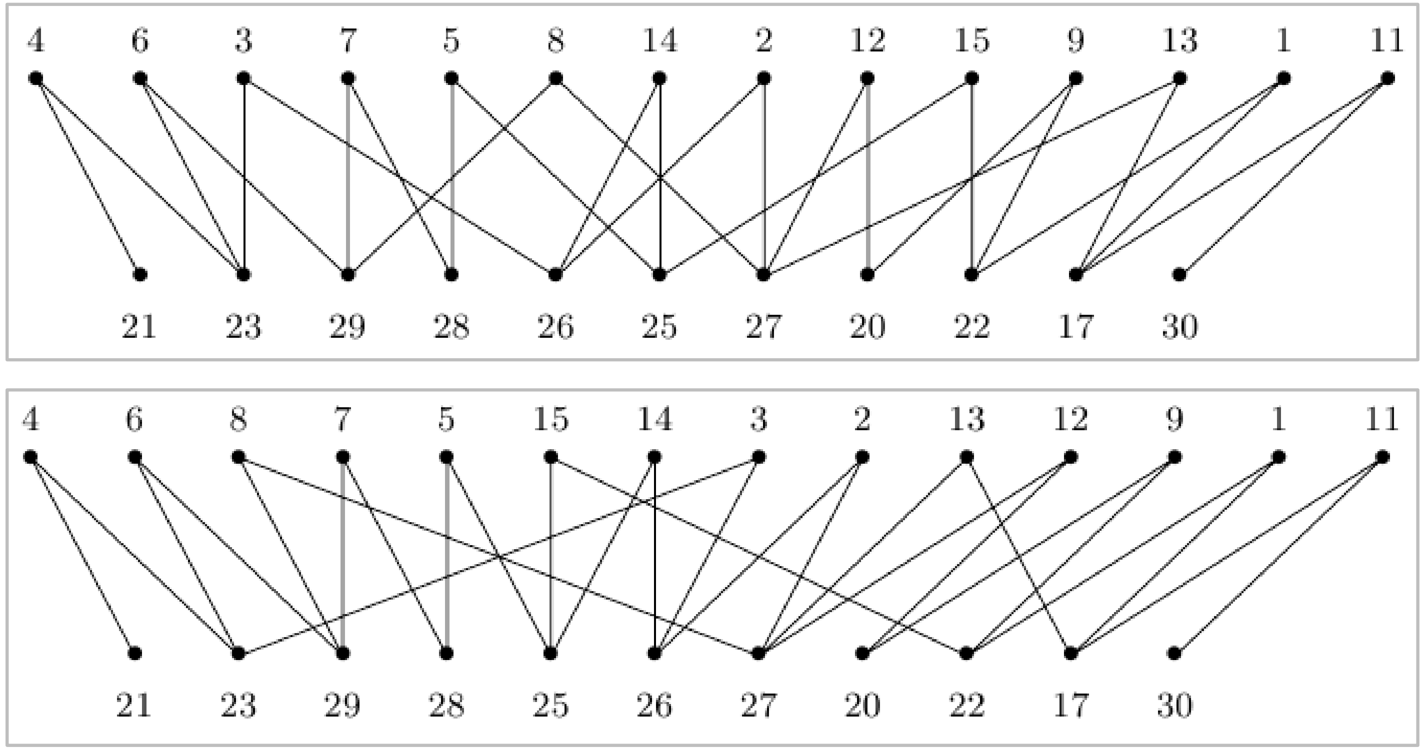


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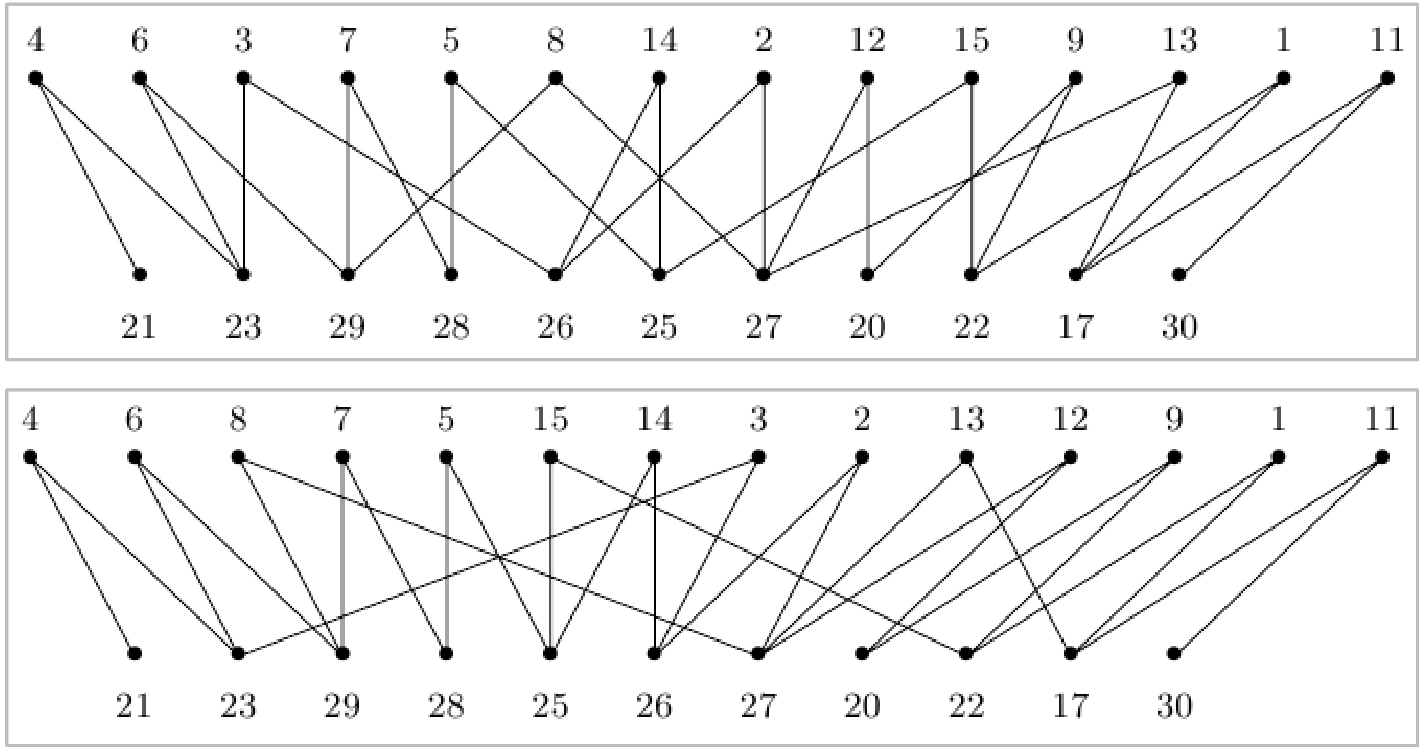


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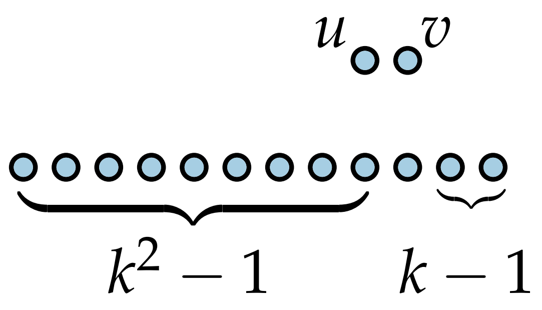

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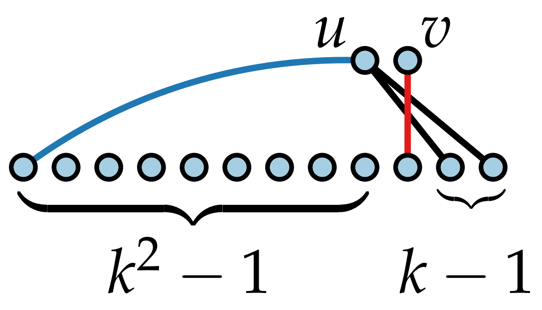

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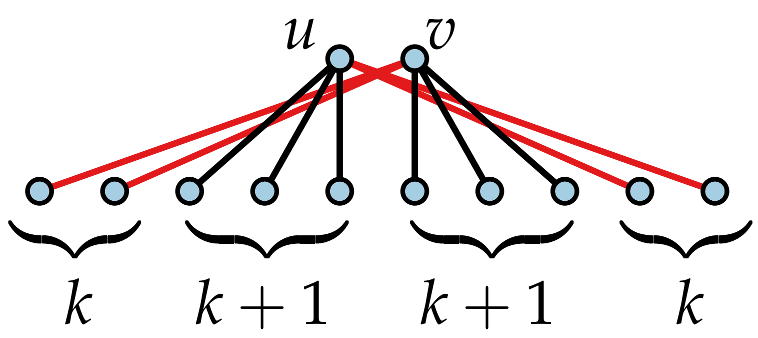


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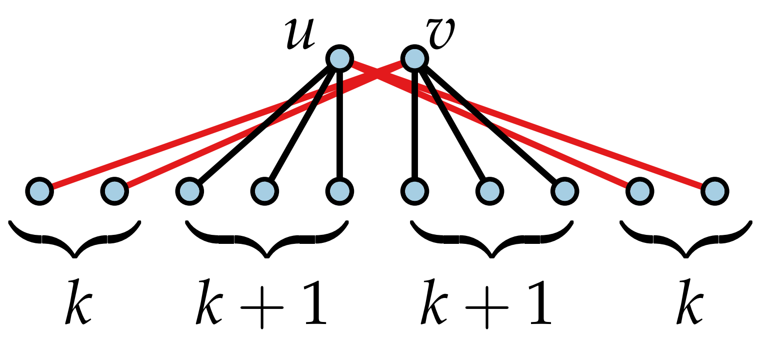


$2k(k+1) + k^2$ vs. $(k+1)^2$

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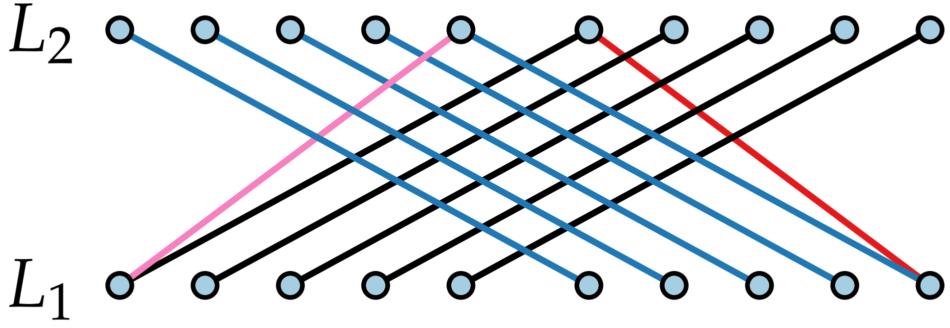

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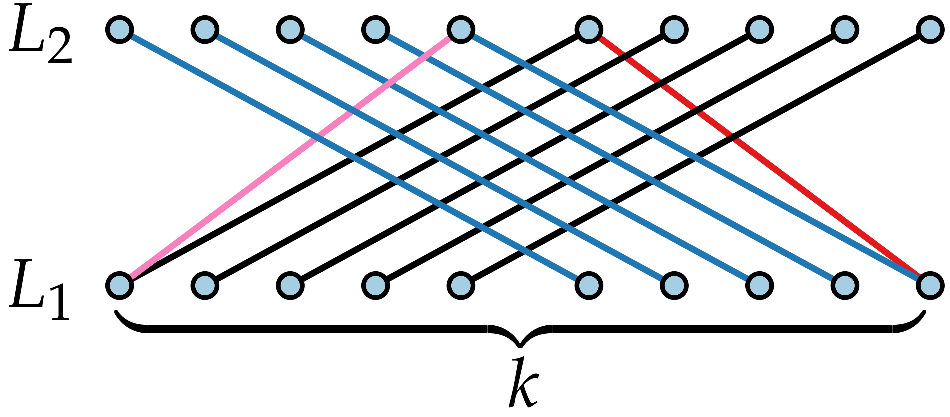

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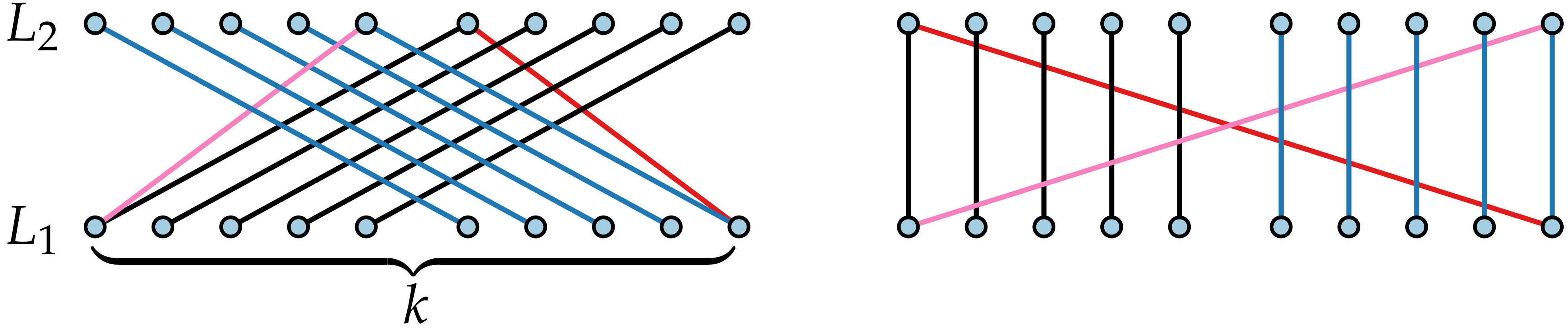

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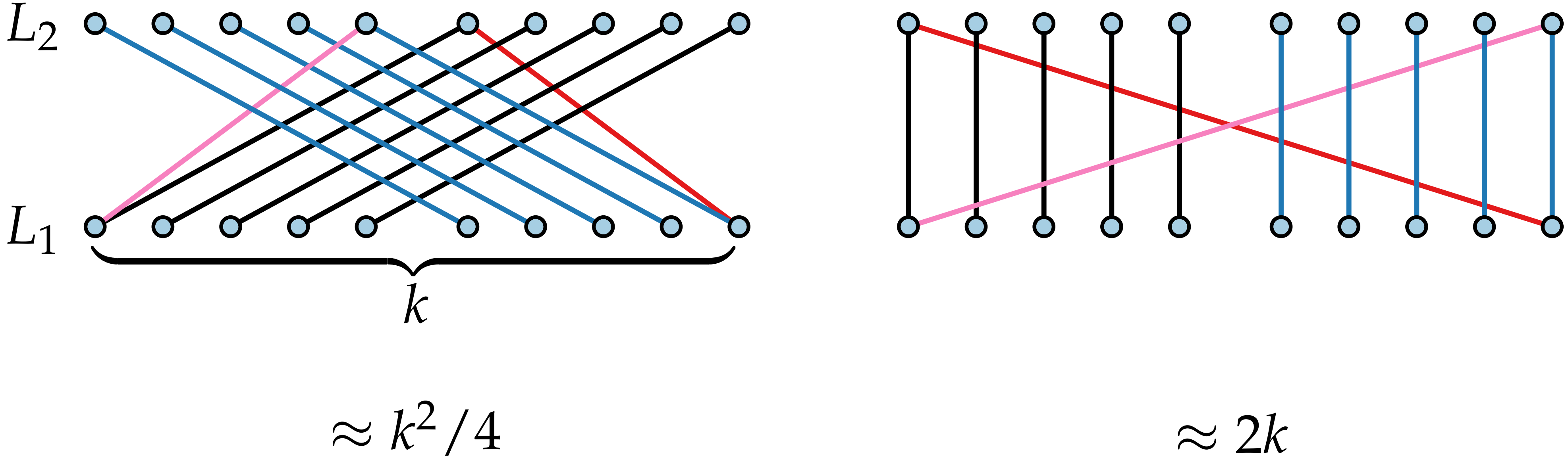
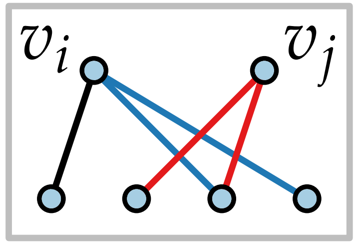


# Integer Linear Program

[Jünger & Mutzel, '97]

- Constant $c_{ij}$ := # crossings between edges incident to $v_i$ or $v_j$ when $\pi_2(v_i) < \pi_2(v_j)$



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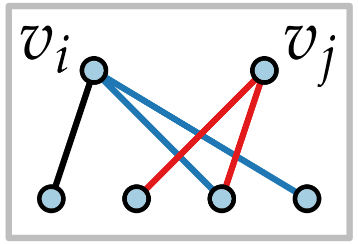

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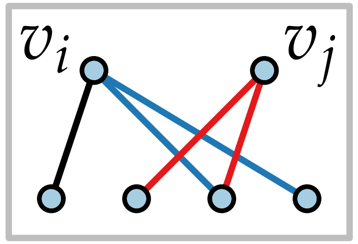


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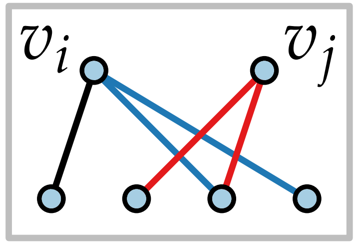


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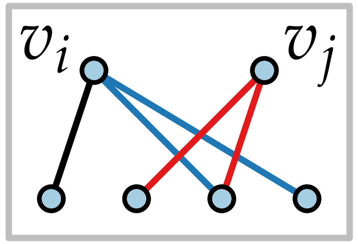


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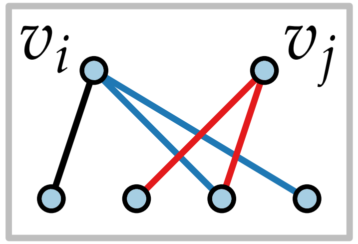


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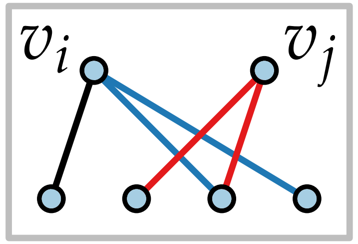


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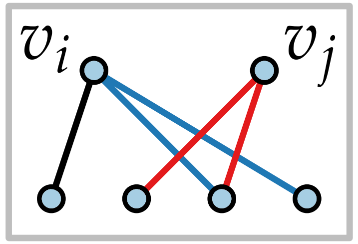


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- Minimize the number of crossings:

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**Properties.**

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**Properties.**

- Branch-and-cut technique for DAGs of limited size
- Useful for graphs of small to medium size
- Finds optimal solution

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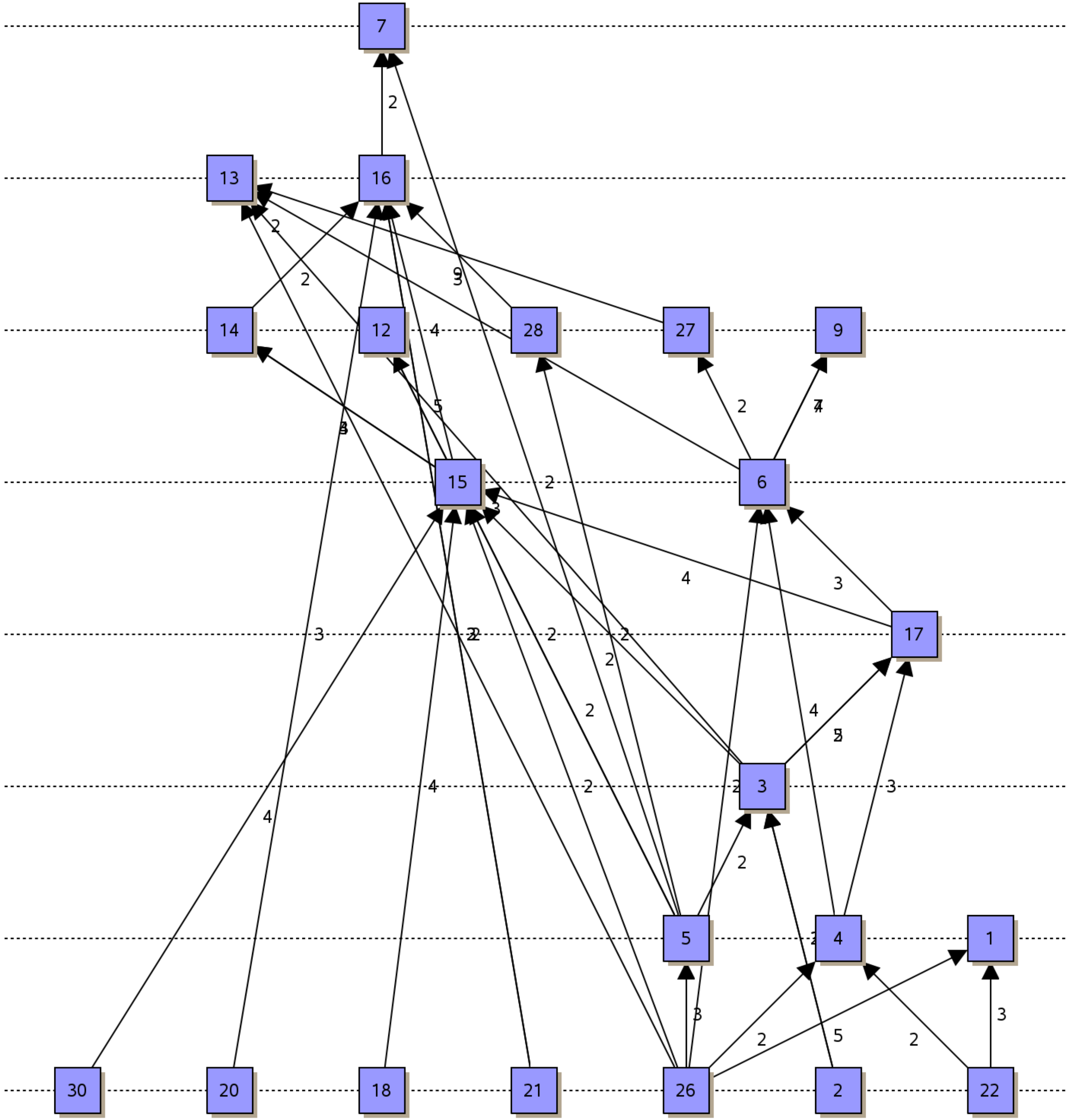
**Properties.**

- Branch-and-cut technique for DAGs of limited size
- Useful for graphs of small to medium size
- Finds optimal solution
- Solution in polynomial time is not guaranteed

# Iterations on Example



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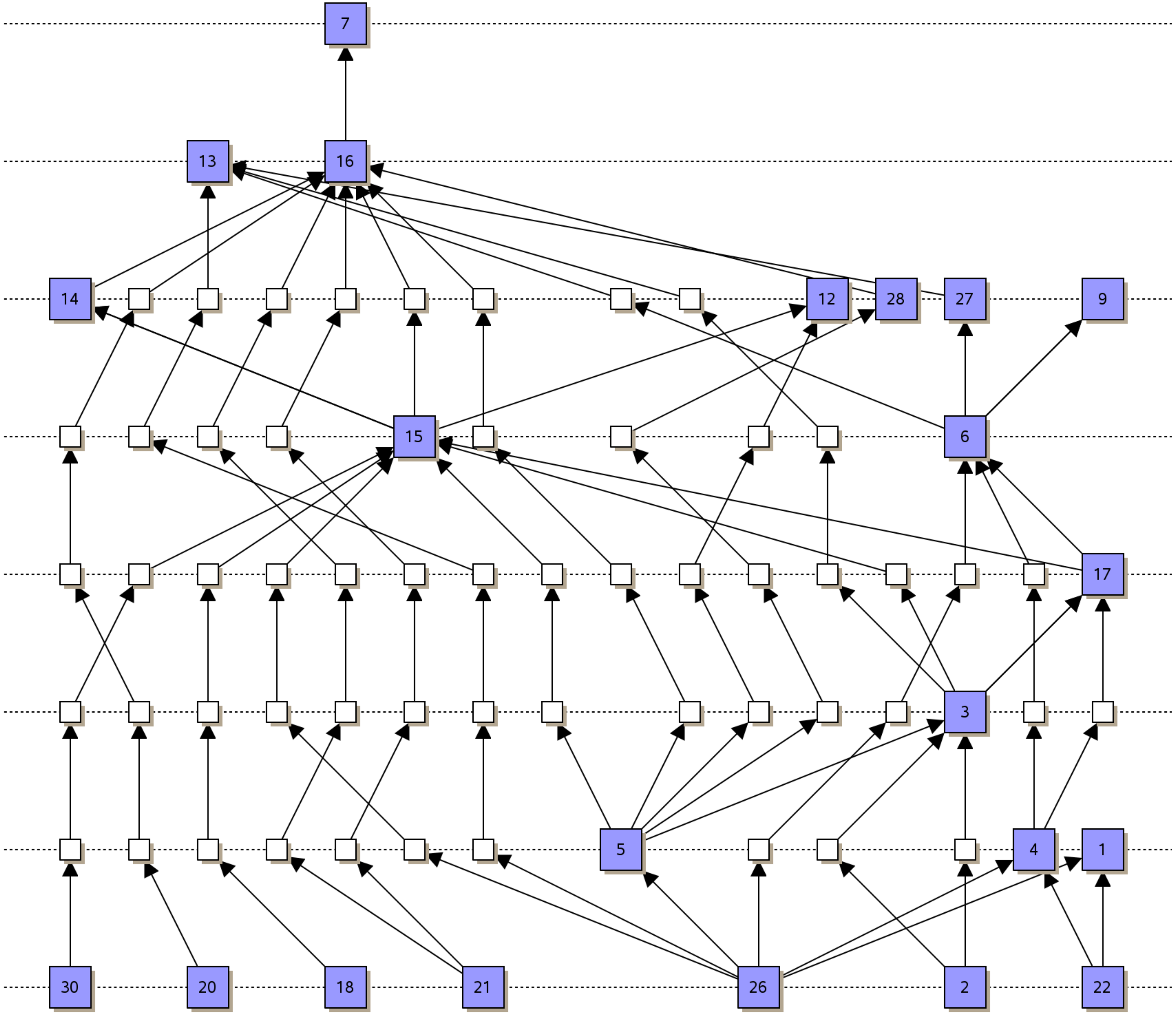

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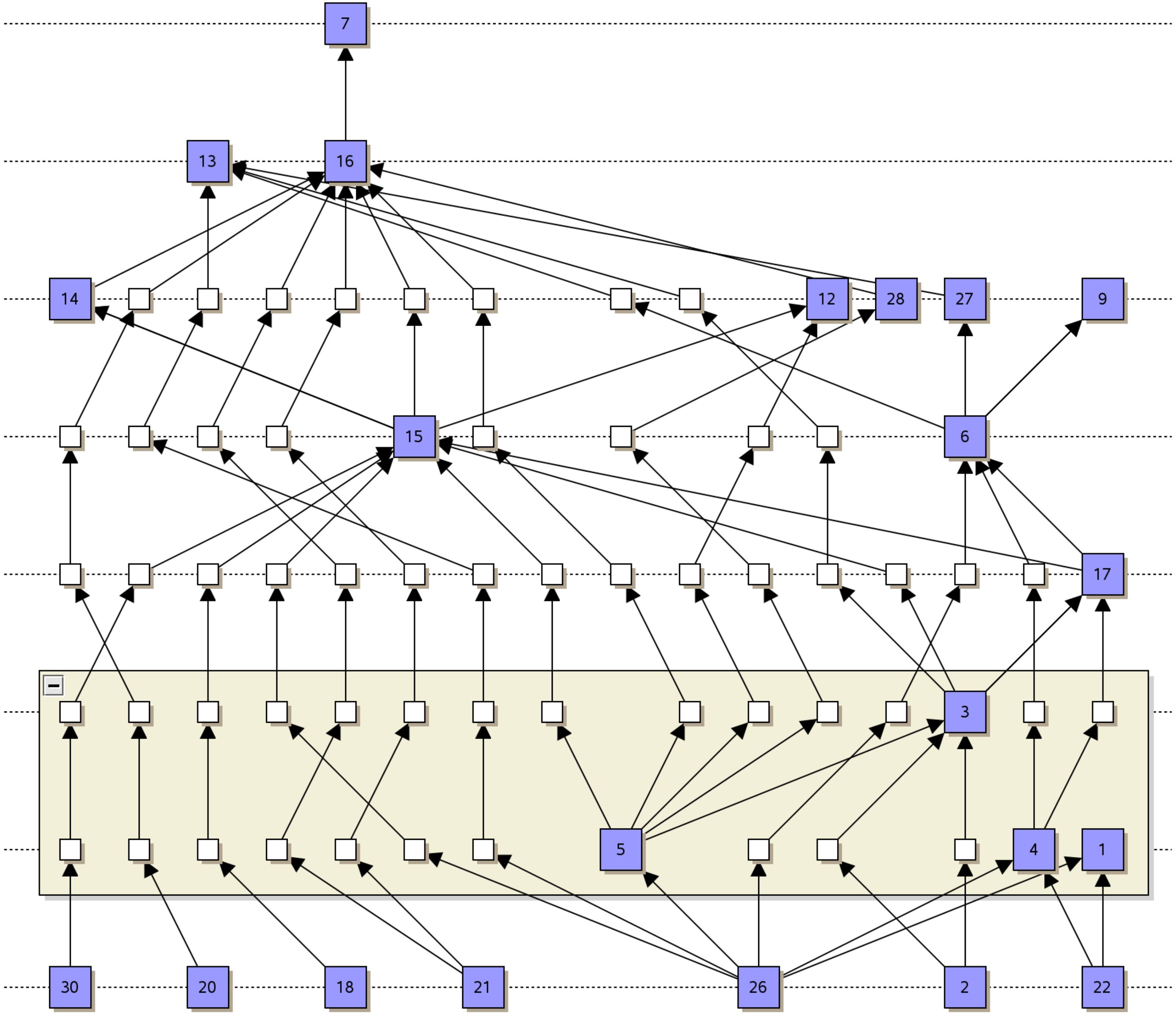

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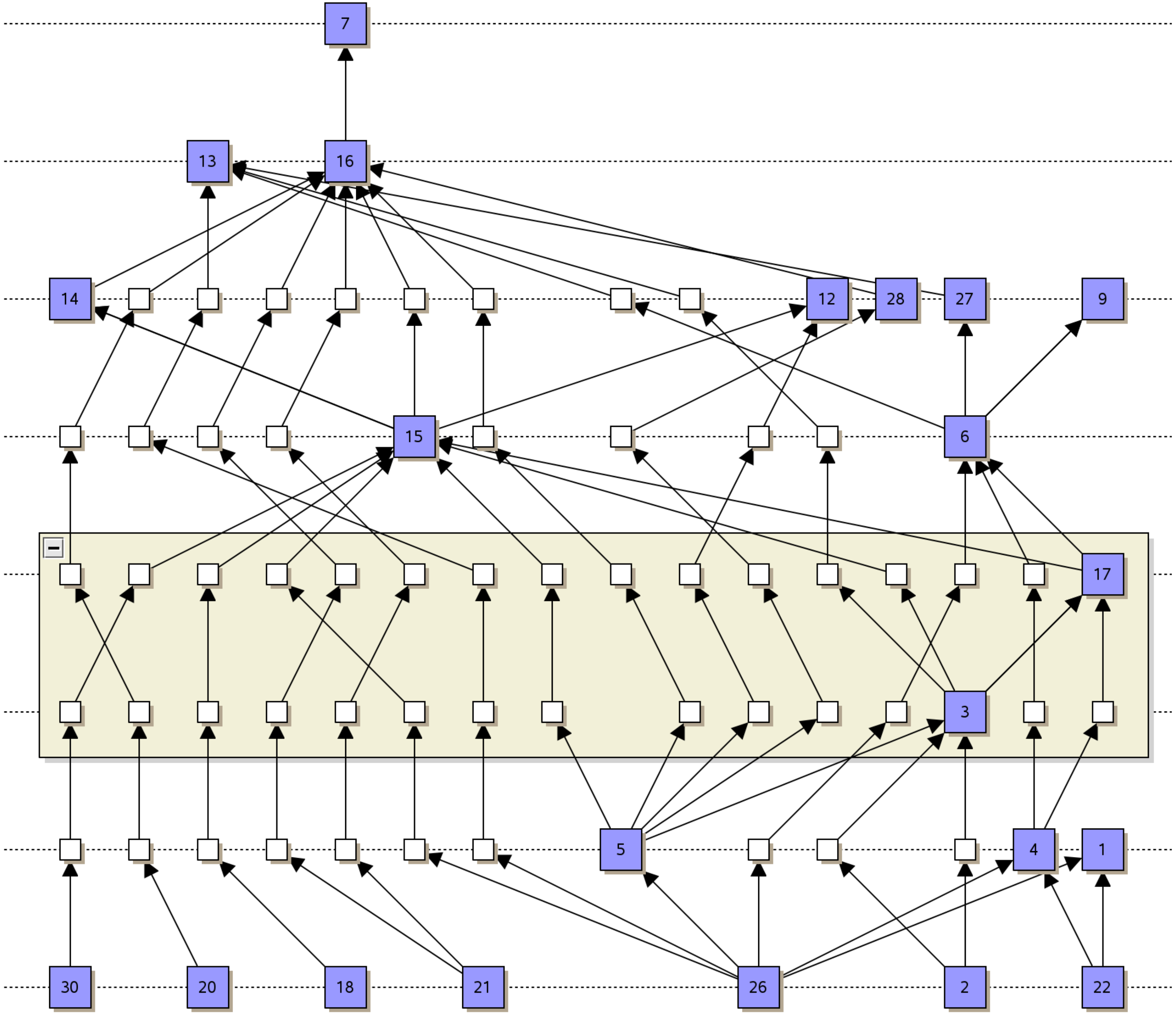

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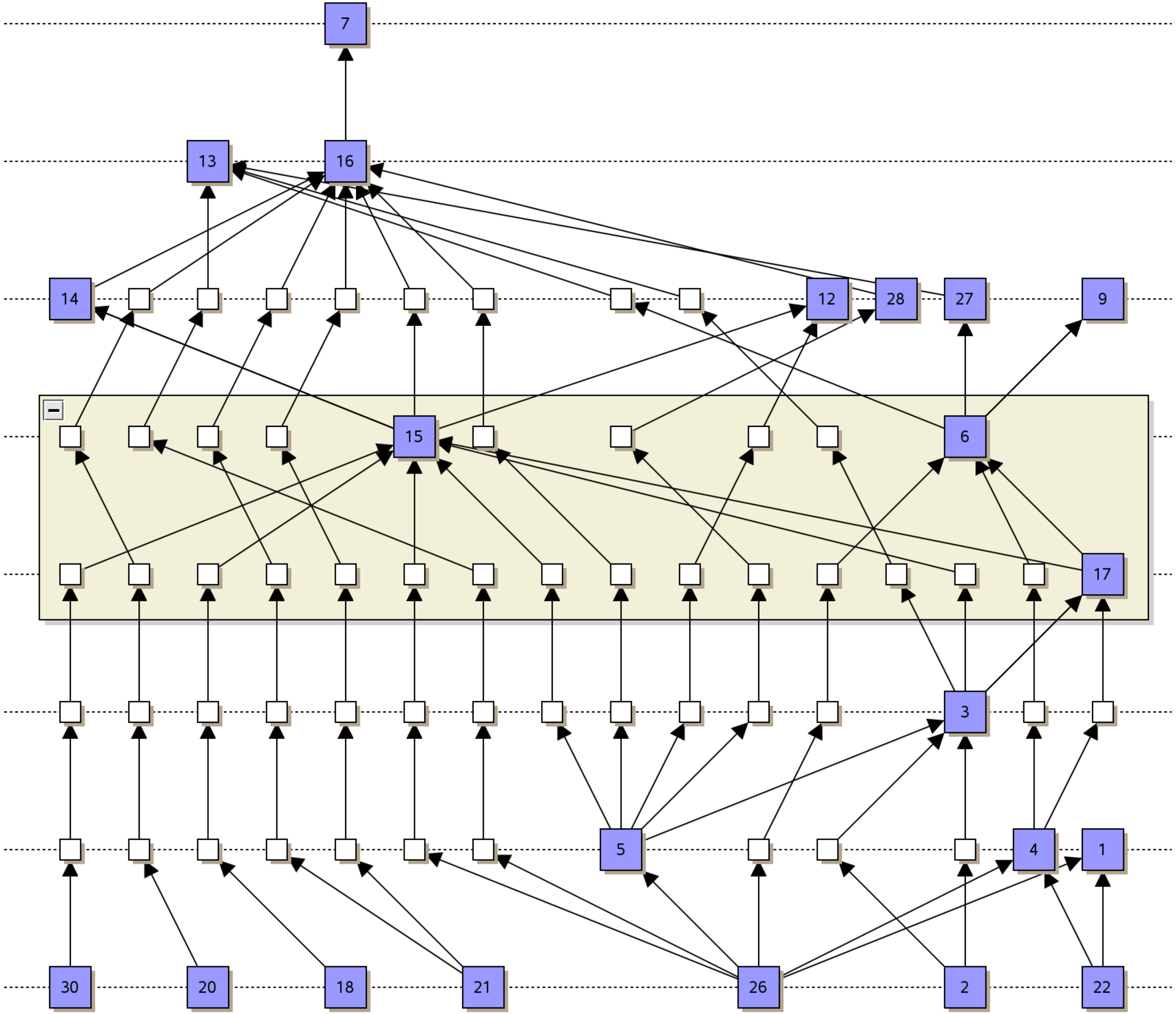

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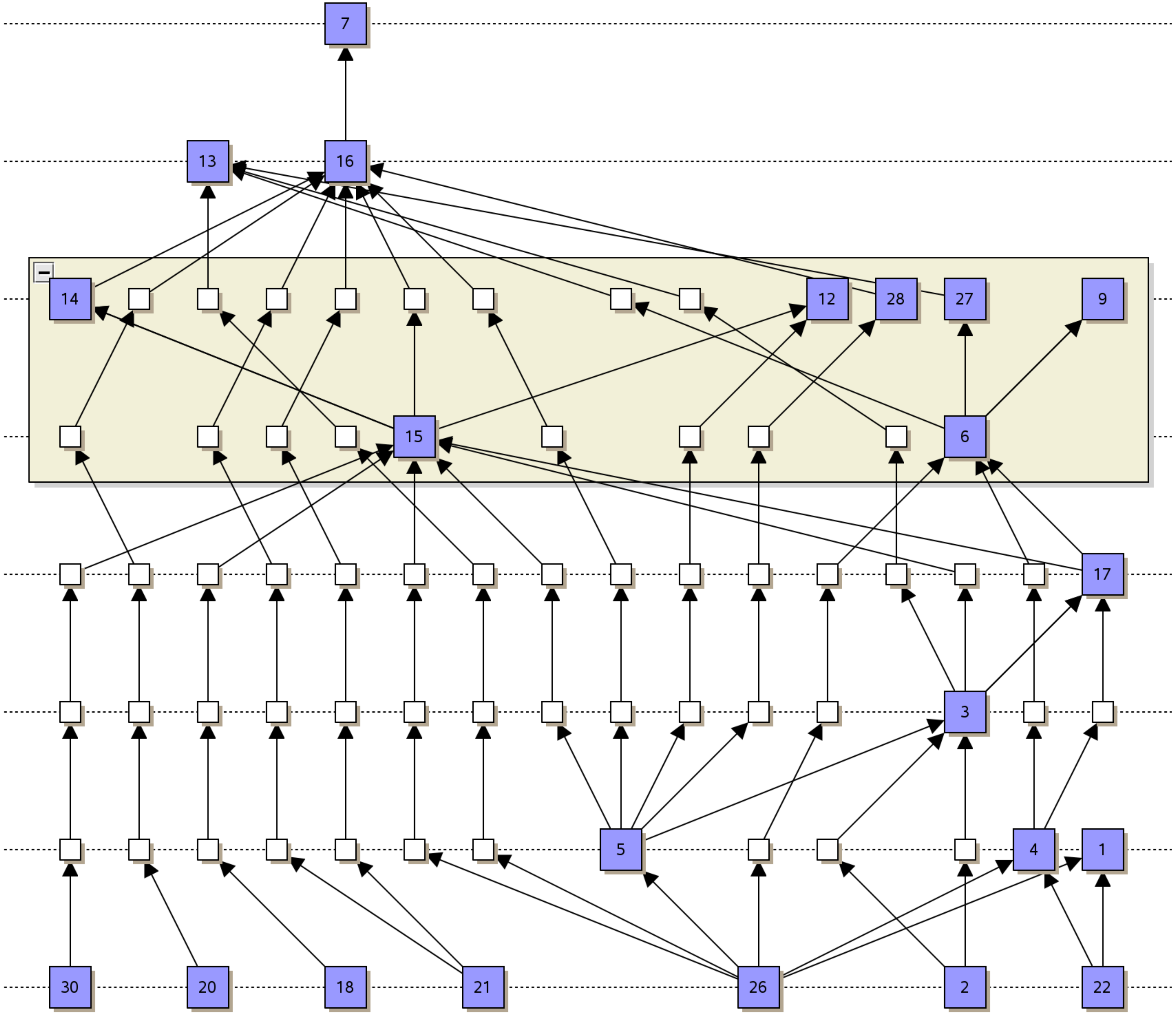

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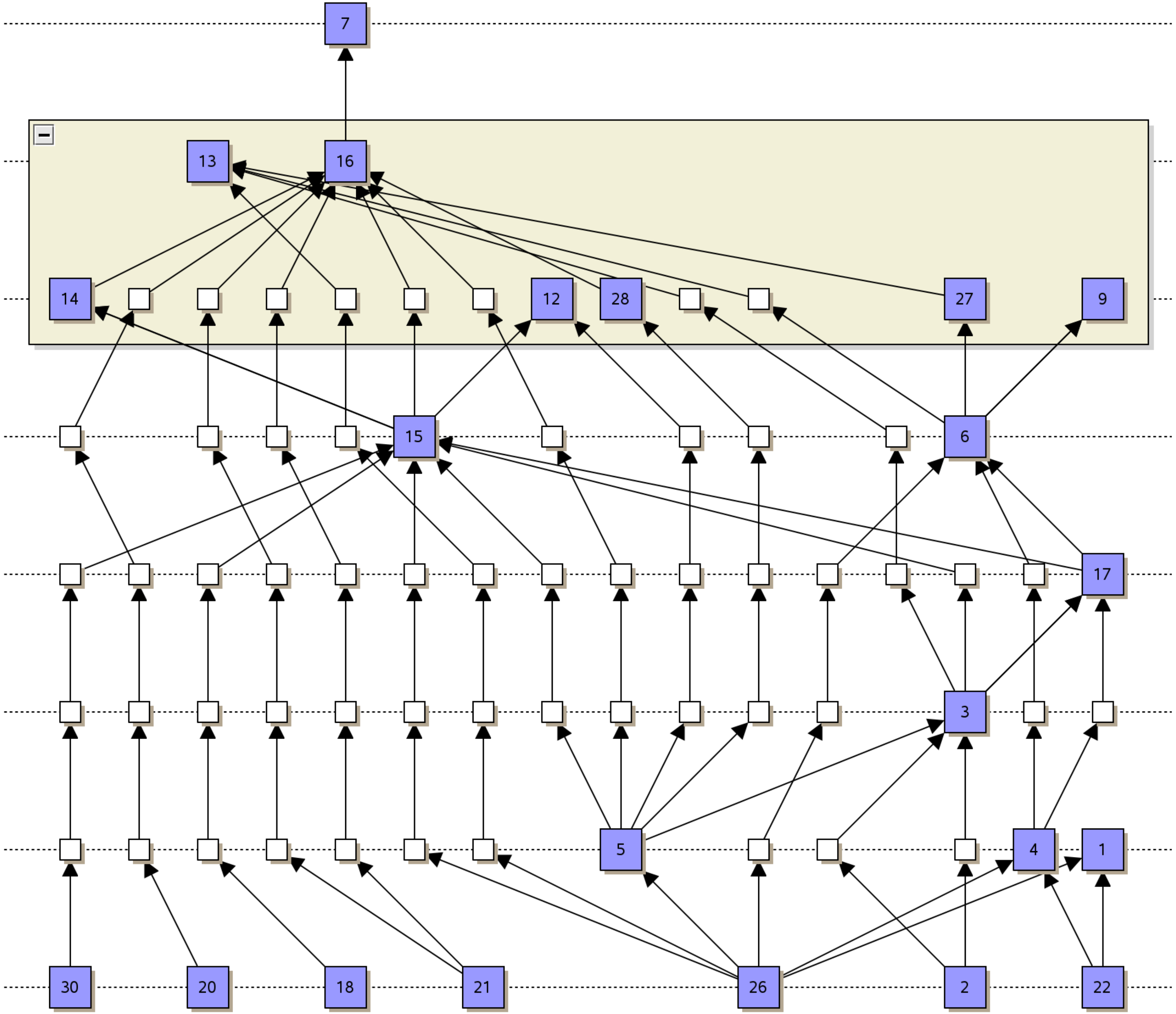

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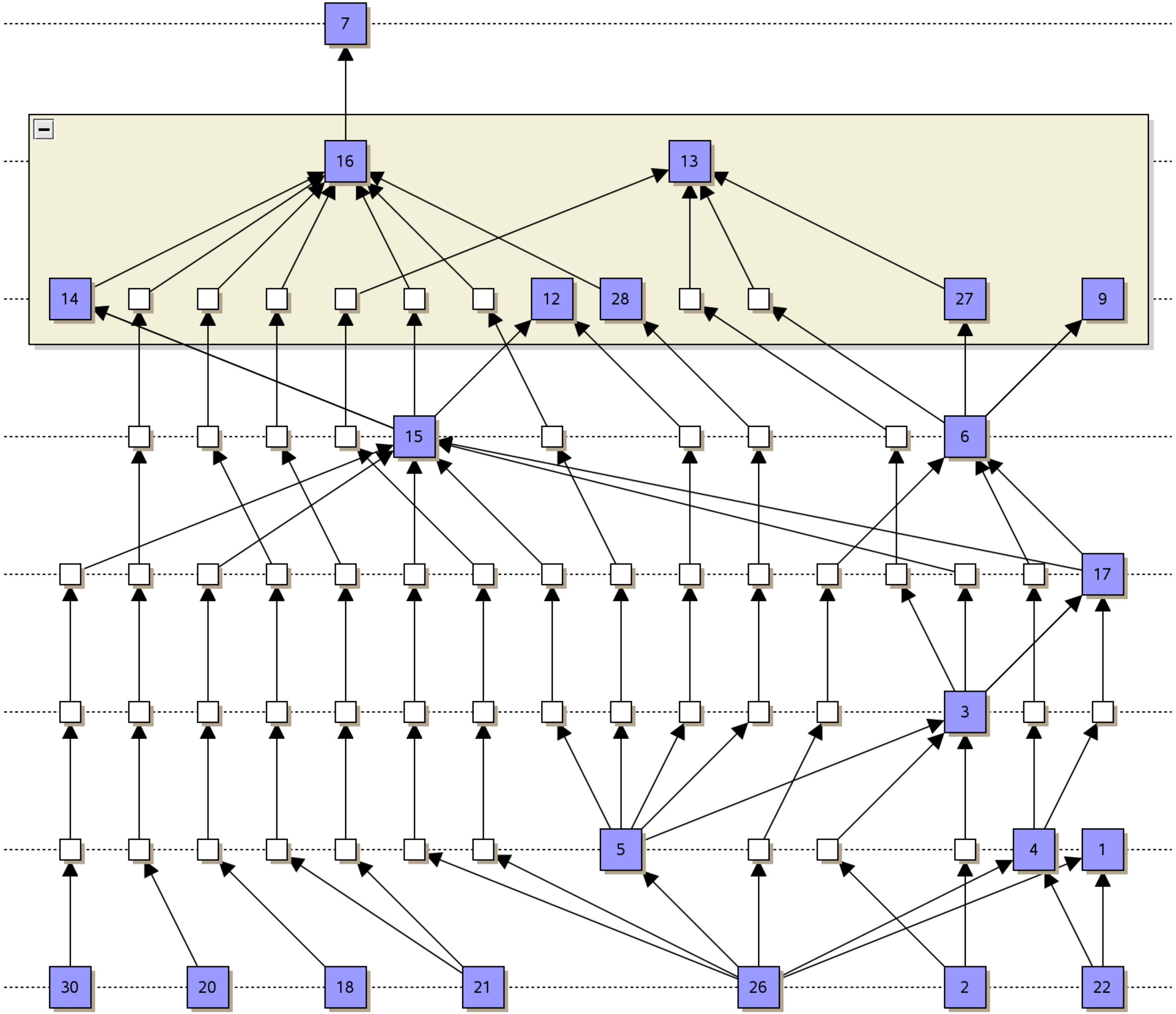

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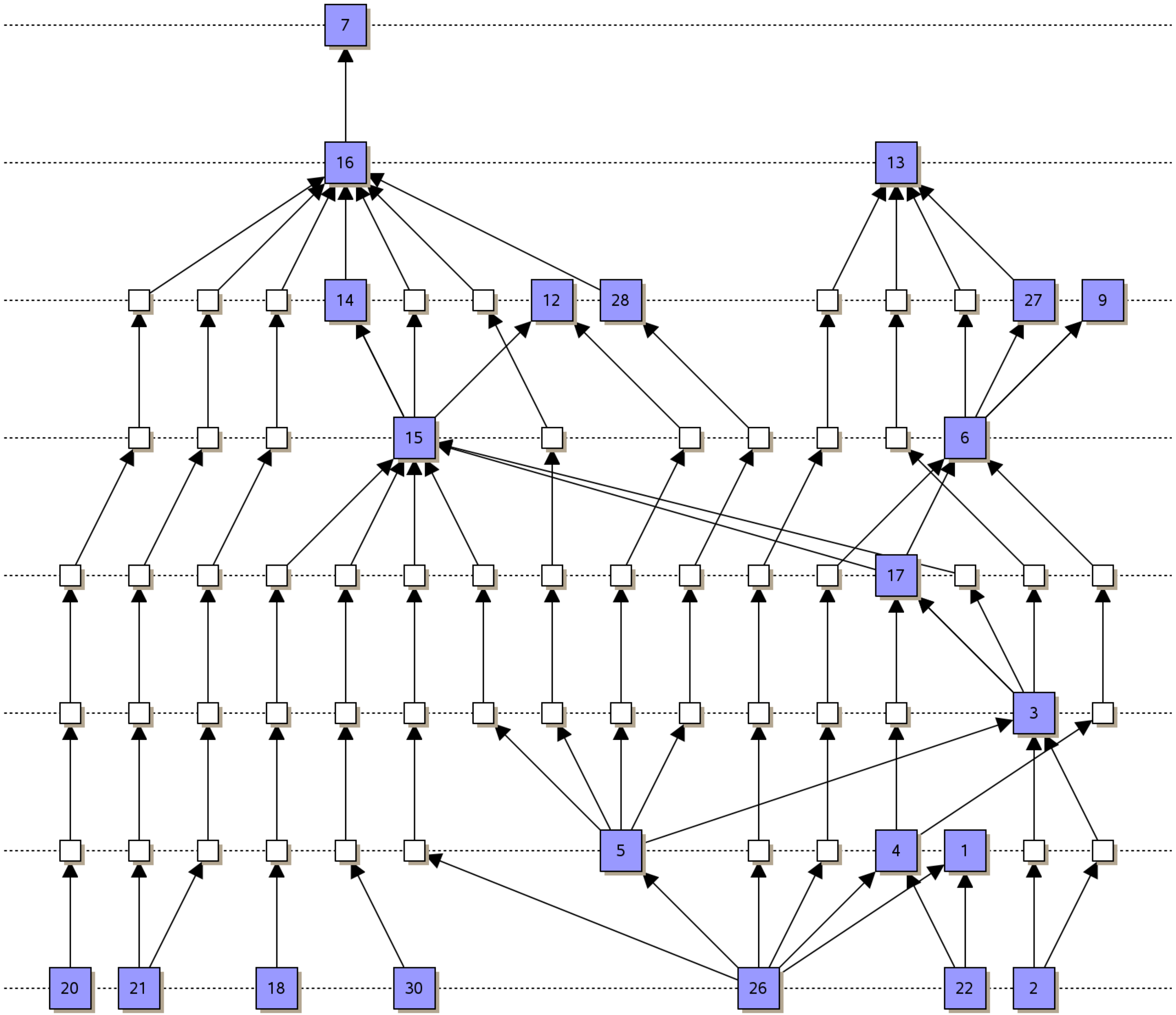
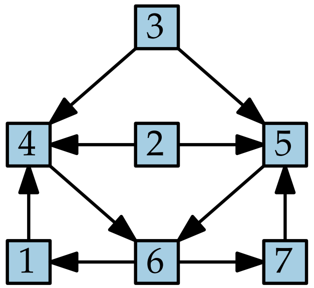
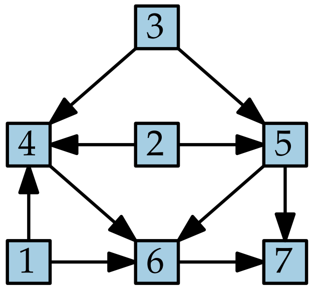


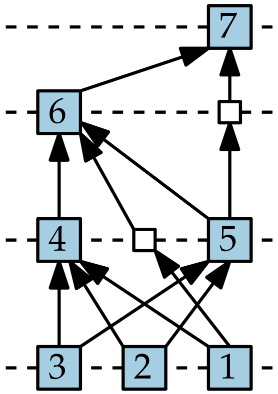
Universität Trier

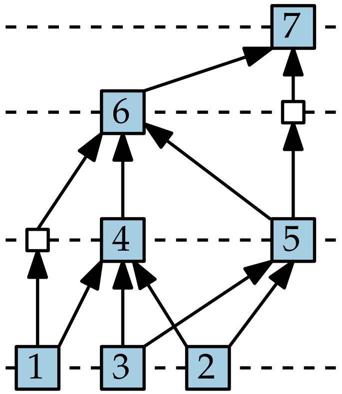
# Visualization of Graphs

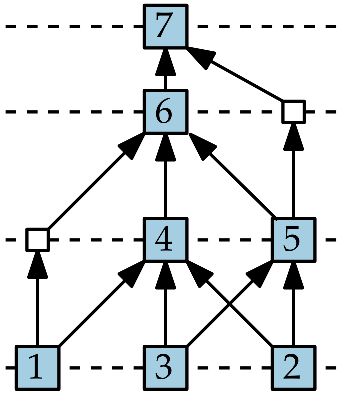


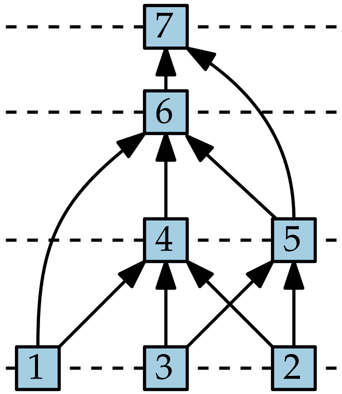




## Lecture 8:
## Hierarchical Layouts:
## Sugiyama Framework



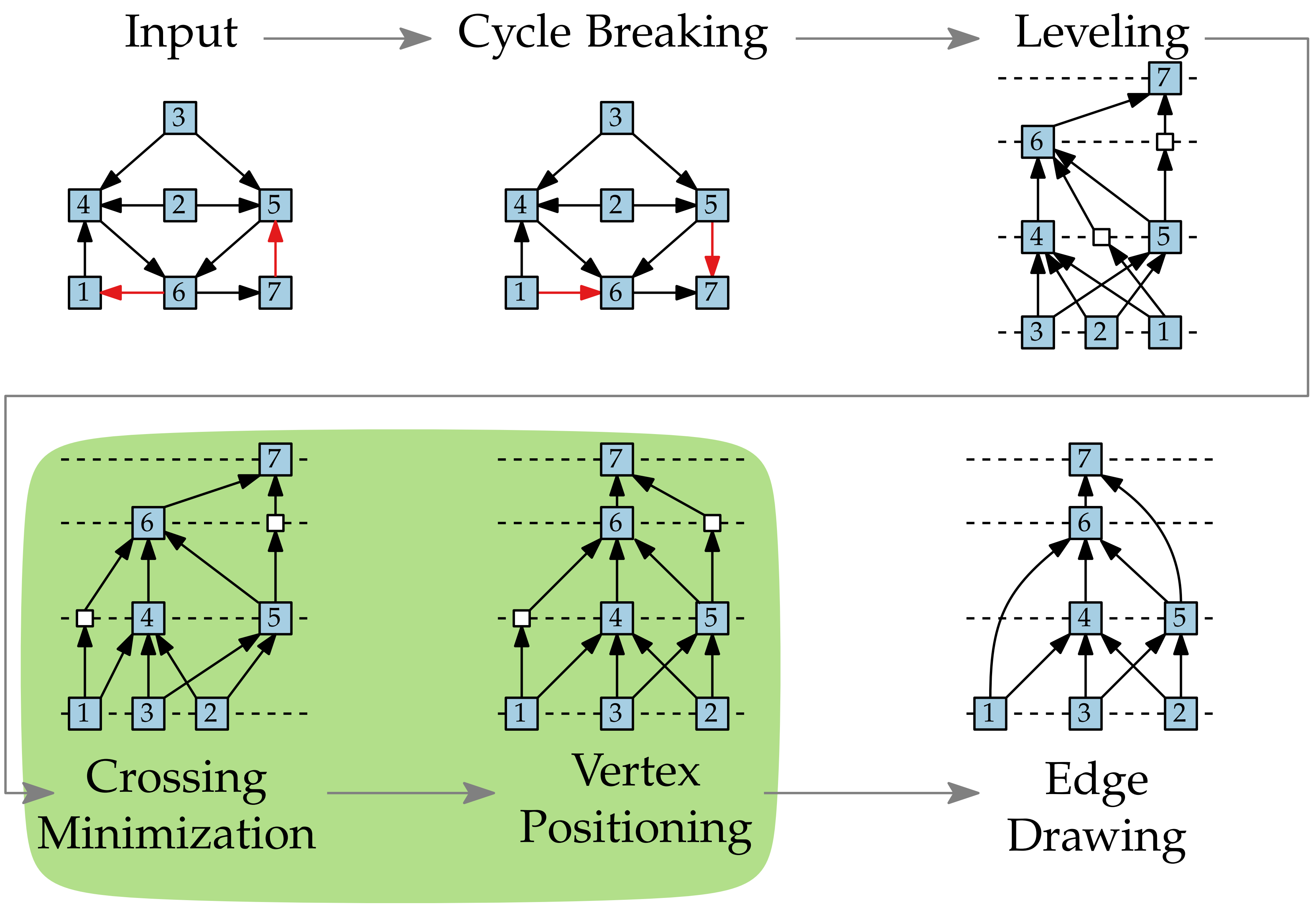




### Part V:
### Vertex Positioning & Drawing Edges

Philipp Kindermann

# Step 4: Vertex Positioning



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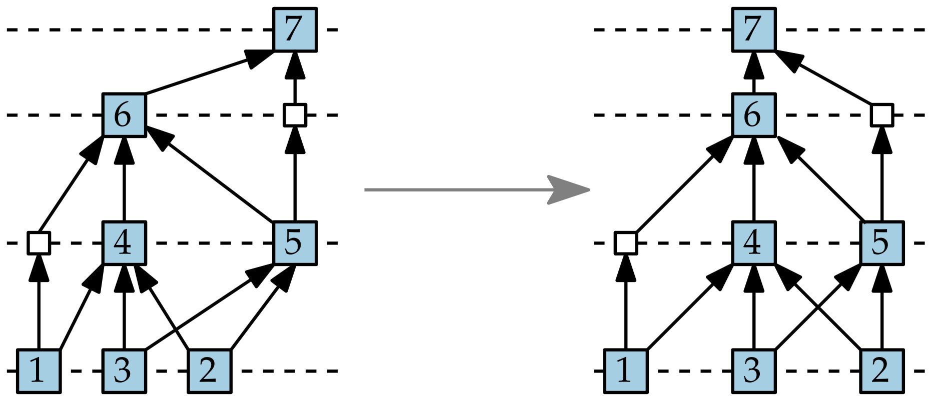

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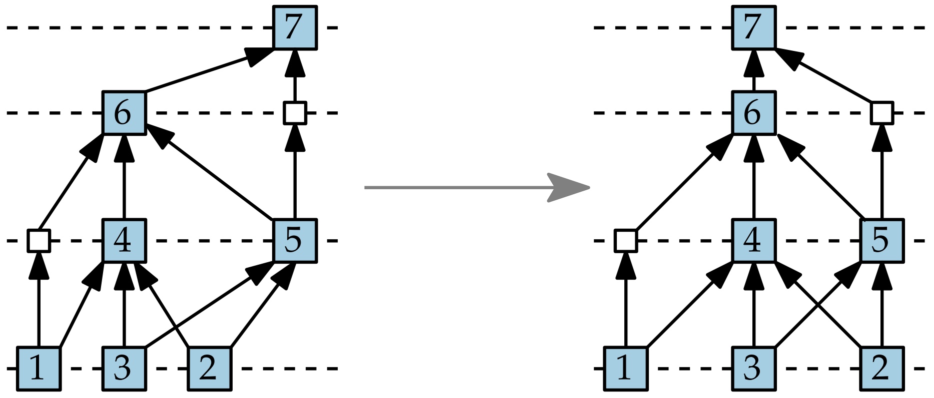


**Goal.**

Paths should be close to straight, vertices evenly spaced

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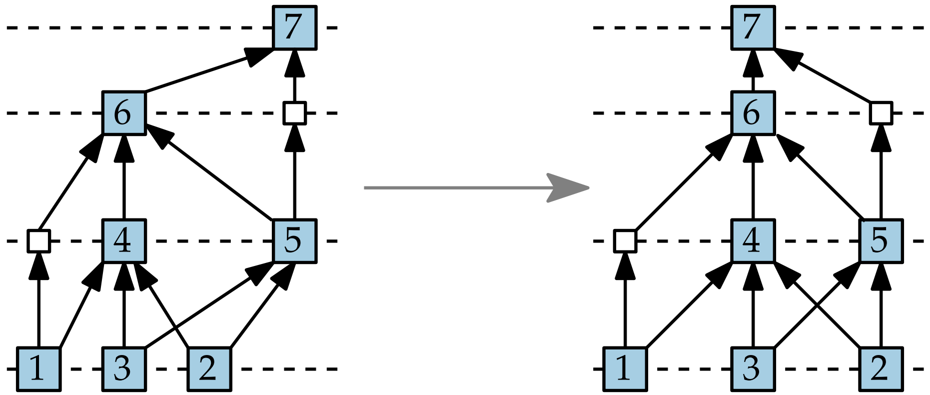


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Paths should be close to straight, vertices evenly spaced

- **Exact:** Quadratic Program (QP)

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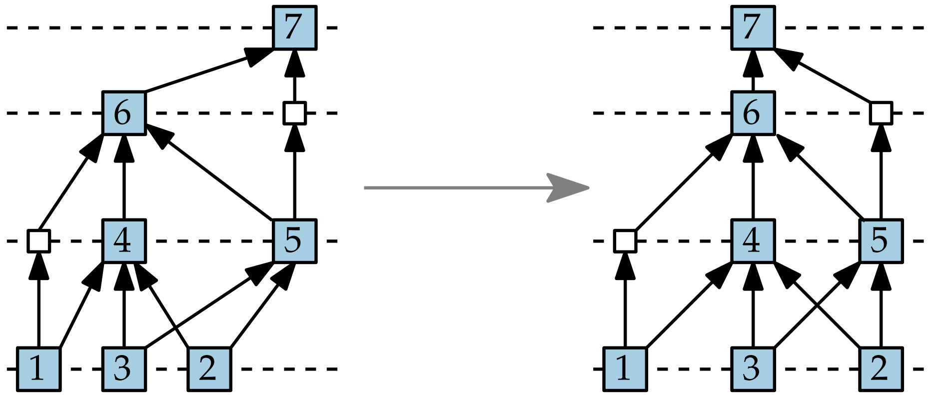


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- **Exact:** Quadratic Program (QP)
- **Heuristic:** Iterative approach

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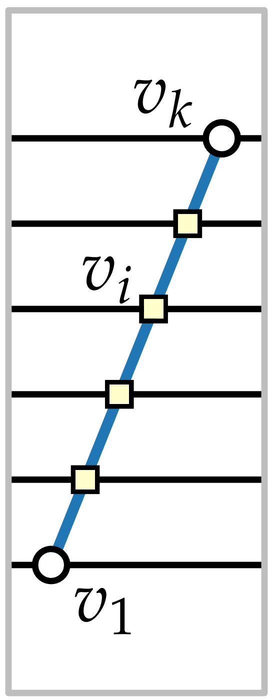

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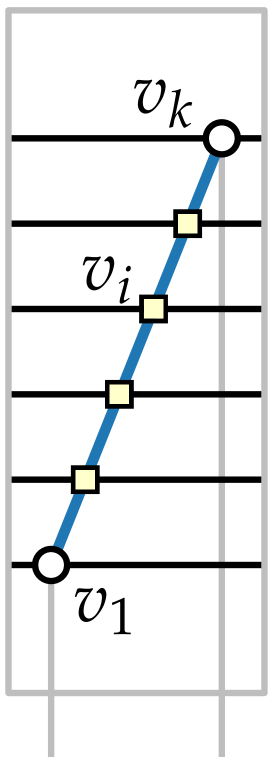

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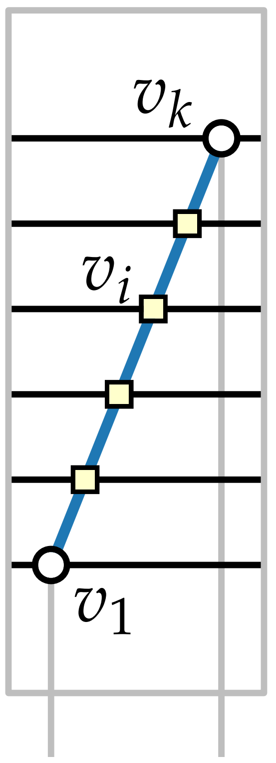
$$\overline{x(v_i)} =$$



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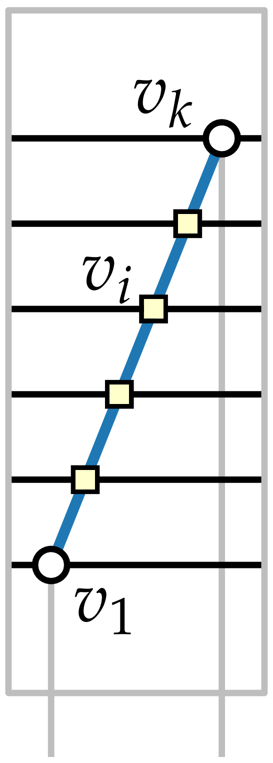

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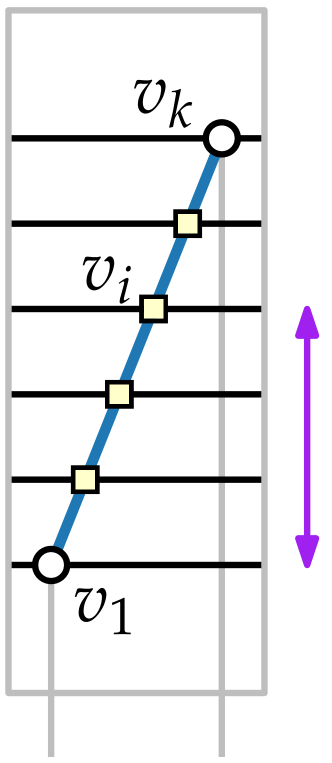

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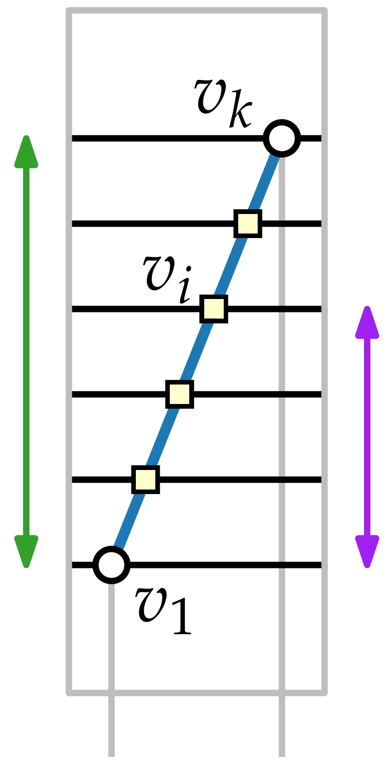

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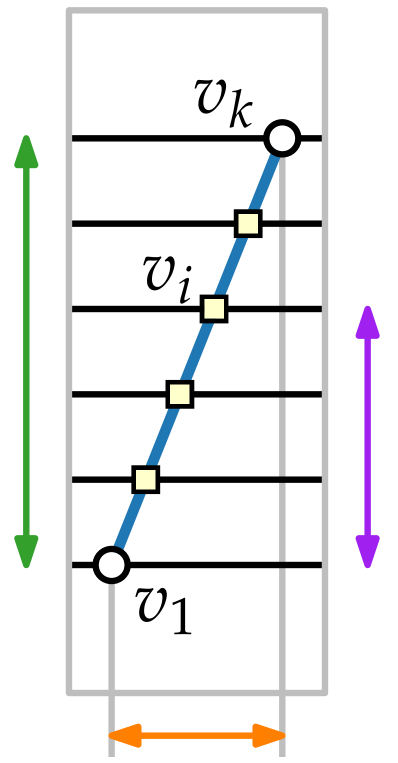

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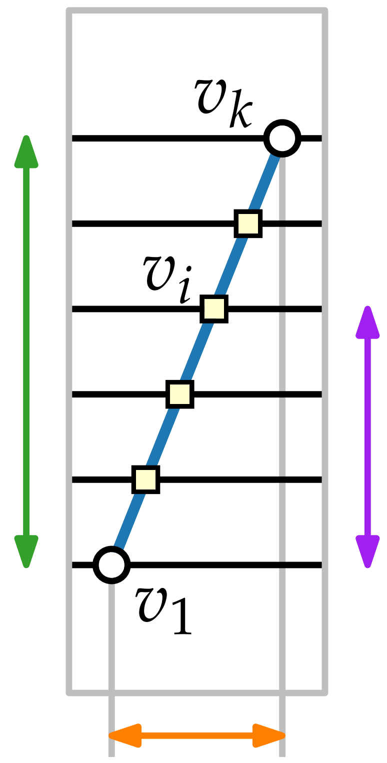

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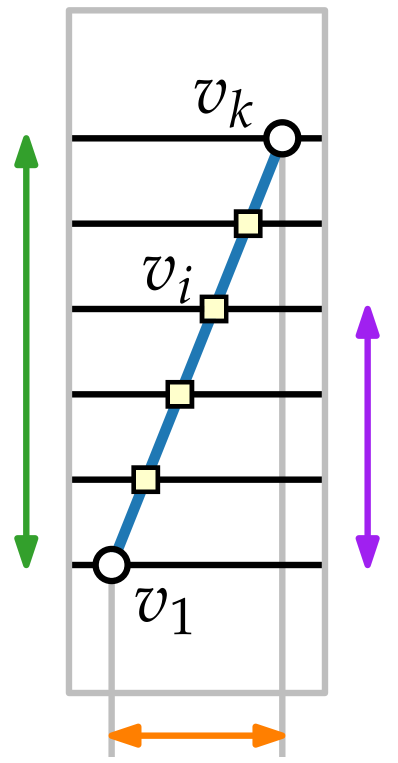
$$\mathrm{dev}(p_e) :=$$



# Quadratic Program

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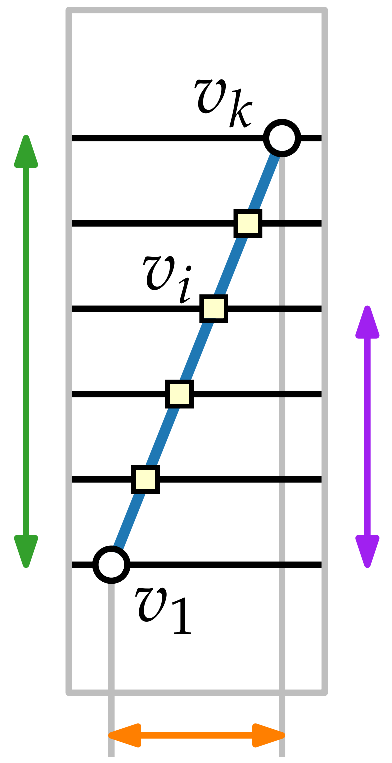

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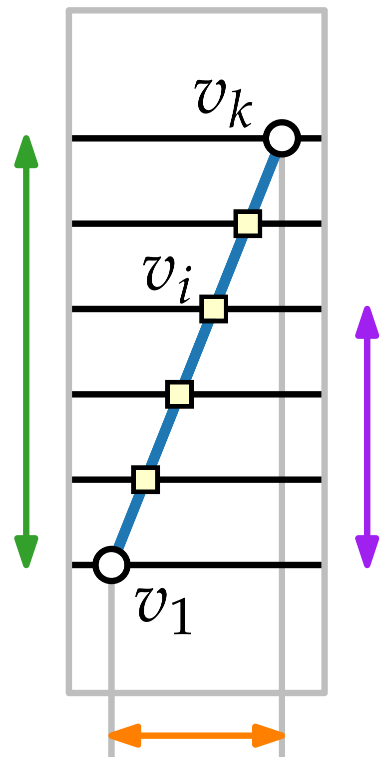

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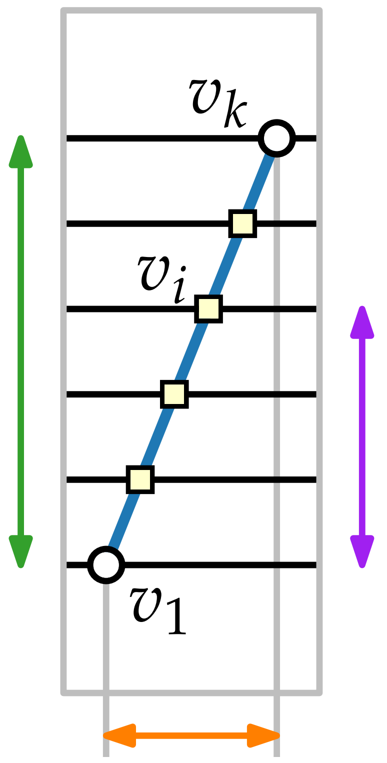

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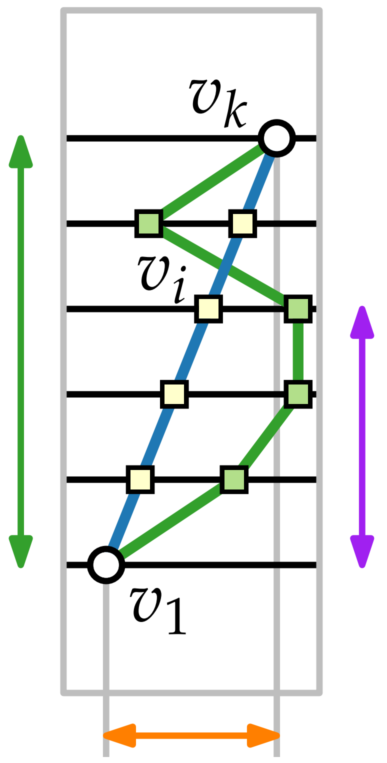

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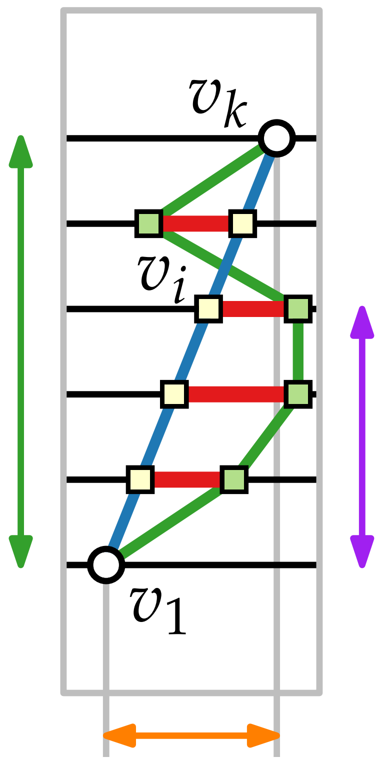

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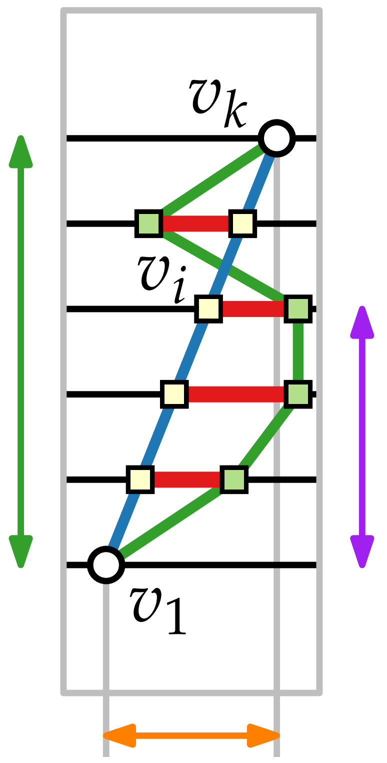

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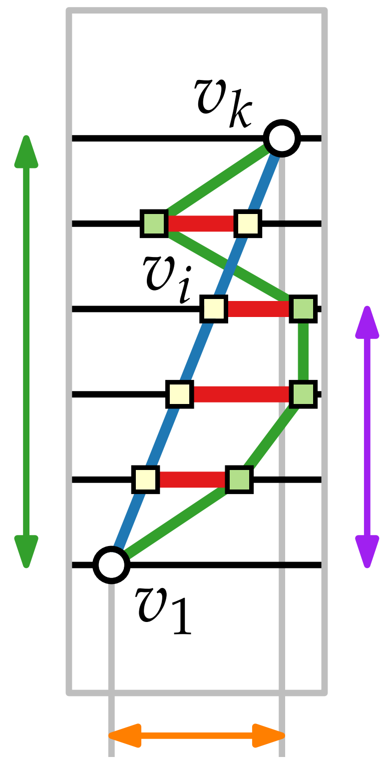

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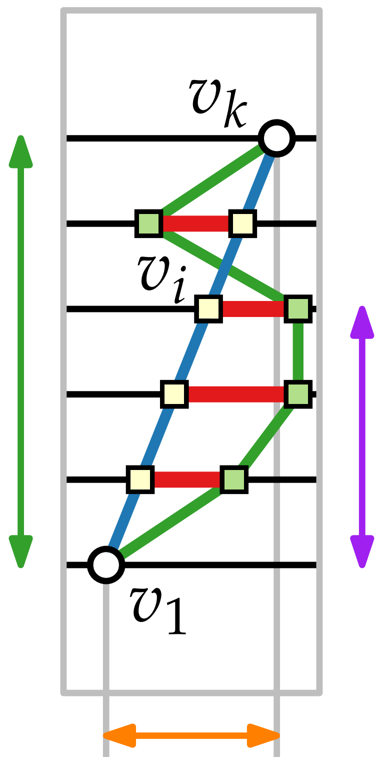

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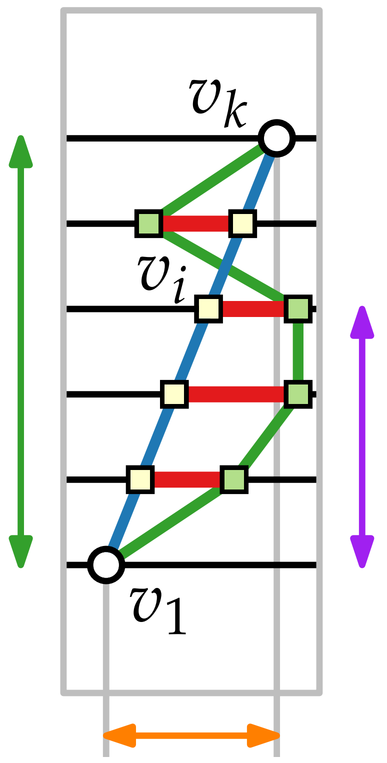

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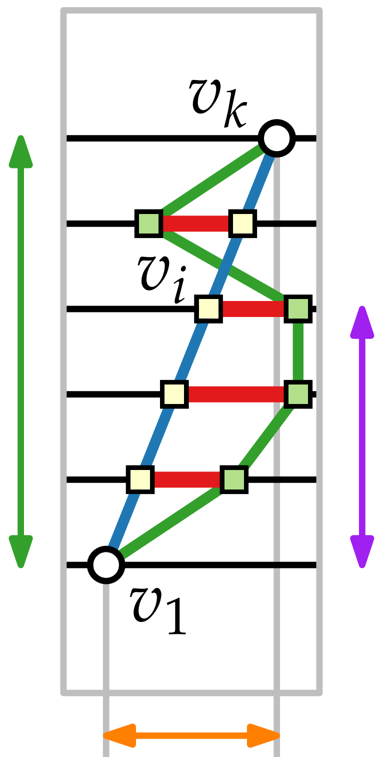


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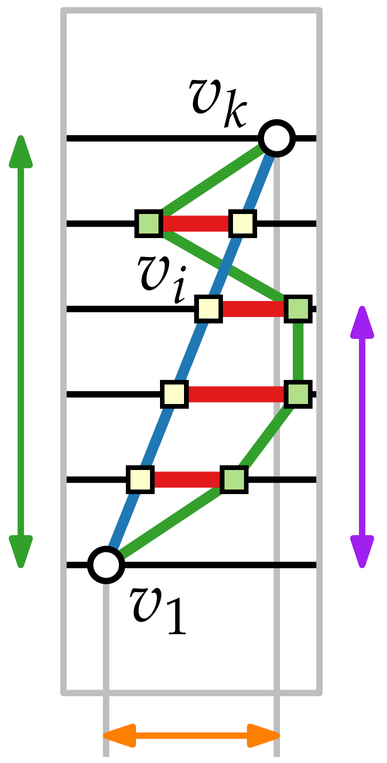


- QP is time-expensive

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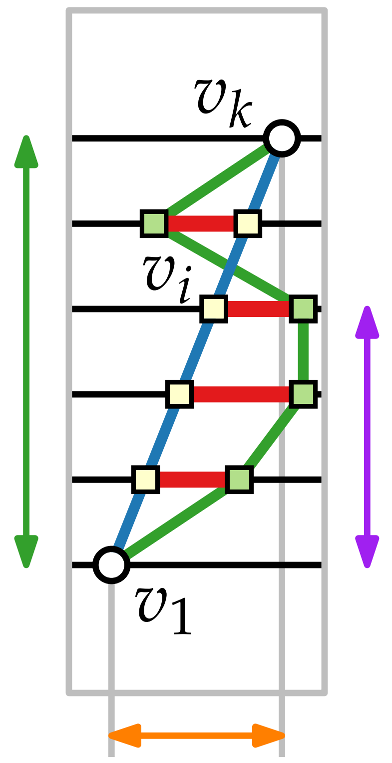


- QP is time-expensive
- width can be exponential

# Iterative Heuristic

- Compute an initial layout

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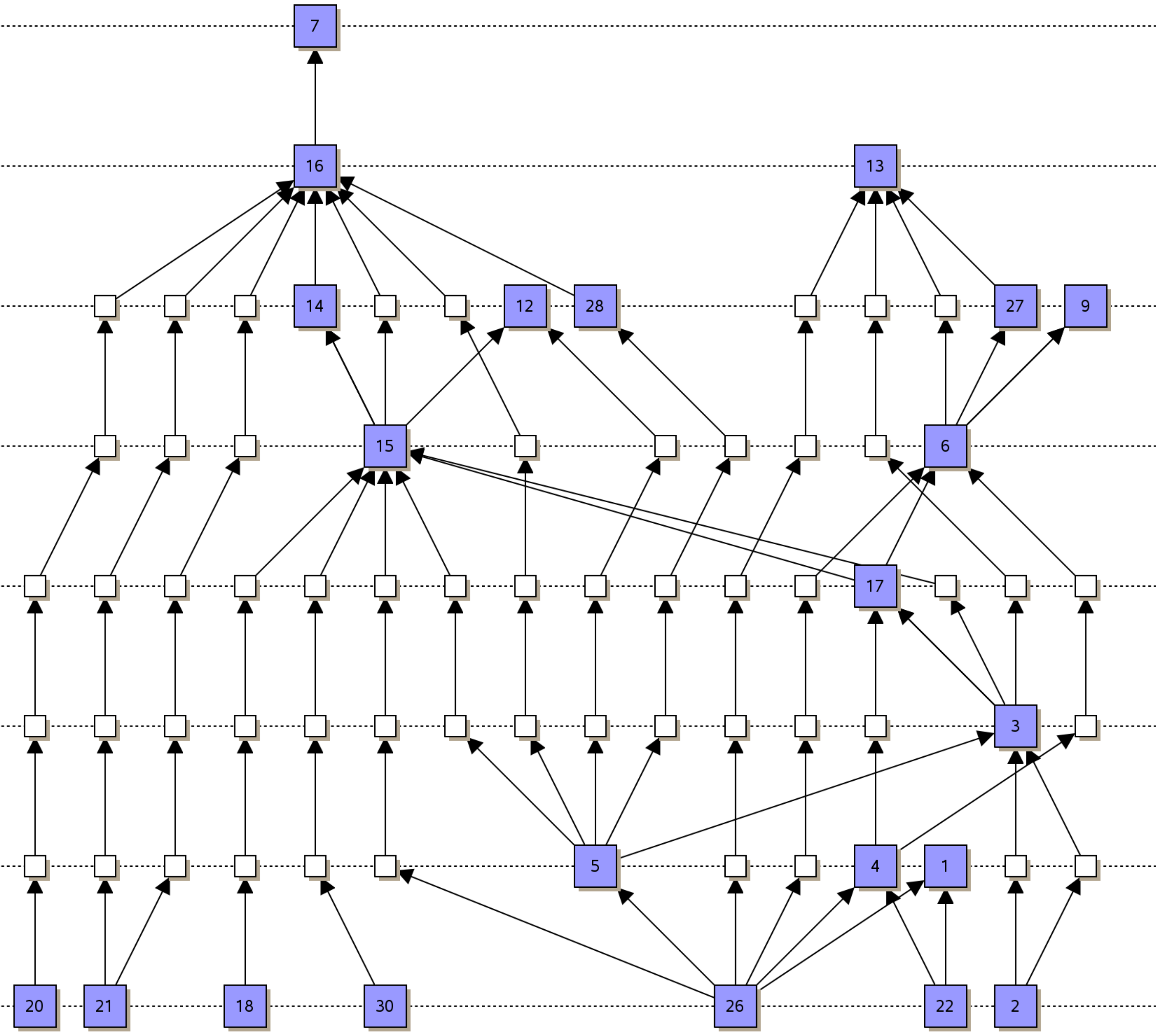
- Compute an initial layout
- Apply the following steps as long as improvements can be made:

  1. Vertex positioning,
  2. edge straightening,
  3. Compactifying the layout width.

# Example



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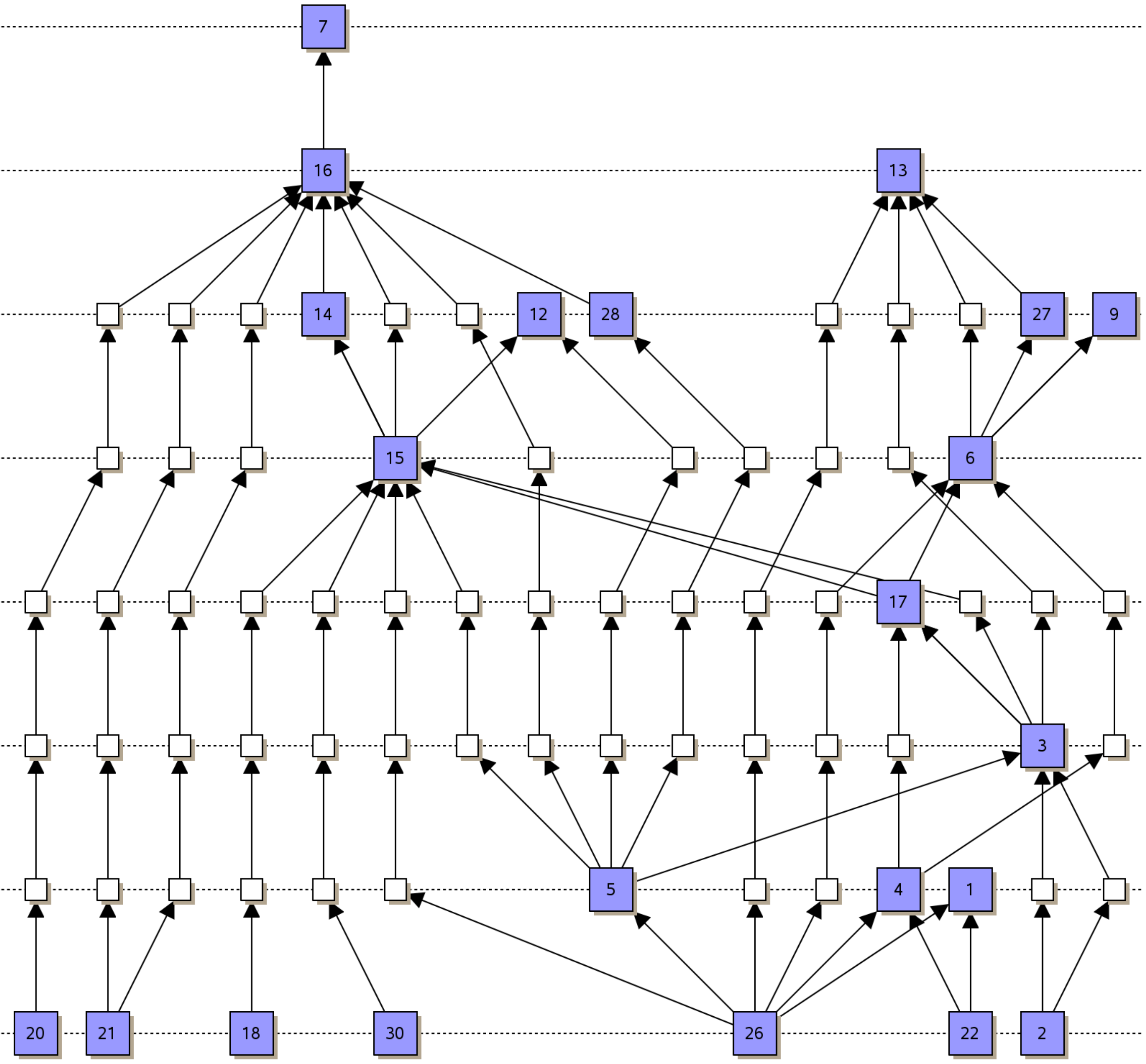

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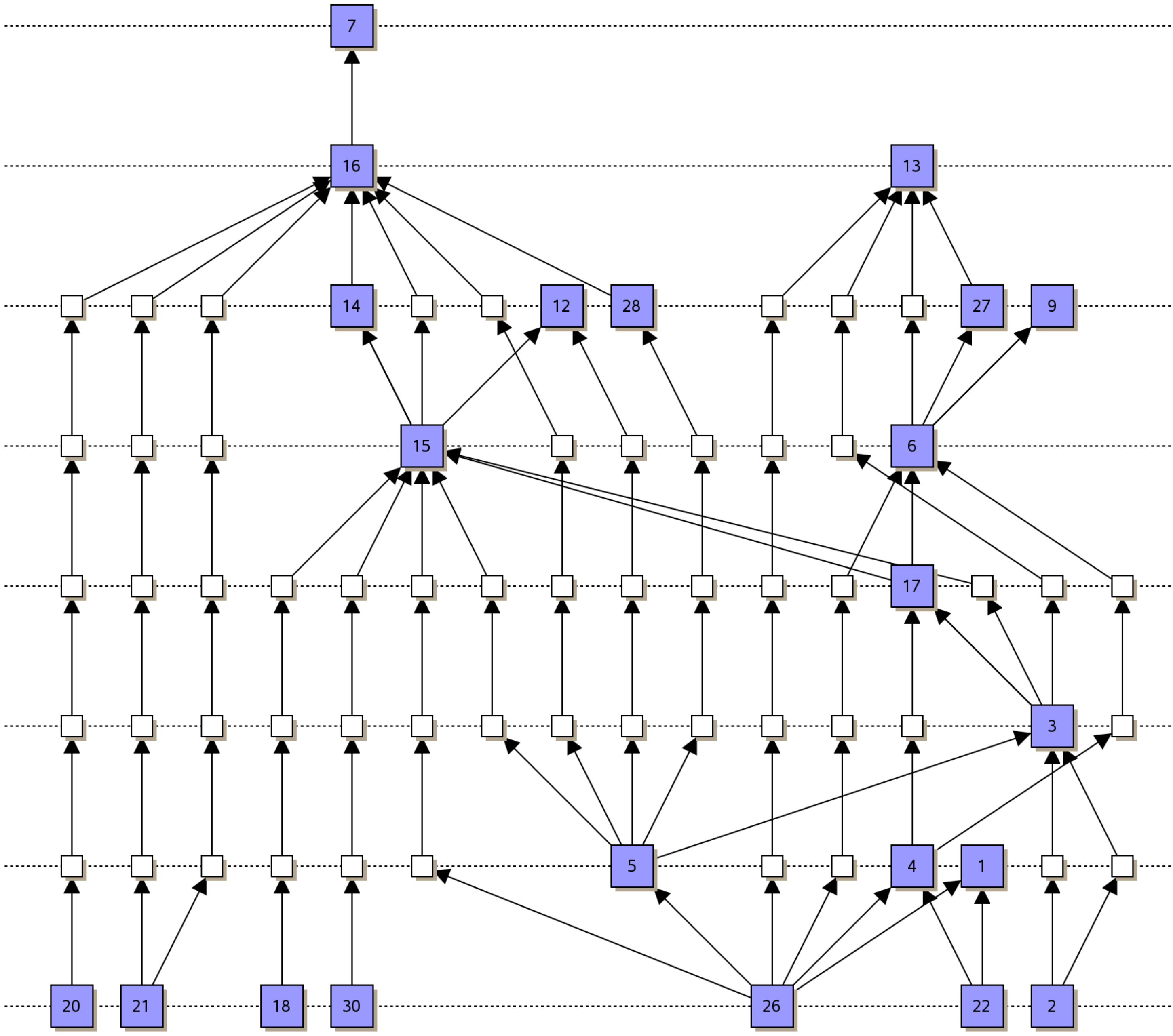

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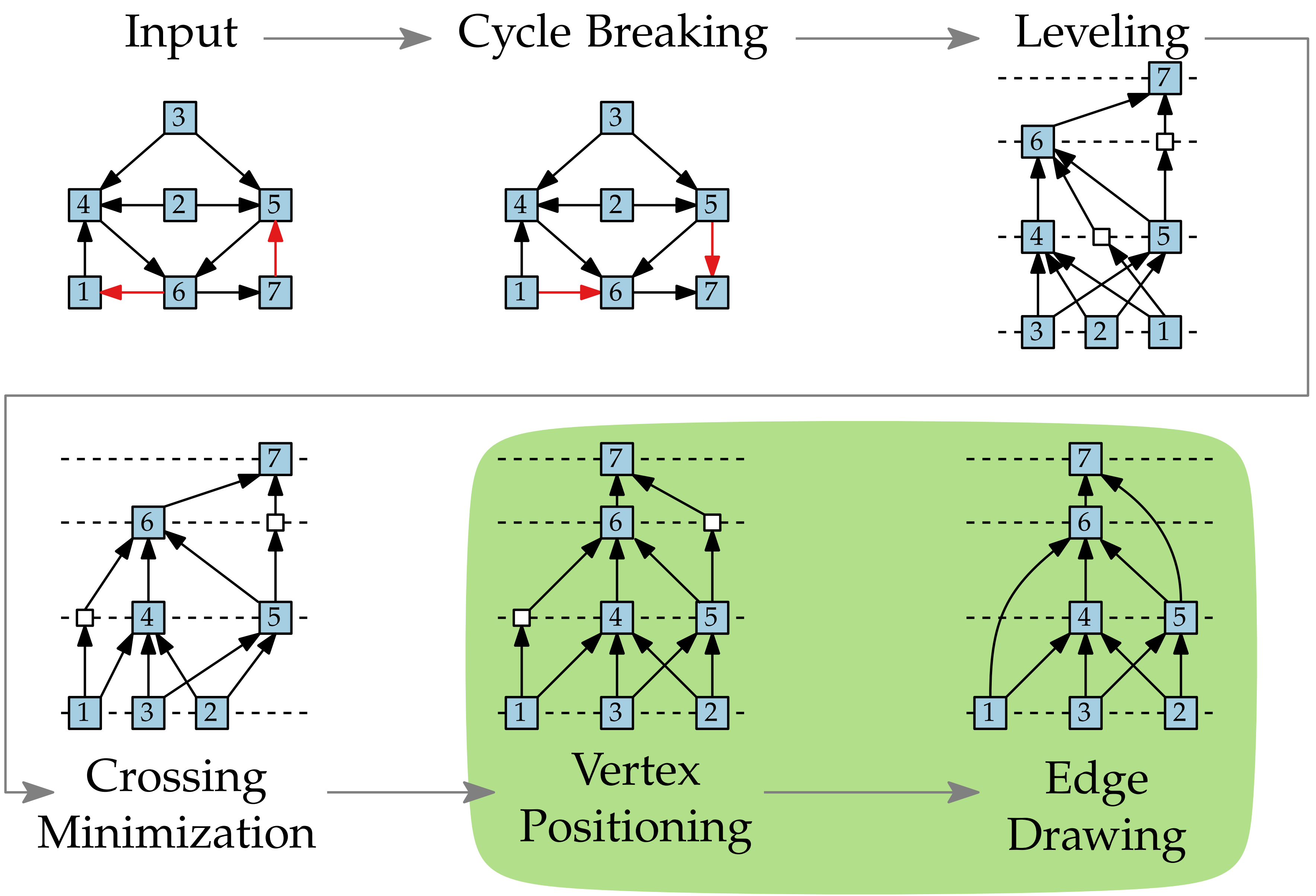
# Step 5: Drawing Edges



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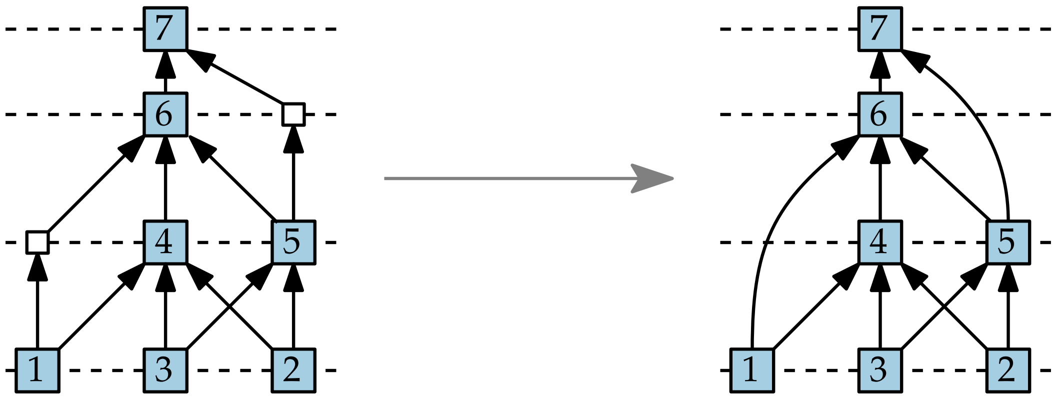
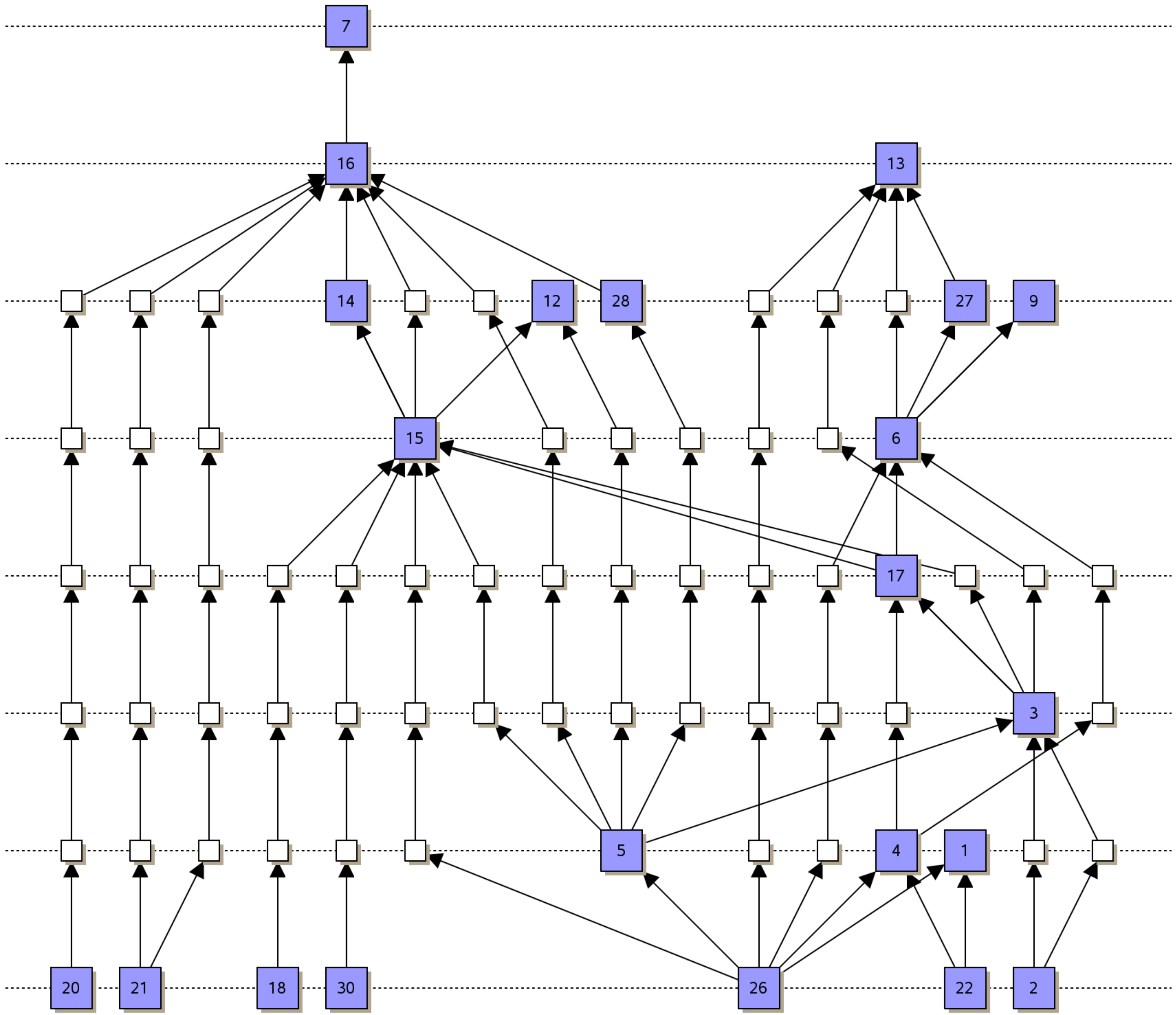



**Possibility.**
Substitute polylines by Bézier curves

# Example



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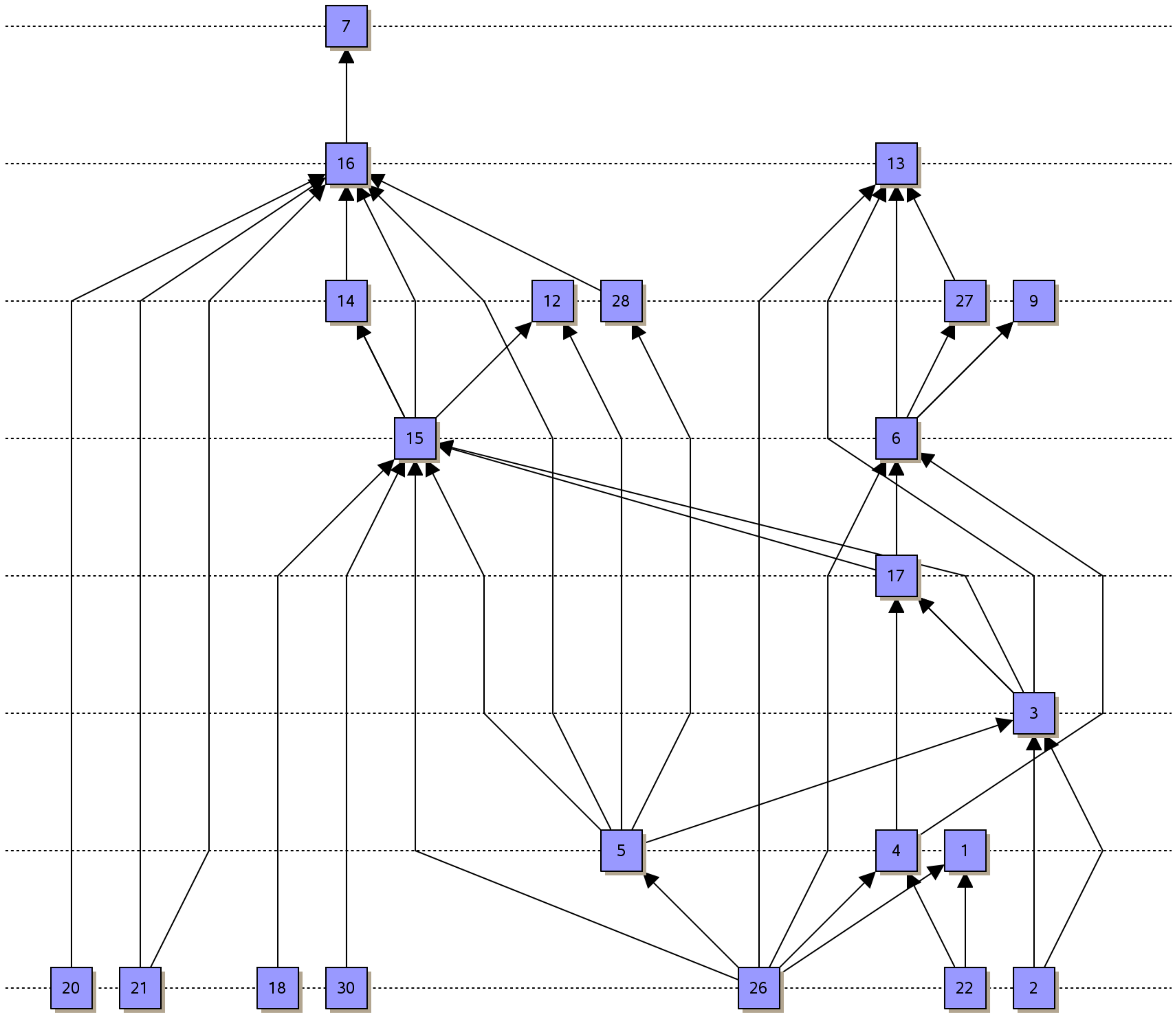

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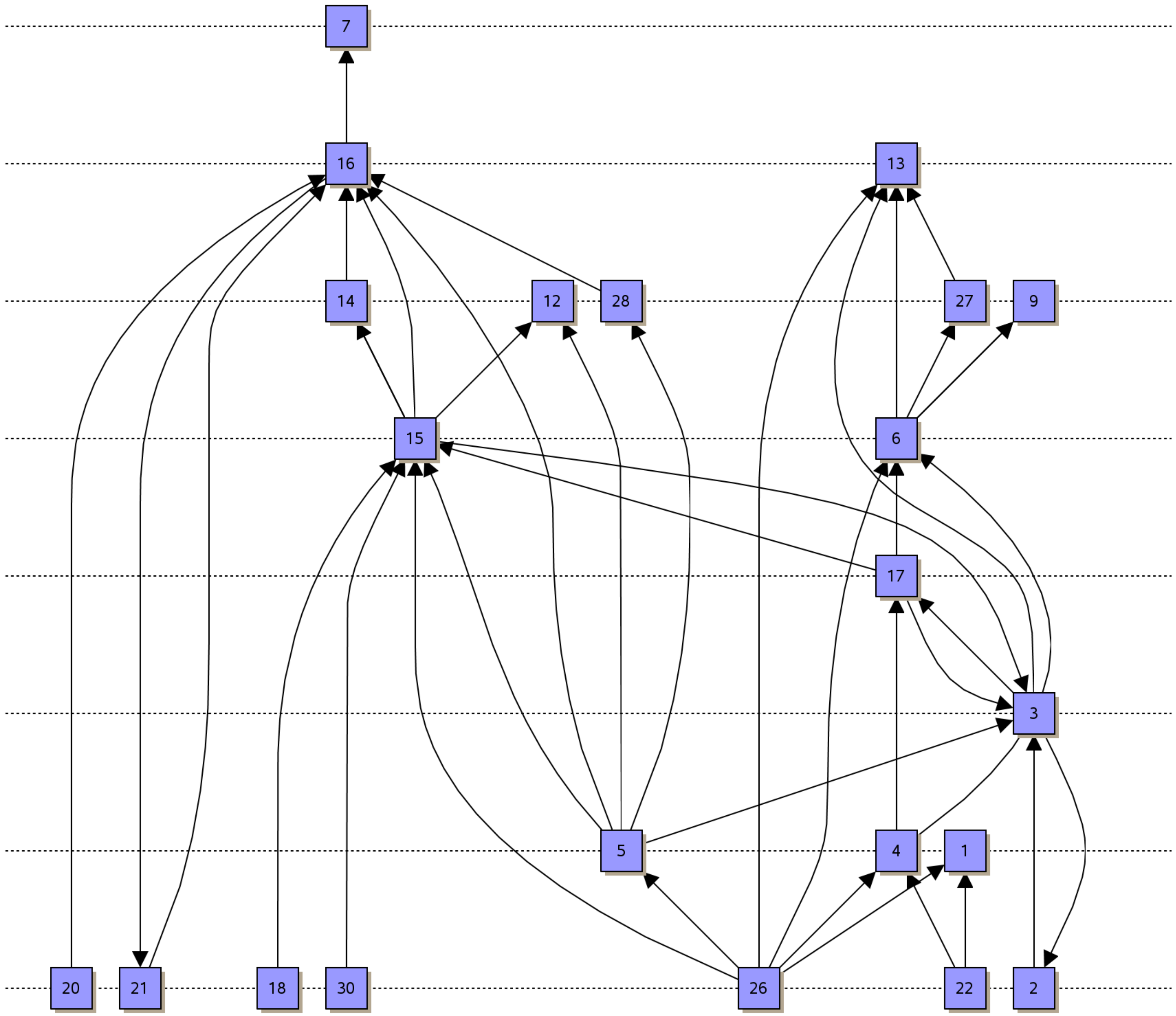
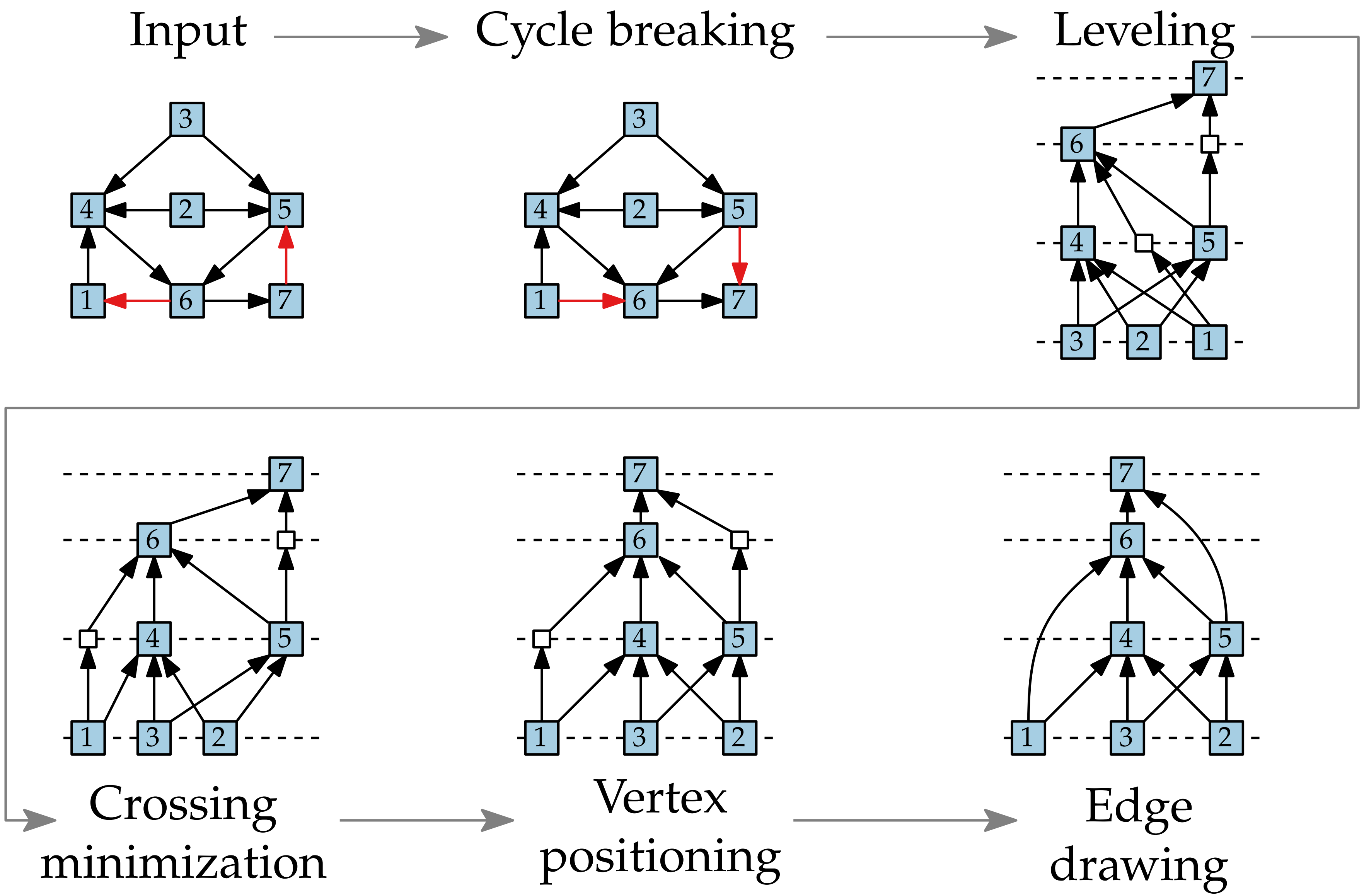


# Classical Approach – Sugiyama Framework

[Sugiyama, Tagawa, Toda ’81]



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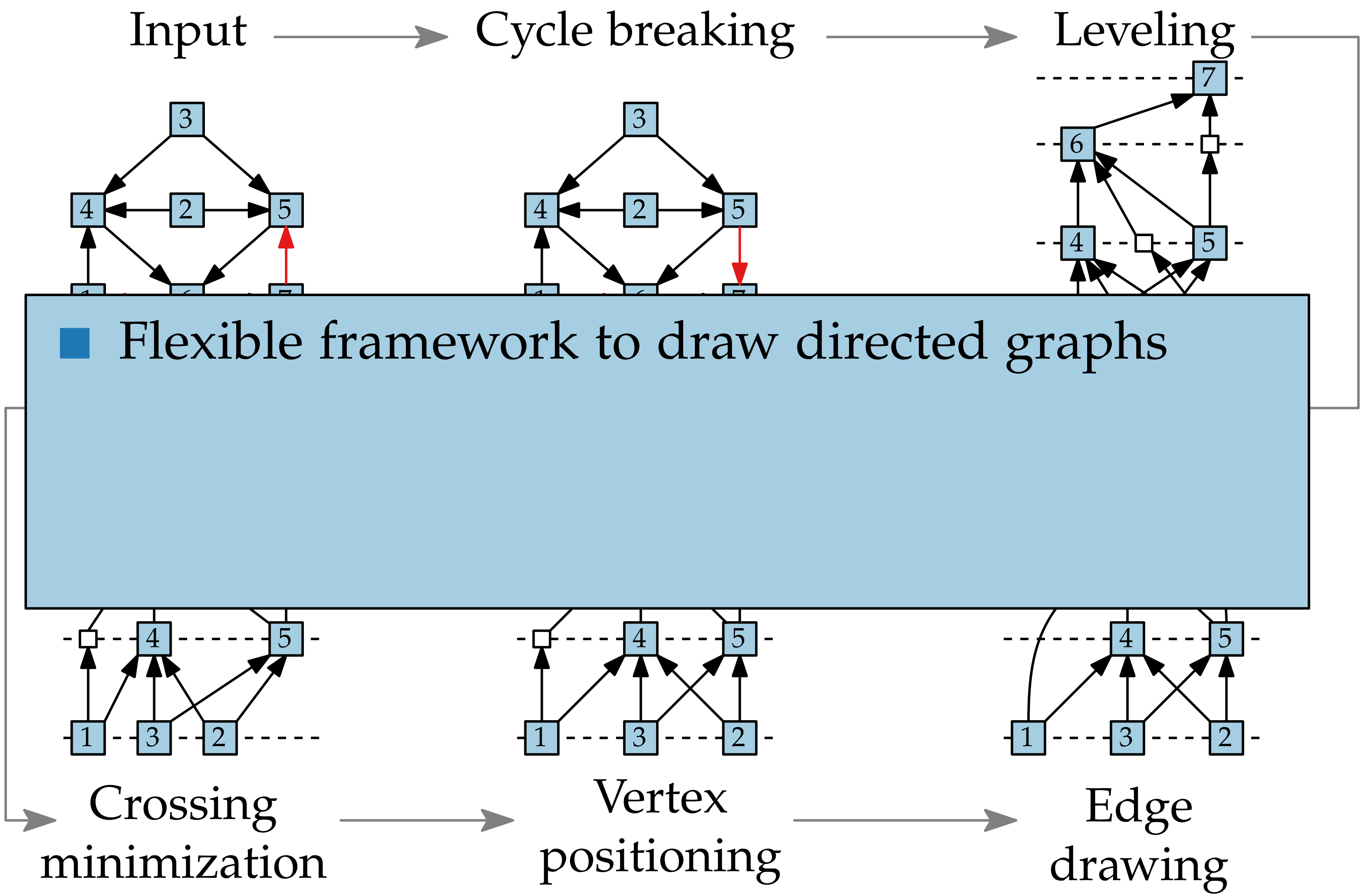


- Flexible framework to draw directed graphs

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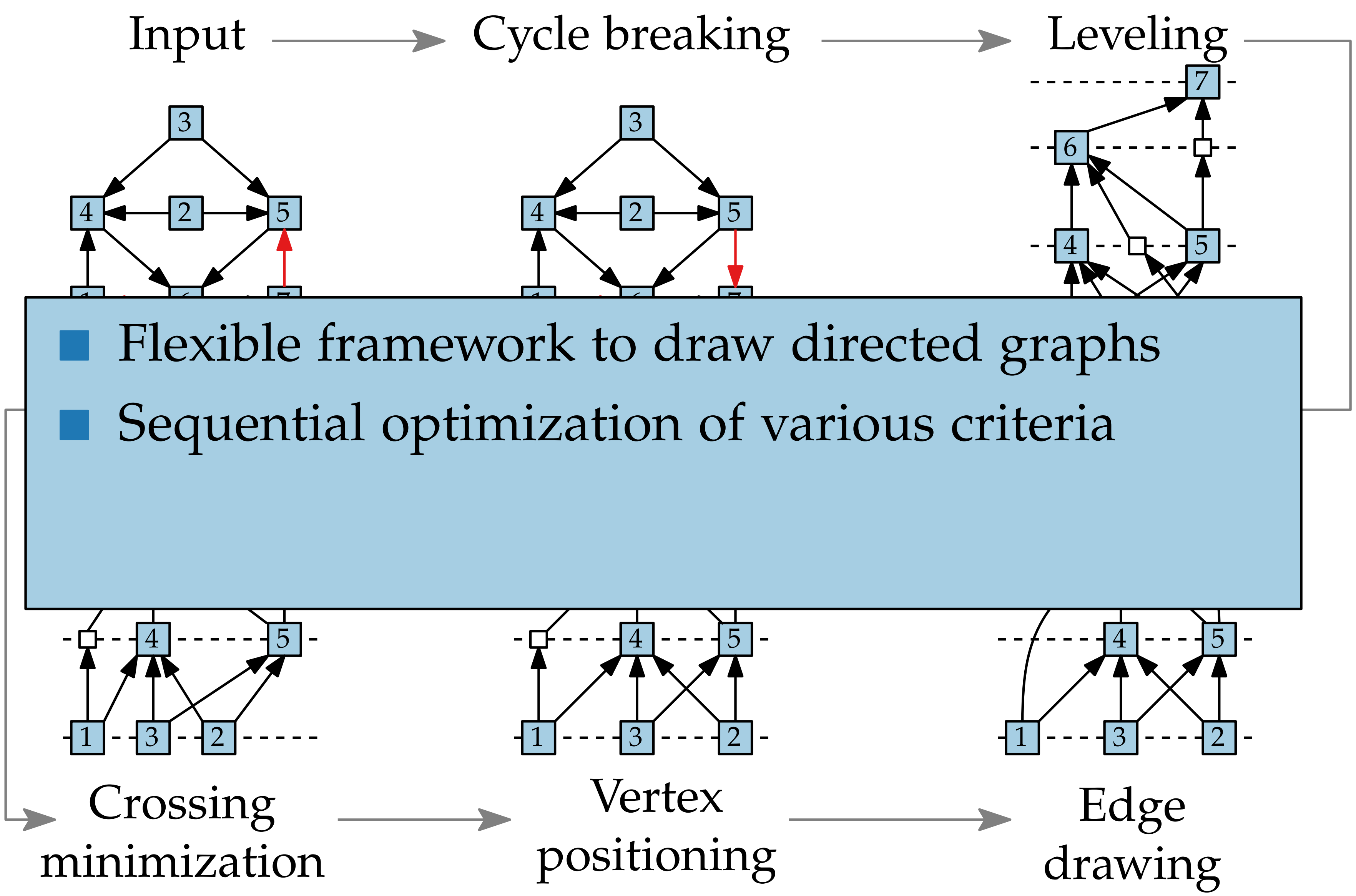


- Flexible framework to draw directed graphs
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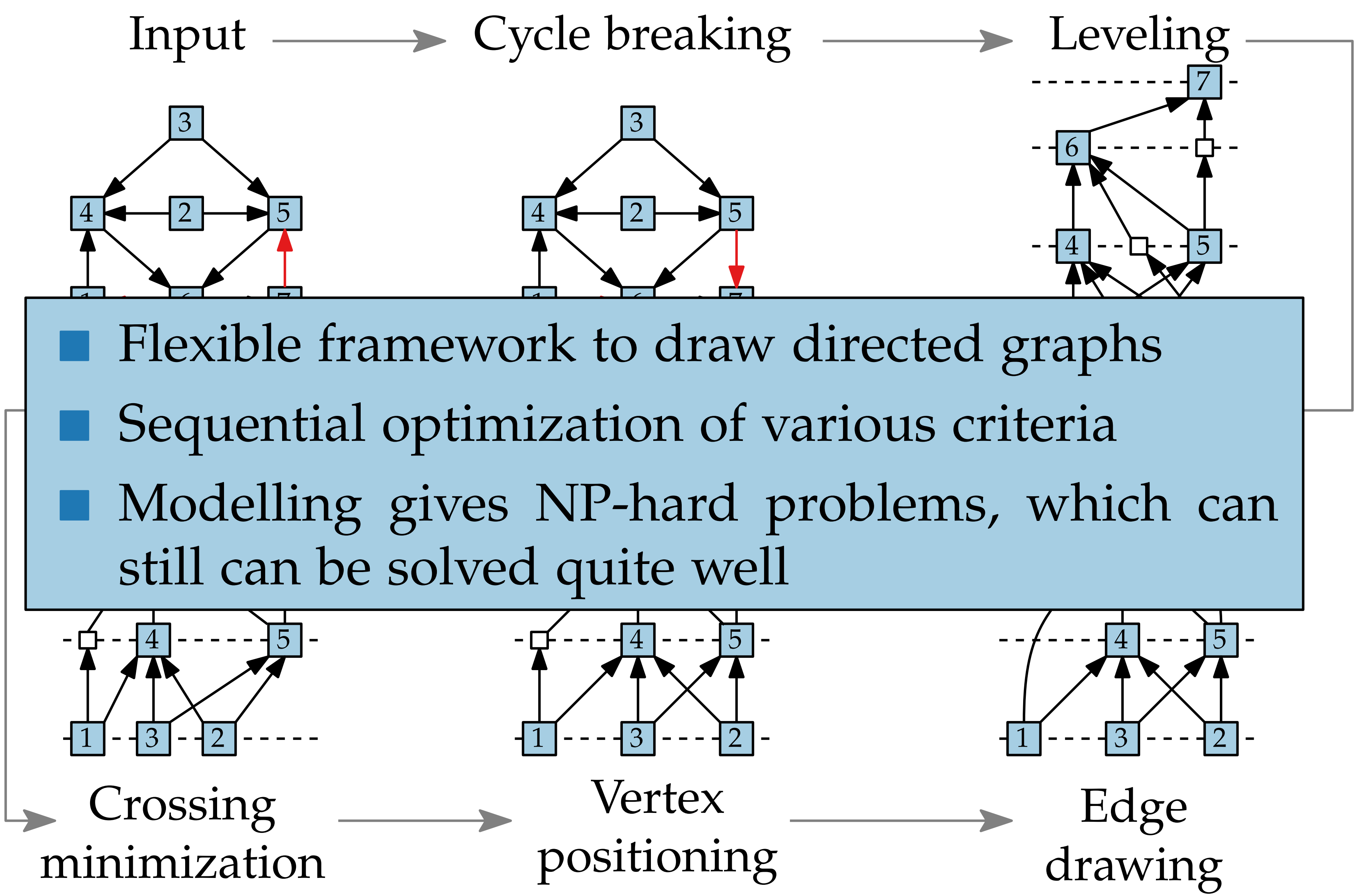


- Flexible framework to draw directed graphs
- Sequential optimization of various criteria
- Modelling gives NP-hard problems, which can still can be solved quite well