CHAPTER 1

GRAPH THEORETIC FOUNDATIONS

1.1. Introduction

A graph can be thought of as a diagram consisting of a collection of vertices together with edges joining certain pairs of vertices. A planar graph is a particular diagram which can be drawn on the plane so that no two edges intersect geometrically except at a vertex at which they are both incident.

First consider the example depicted in Fig. 1.1(a), which consists of six vertices (drawn by small black cycles) and 12 edges (drawn by straight lines there). Is the graph planar? That is, can you draw the graph on the plane by locating vertices and drawing edges appropriately in such a way that no two edges intersect except at a common endvertex? The drawing in Fig. 1.1(a), as it is, has two intersections in the circles drawn by dotted lines. However, one can avoid them if the vertex v_6 is located in the exterior of the square $v_1v_2v_3v_4\ v_1$ (drawn by a thick line), as shown in Fig. 1.1(b). Thus the graph is known to be planar. Next consider the graph depicted in Fig. 1.2(a), known as the complete graph K_5 on five vertices. Is K_5 planar? If we suppose so, then one may assume without loss of generality that $v_1v_2v_3v_4v_5$ is drawn on the plane as a regular pentagon. (Look on the plane as flexible rubber, and deform it as desired.) One may also assume that the edge (v_1, v_3) is drawn in the interior of the pentagon. Then both the edges (v_2, v_5) and (v_2, v_4) must be drawn in the exterior, and consequently edge (v_3, v_5) must be drawn in the interior, as shown in Fig. 1.3(a). Then an intersection must occur whether the edge (v_1, v_4) is drawn in the interior or exterior. Thus K_5 cannot be drawn on the plane without edge-crossing, so is nonplanar. Another example of nonplanar graphs is the "complete bipartite graph" $K_{3,3}$ depicted in Fig. 1.2(b). One may assume that edge (u_1, v_2) is drawn in the interior of the hexagon $u_1v_1u_2v_2u_3v_3$, and hence edge (v_1, u_3) in the exterior. Then (u_2, v_3) cannot be drawn without producing an intersection. Thus $K_{3,3}$ is also known to be nonplanar.

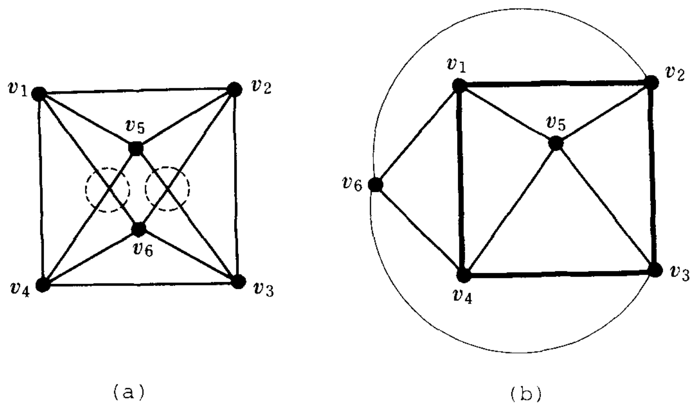


Fig. 1.1. (a) A graph G; and (b) A plane embedding of G.

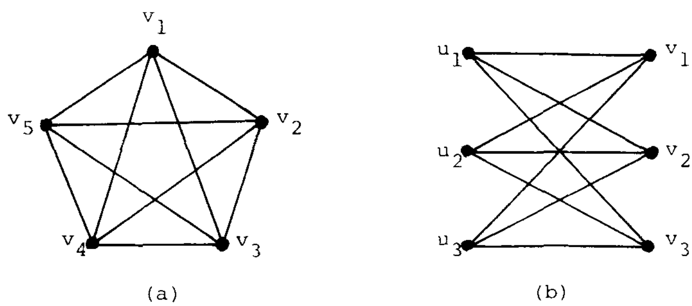
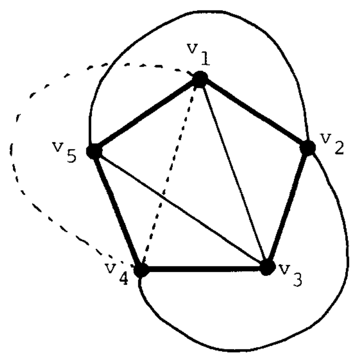


Fig. 1.2. Kuratowski's graphs: (a) Complete graph K_5; (b) Complete bipartite graph $K_{3,3}$.

As above, not all graphs are planar. However planar graphs arise quite naturally in real-world applications, such as road or railway maps, electric printed circuits, chemical molecules, etc. Planar graphs play an important role in these problems, partly due to the fact that some practical problems can be efficiently solved for planar graphs even if they are intractable for general graphs. Moreover, a number of interesting and applicable results are known concerning the mathematical and algorithmic properties of planar graphs. Thus the theory of planar graphs has emerged as a worthwhile mathematical discipline in its own right.

1.2. Some basic definitions

Let us formally define the notion of a graph. A *graph* $G = (V, E)$ is a structure which consists of a finite set of *vertices* V and a finite set of *edges* E;



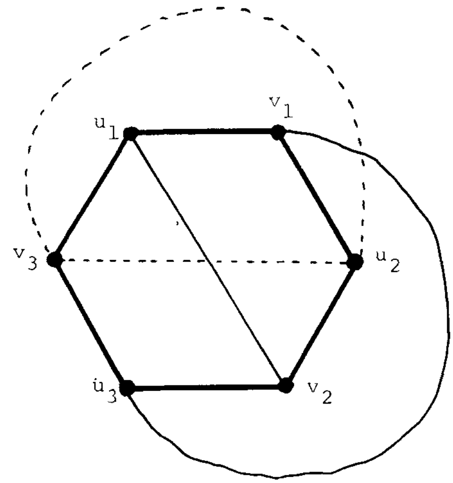


Fig. 1.3. Partial embedding of (a) K_5 and (b) $K_{3,3}$.

each edge is an unordered pair of distinct vertices. Throughout the book n denotes the number of vertices of G, that is, $n = |V|$, while m denotes the number of edges, that is, $m = |E|$. Any edge of the form (u, v) is said to *join* the vertices u and v. Our graph G is a so-called *simple finite graph*, that is, G has no "multiple edges" or "loops" and sets V and E are finite. *Multiple edges* join the same pair of vertices, while a *loop* joins a vertex to itself. The graph, in which loops and multiple edges are allowed, is called a *multigraph*. The graph, in which E is defined to be a set of ordered pairs of distinct vertices, is a *directed graph* (*digraph* for short).

If $(u, v) \in E$, then two vertices u and v of a graph G are said to be *adjacent*; u and v are then said to be *incident* to edge (u, v); u is a *neighbour* of v. The *neighbourhood* $N(v)$ is the set of all neighbours of v. Two distinct edges are *adjacent* if they have a vertex in common. The *degree* of a vertex v of G is the number of edges incident to v, and is written as $d(G, v)$ or simply $d(v)$. In the graph G depicted in Fig. 1.1(a) vertices v_1 and v_2 are adjacent; $N(v_1) = \{v_2, v_4, v_5, v_6\}$, and hence $d(v_1) = 4$.

We say that $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. If $V' = V$ then G' is called a *spanning* subgraph of G. If G' contains all the edges of G that join two vertices in V' then G' is said to be *induced by* V'. If V' consists of exactly the vertices on which edges in E' are incident, then G' is said to be *induced by* E'. Fig. 1.4(a) depicts a spanning subgraph of G in Fig. 1.1(b); Fig. 1.4(b) depicts a subgraph induced by $V' = \{v_1, v_2, v_4, v_5\}$; Fig. 1.4(c) depicts a subgraph induced by $\{(v_1, v_2), (v_1, v_4), (v_1, v_5), (v_2, v_5)\}$.

We shall often construct new graphs from old ones by deleting some vertices or edges. If $V' \subset V$ then $G - V'$ is the subgraph of G obtained by deleting the vertices in V' and all edges incident on them, that is, $G - V'$ is a subgraph induced by $V - V'$. Similarly if $E' \subset E$ then $G - E' = (V, E - E')$. If $V' = \{v\}$ and $E' = \{(u, v)\}$ then this notation is simplified to $G - v$ and $G - (u, v)$.

We also denote by G/e the graph obtained by taking an edge e and *contracting* it, that is, removing e and identifying its ends u and v in such a way that the resulting vertex is adjacent to those vertices (other than u and v) which were originally adjacent to u or v. For $E' \subset E$ we denote by G/E' the graph which results from G after a succession of such contractions for the edges in E'. The graph G/E' is called a *contraction* of G.

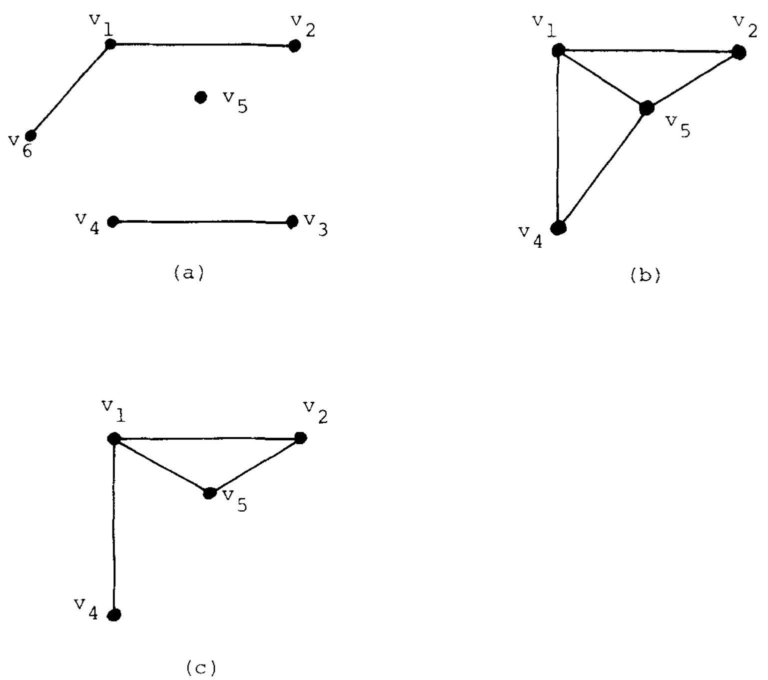


Fig. 1.4. Subgraphs of G in Fig. 1.1(b): (a) Spanning subgraph; (b) Vertex-induced subgraph; (c) Edge-induced subgraph.

A v_0-v_l *walk* in G is an alternating sequence of vertices and edges of G, $v_0, e_1, v_1, \ldots, v_{l-1}, e_l, v_l$, beginning and ending with a vertex, in which each edge is incident on two vertices immediately preceding and following it. The number l of edges is called its *length*. If the vertices $v_0, v_1, \ldots, v_l$ are distinct (except, possibly, $v_0 = v_l$), then the walk is called a *path* and is usually denoted by $v_0 v_1 \cdots v_l$. A path or walk is *closed* if $v_0 = v_l$. A closed path containing at least one edge is called a *cycle*. A cycle of length 3, 4, 5, . . . , is called a triangle, quadrilateral, pentagon, etc. One example of walks in G depicted in Fig. 1.1(b) is

$$v_1, (v_1, v_2), v_2, (v_2, v_3), v_3, (v_3, v_5), v_5,$$

$$(v_5, v_2), v_2, (v_2, v_3), v_3, (v_3, v_4), v_4,$$

which is not closed, that is *open*. One example of cycles is $v_1v_2v_3v_4v_1$, a quadrilateral.

The dual concept of a cycle is a "cutset" which we now define. A *cut* of a graph G is a set of edges of G whose removal increases the number of components. A *cutset* is defined to be a cut no proper subset of which is a cut, that is, a cutset is a minimal cut. Fig. 1.5 illustrates these concepts; $\{a, b, c, d, e\}$ is a cut but not a cutset; both $\{a, b, c\}$ and $\{d, e\}$ are cutsets.

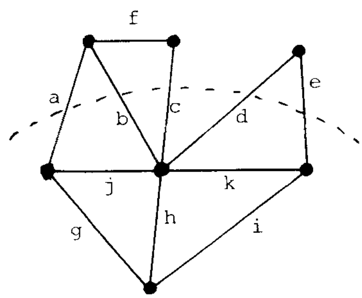


Fig. 1.5. Cut and cutset.

A graph G is *connected* if for every pair $\{u, v\}$ of distinct vertices there is a path between u and v. A (*connected*) *component* of a graph is a *maximal connected subgraph*. A *cutvertex* is a vertex whose deletion increases the number of components. Similarly an edge is a *bridge* if its deletion increases the number of components. G is 2-*connected* if G is connected and has no cutvertex. A *block* of G is a maximal 2-connected subgraph of G. A *separation pair* of a 2-connected graph G is two vertices whose deletion disconnects G. G is 3-*connected* if G has no cutvertex or separation pair. In general, a *separating set* of a connected graph G is a set of vertices of G whose deletion disconnects G. The graph in Fig. 1.4(a) is disconnected, and has three components; the graph in Fig. 1.4(c) which is not 2-connected has a cutvertex v_1, a bridge (v_1, v_4) and two blocks; the graph in Fig. 1.4(b) has no cutvertex but has a separation pair $\{v_1, v_5\}$, so is 2-connected but not 3-connected; G in Fig. 1.1(b) has no cutvertex or separation pair, so is 3-connected.

If G has a separation pair $\{x, y\}$, then we often split G into two graphs G_1

and G_2, called *split graphs*. Let $G_1' = (V_1, E_1')$ and $G_2' = (V_2, E_2')$ be two subgraphs satisfying the following conditions (a) and (b):

(a) $V = V_1 \cup V_2, V_1 \cap V_2 = \{x, y\}$;

(b) $E = E_1' \cup E_2', \quad E_1' \cap E_2' = \varnothing, \quad |E_1'| \geqslant 2, \quad |E_2'| \geqslant 2.$

Define G_1 to be the graph obtained from G_1' by adding a new edge (x, y) if it does not exist; similarly define G_2. (See Fig. 1.6.)

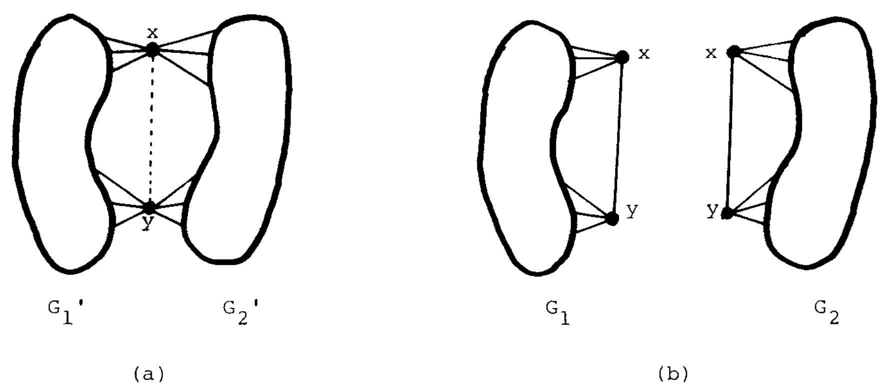


Fig. 1.6. (a) A graph G with a separation pair $\{x, y\}$, where edge (x, y) may not exist; (b) Split graphs G_1 and G_2.

Before ending this section, we will define some special graphs. A graph without any cycles is a *forest*; a *tree* is a connected forest. Fig. 1.4(a) is a forest having three components.

A graph in which every pair of distinct vertices are adjacent is called a *complete graph*. The complete graph on n vertices is denoted by K_n. K_5 has been depicted in Fig. 1.2(a).

Suppose that the vertex set V of a graph G can be partitioned into two disjoint sets V_1 and V_2, in such a way that every edge of G joins a vertex of V_1 to a vertex of V_2; G is then said to be a *bipartite graph*. If every vertex of V_1 is joined to every vertex of V_2, then G is called a *complete bipartite graph* and is denoted by $K_{s,r}$ where $s = |V_1|$ and $r = |V_2|$. Fig. 1.2(b) depicts a complete bipartite graph $K_{3,3}$ with partite sets $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$.

1.3. Planar graphs

Let us formally define a planar graph. Draw a graph G in the given space (e.g. plane) with points representing vertices of G and curves representing edges. G

can be *embedded* (or has an *embedding*) in the space if G can be drawn in such a way that no two edges intersect except at an endvertex in common. A graph G is *planar* if G has an embedding in the plane. Considering the stereographic projection depicted in Fig. 1.7, one can easily establish that a graph is planar if and only if it can be embedded on the surface of a sphere. A *plane graph* is a planar graph which is embedded in the plane. Thus Fig. 1.1(a) is a planar graph, while Fig. 1.1(b) is a plane graph.

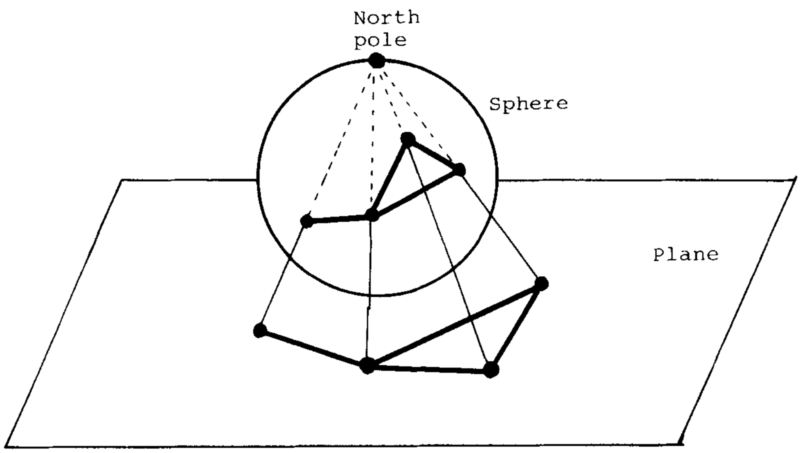
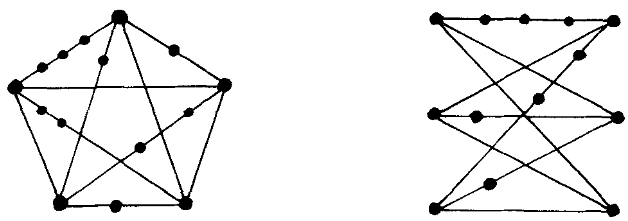


Fig. 1.7. Stereographic projection.

Delete from the plane all curves and points corresponding, respectively, to the edges and vertices of a plane graph G, then the remainder fall into connected components, called *faces*. Note that one face is unbounded; it is called an *outer* (or *infinite*) *face*. If G is embedded on the sphere, then no face of G can be regarded as an outer one. On the other hand, clearly a plane graph G can always be embedded in the plane so that a given face of G becomes the outer face. (Consider again the stereographic projection in Fig. 1.7, and rotate the sphere so that the north pole is in that face.) The *boundary* of a face F is the set of edges in the closure of the face. Thus the boundary is a walk in general, and is a cycle if G is a 2-connected graph with at least three vertices. A cycle in a plane graph G is called a *facial cycle* if it is a boundary of a face.

In general a planar graph has many embeddings in the plane. We shall now define an equivalence relation among these embeddings. Two embeddings of a planar graph are *equivalent* when the boundary of a face in one embedding always corresponds to the boundary of a face in the other. If G is a disconnected plane graph, one can obtain a new nonequivalent embedding simply by replacing a connected component within another face. Similarly, if G has a cutvertex v, one may obtain a new nonequivalent embedding by replacing a component of $G - v$ (together with the edges joining v and vertices in the component) in another face incident to v. Thus we shall assume that G is 2-connected if the embedding is unique. We say that the plane embedding of a graph is *unique* when the embeddings are all equivalent. Whitney [Whi33b] proved that the embedding of a 3-connected planar graph is unique. Before proving the result, we need two more terms. A graph G' is said to be a *subdivision* of a graph G if G' is obtained from G by subdividing some of the edges, that is, by replacing the edges by paths having at most their endvertices in common. Fig. 1.8 depicts subdivisions of K_5 and $K_{3,3}$. If C is a cycle of G, then a *C-component* (or *bridge*) of G is either an edge (together with its ends) not in C joining two vertices of C or it is a connected component of $G - V(C)$ together with all edges (and their ends) of G joining this component to C. Fig. 1.9 illustrates a cycle C and two C-components H and H'.



(a) (b)

Fig. 1.8. Subdivisions of (a) K_5 and (b) $K_{3,3}$.

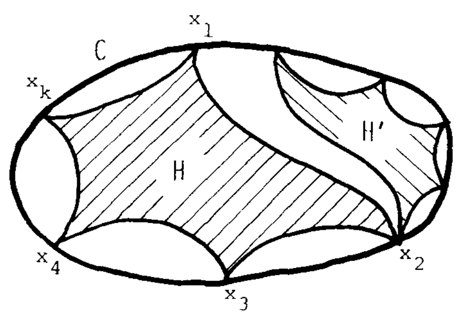


Fig. 1.9. Facial cycle C with two C-components H and H'.

Theorem 1.1. *The embedding of a 2-conneted planar graph G is unique if and only if G is a subdivision of a 3-connected graph.*

Proof. *Necessity*: Suppose that a 2-connected planar graph G is not a subdivision of a 3-connected graph. Then there is a separation pair having split graphs G_1 and G_2 such that both G_1' and G_2' are not paths. (See Fig. 1.6.) Then a new embedding of G is obtained by a reflection or twist of G_1' or G_2'. The boundary of the outer face in the original embedding is no longer a face boundary in the new embedding. Thus the embedding is not unique.

Sufficiency: Suppose that the embedding of a 2-connected planar graph G is not unique. Thus, according to the definition, the original embedding $Em(G)$ of G has a face F with the facial cycle C such that C is no longer a facial cycle in another embedding $Em'(G)$ of G. Clearly G has two C-components H and H'; one in the interior and the other in the exterior of C in $Em'(G)$. One may assume that C is the boundary of the outer face in $Em(G)$. Let $x_1, x_2, \ldots, x_k$ be the vertices of C contained in H occurring in cyclic order. One may assume that all the vertices of C contained in H' are in the subpath of C joining x_1 and x_2 and containing no other x_i. (See Fig. 1.9.) Then $\{x_1, x_2\}$ is a separation pair, for which both G_1' and G_2' are not paths. Therefore G is not a subdivision of a 3-connected graph. Q.E.D.

Theorem 1.1 immediately implies that every 3-connected planar graph has a unique plane embedding.

1.4. Euler's formula

There is a simple formula relating the numbers of vertices, edges and faces in a connected plane graph. It is known as Euler's formula because Euler established it for those plane graphs defined by the vertices and edges of polyhedra. In this section we discuss Euler's formula and its immediate consequences.

Theorem 1.2. (*Euler 1750*). *Let G be a connected plane graph, and let n, m, and f denote respectively the number of vertices, edges and faces of G. Then $n - m + f = 2$.*

Proof. We employ an induction on m, the result being obvious for $m = 0$ or 1. Assume that the result is true for all connected plane graphs having fewer than m edges, where $m \geqslant 2$, and suppose G has m edges. Consider first the case G is a tree. Then G has a vertex v of degree one. The connected plane graph $G - v$ has $n - 1$ vertices, $m - 1$ edges and f faces, so that by the inductive hypothesis, $(n - 1) - (m - 1) + f = 2$, which implies that $n - m + f = 2$. Consider next the case when G is not a tree. Then G has an edge e on a cycle. In this case the connected plane graph $G - e$ has n vertices, $m - 1$ edges, and $f - 1$ faces, so the desired formula immediately follows from the inductive hypothesis.

Q.E.D.

A *maximal planar* graph is one to which no edge can be added without losing planarity. Thus in any embedding of a maximal planar graph G with $n \geqq 3$, the boundary of every face of G is a triangle. Although a general graph may have up to $n(n - 1)/2$ edges, it is not true for planar graphs.

Corollary 1.1. *If G is a planar graph with n ($\geqslant 3$) vertices and m edges, then $m \leqslant 3n - 6$. Moreover the equality holds if G is maximal planar.*

Proof. We can assume without loss of generality that G is a maximal planar graph; otherwise add new edges without increasing n so that the resulting graph is maximal planar. Consider a plane embedding of G. Every face is bounded by exactly three edges, and each edge is on the boundaries of two faces. Therefore, counting up the edges around each face, we have $3f = 2m$. Applying theorem 1.2, we obtain $m = 3n - 6$. Q.E.D.

A stronger bound can be obtained for the number of edges of bipartite planar graphs.

Corollary 1.2. *If G is a planar bipartite graph with n ($\geqslant 3$) vertices, and m edges, then $m \leqslant 2n - 4$.*

Proof. Noting that $4f \leqslant 2m$ and applying Theorem 1.2, we have the desired equation. Q.E.D.

Corollaries 1.1 and 1.2 immediately imply that K_5 and $K_{3,3}$ are nonplanar. The following result is often useful.

Corollary 1.3. *Let G be a planar graph with n ($\geqslant 3$) vertices. Let d_M be the maximum degree of a vertex, and let n_i denote the number of vertices having degree i, where $i = 1, 2, \ldots, d_M$. Then*

$$5n_1 + 4n_2 + 3n_3 + 2n_4 + n_5 \geqslant n_7 + 2n_8 + \cdots + (d_M - 6)n_{d_M} + 12.$$

Proof. Clearly $n = \Sigma\, n_i$ and $2m = \Sigma\, in_i$ where summations are over $i = 1, \ldots, d_M$. Since $6n \geqslant 2m + 12$ by Corollary 1.1, it follows that

$$6\,\Sigma\, n_i \geqslant \Sigma\, in_i + 12,$$

from which the desired inequality immediately follows. Q.E.D.

The following corollary immediately follows from the corollary above.

Corollary 1.4. *Every planar graph contains a vertex of degree at most* 5.

1.5. Kuratowski's theorem

One of the most beautiful theorems in graph theory is Kuratowski's, which gives a characterization of planar graphs in terms of "forbidden graphs". In this section we present and prove the theorem.

Since both K_5 and $K_{3,3}$ are nonplanar as seen twice in Sections 1.1 and 1.4, every planar graph does not contain a subdivision of K_5 and $K_{3,3}$ depicted in Fig. 1.8. Surprisingly the converse is also true.

Theorem 1.3. (*Kuratowski* 1930) *A graph is planar if and only if it does not contain a subdivision of* K_5 *and* $K_{3,3}$.

Since we have already seen the necessity of Theorem 1.3, we shall prove only the sufficiency. The following proof is adapted from [Tho80]. First we give the following lemmas.

Lemma 1.1. [*Tho*80] *If* G *is a* 3-*connected graph having five or more vertices, then* G *contains an edge* e *such that the graph* G/e *obtained from* G *by contracting* e *is* 3-*connected*.

Proof. Suppose that the claim is not true. Then G/e is not 3-connected for any edge $e = (x, y)$ of G. Since G/e has no cutvertex, it must have a separation pair of vertices, one of which is the vertex obtained by identifying x and y. Thus G has a separation set of the form $\{x, y, z\}$. Assume that e and z are chosen in a way that the largest component H of $G - \{x, y, z\}$ is largest possible. Let H' be another component of $G - \{x, y, z\}$, and let $e' = (z, u)$ be an edge joining z and a vertex u in H'. Since G/e' is not 3-connected, G has a separating set of the form $\{z, u, v\}$. (See Fig. 1.10.) Clearly either vertex v is in H' or $v = x$ or y. If v is in H', then clearly $G - \{z, u, v\}$ has a component including all the vertices in H together with x and y, contradicting the assumption about the maximality of H. Thus $v = x$ or y, and we may assume $v = x$. Then the subgraph $H + y$ of G induced by the vertices of H together with y must be connected; for, otherwise, $\{z, x\}$ would be a separation pair of G. Therefore $H + y$ is contained in a single component of $G - \{z, u, v\}$, contradicting the assumption about the maximality of H. Q.E.D.

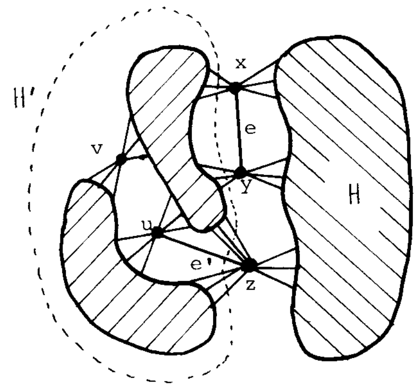


Fig. 1.10. Illustration for proof of Lemma 1.1.

Lemma 1.2. *Let e be an edge in a graph G. If G/e contains a subdivision of* K_5 *or* $K_{3,3}$*, then G also contains a subdivision of* K_5 *or* $K_{3,3}$.

Proof. Let $e = (x, y)$ and let z be the vertex obtained by identifying x and y. Let S be the subdivision of K_5 or $K_{3,3}$ contained in G/e. Each edge (z, v) incident on z in G/e may have two counterparts in G: edges (x, v) and (y, v). Arbitrarily choosing one of them for each edge incident on z, construct a subgraph T of G induced by the edges in S together with e. If z has degree 2 in S, then clearly T and hence G contain a subdivision of K_5 or $K_{3,3}$. Thus we may assume that z has degree 3 or more. Furthermore if x or y has degree 2 in T, then clearly T and hence G contain a subdivision of K_5 or $K_{3,3}$. Hence S must be a subdivision of K_5 and both x and y must have degree 3 in T as shown in Fig. 1.11. As indicated by thick lines, T and hence G contain a subdivision of $K_{3,3}$. Q.E.D.

We are now ready to prove the sufficiency of Theorem 1.3.

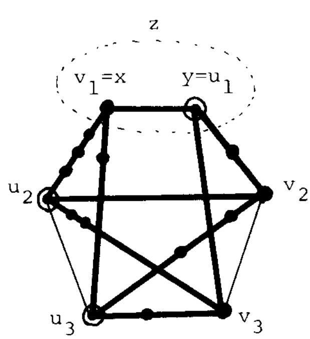


Fig. 1.11. Subgraph T in G, which contains a subdivision of $K_{3,3}$ denoted by thick lines.

Sufficiency of Theorem 1.3. We prove by induction on n the sufficiency: G is planar if G does not contain a subdivision of K_5 and $K_{3,3}$. Since $K_5 - e$ is planar for any edge in K_5, the statement is true if $n \leqslant 5$. We now assume that G has $n \geqslant 6$ vertices and that the statement is true for graphs having fewer than n vertices. There are two cases to consider.

Case 1: G is not 3-connected. Clearly a graph is planar if and only if each of its blocks is planar, so we may assume that G is 2-connected. Therefore G has a separation pair $\{x, y\}$. Let G_1 and G_2 be the split graphs with respect to the pair. (See Fig. 1.6.) Clearly both G_1 and G_2 contain fewer vertices than G, and contain no subdivision of K_5, or $K_{3,3}$. Therefore by inductive hypothesis both G_1 and G_2 are planar. Furthermore both have plane embeddings such that edge (x, y) lies on the outer face boundary. These two plane embeddings can easily be fitted together at x and y to give a plane embedding of G. That is, G is planar.

Case 2: G is 3-connected. Lemma 1.1 implies that G has an edge $e = (x, y)$ such that G/e is 3-connected. Let z be the vertex obtained by identifying x and y. By Lemma 1.2 G/e has no subdivision of K_5 or $K_{3,3}$, and hence by the inductive hypothesis G/e is planar. Consider a plane graph G/e and the plane subgraph $G/e - z$. Since G/e is 3-connected, by Theorem 1.1 the plane embedding of G/e is unique. Let F be the face of the plane graph $G/e - z$ which contained point z, and let C be the boundary cycle of F. Clearly all the neighbours of x or y except themselves must be on cycle C. Let $x_1, x_2, \ldots, x_k$ be the neighbours of x occurring on C in cyclic order and let P_i be the subpath of C joining x_i and x_{i+1} and not containing any x_j, $j \neq i$, $i + 1$, where $x_{k+1} = x_1$. (See Fig. 1.12.) If all neighbours of y except x are contained in one of the paths, then a plane embedding of G can be easily obtained from that of G/e, that is, G is planar, as indicated by dashed lines in Fig. 1.12. Thus not all the neighbours of y except x are contained in one of the paths. Then, since y has three or more neighbours including x, there are three possibilities:

(a) y has three or more neighbours of $\{x_1, \ldots, x_k\}$;
(b) y has a neighbour u in $P_i - \{x_i, x_{i+1}\}$ for some i and a neighbour v not in P_i;
(c) y has two neighbours x_i and x_j such that $j \neq i + 1$ and $i \neq j + 1$.

In case (a) the subgraph of G induced by the vertices in C together with x and y contains a subdivision of K_5. On the other hand, in cases (b) and (c), the subgraph of G induced by the vertices in C together with x and y contains a subdivision of $K_{3,3}$. (See Fig. 1.13.) Thus a contradiction occurs in either case. This completes the proof. Q.E.D.

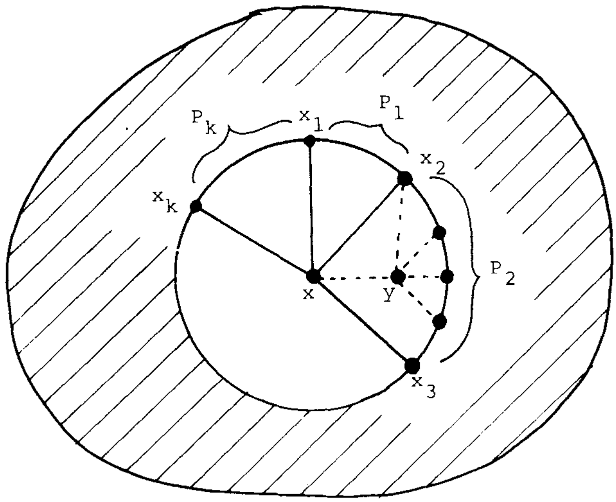


Fig. 1.12. Plane graph $G - y$.

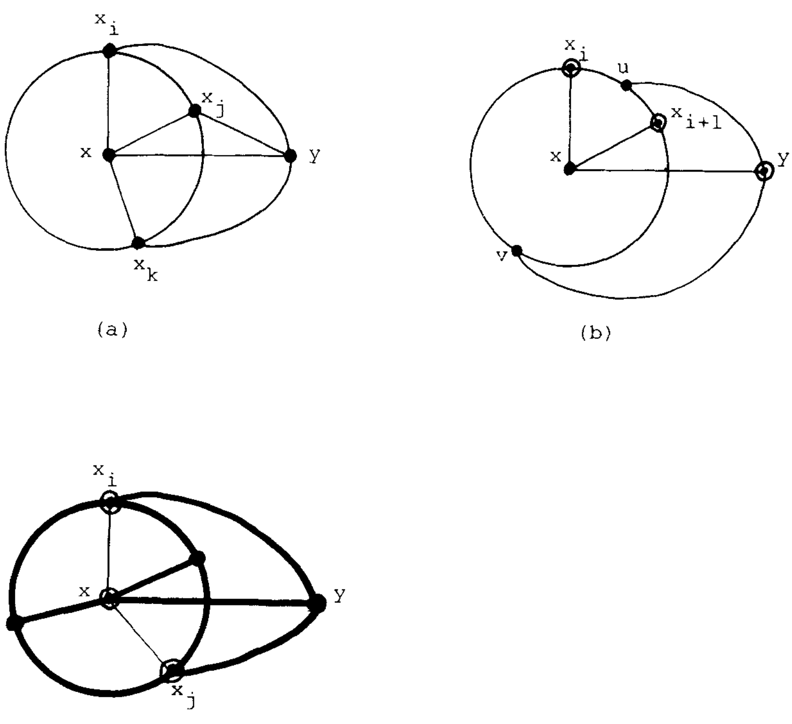


Fig. 1.13. Subdivisions of K_5 and $K_{3,3}$.

1.6. Dual graphs

Several criteria for planarity have been discovered since the original work of Kuratowski. In this section we first define two types of duals of a graph, i.e., geometric and combinatorial duals, and then give two criteria for planarity in terms of combinatorial duals and in terms of the basis of the "cycle space", respectively. Throughout this section "graphs" means multigraphs.

For a plane graph G, we often construct another graph G^* called the (*geometric*) *dual* of G as follows. A vertex v_i^* is placed in each face F_i of G; these are the vertices of G^*. Corresponding to each edge e of G we draw an edge e^* which crosses e (but no other edge of G) and joins the vertices v_i^* which lie in the faces F_i adjoining e; these are the edges of G^*. The construction is illustrated in Fig. 1.14; the vertices v_i^* are represented by small white circles, the edges e^* of G^* by dashed lines. Clearly G^* has a loop if and only if G has a bridge, and G^* has multiple edges if and only if two faces of G have at least two edges in common. Thus G^* is not necessarily a simple graph even if G is simple. Clearly the geometric dual G^* of a plane graph G is also plane. One can easily observe the following lemma.

Lemma 1.3. *Let G be a plane connected graph with n vertices, m edges and f faces, and let the geometric dual G^* have n^* vertices, m^* edges and f^* faces; then $n^* = f$, $m^* = m$, and $f^* = n$.*

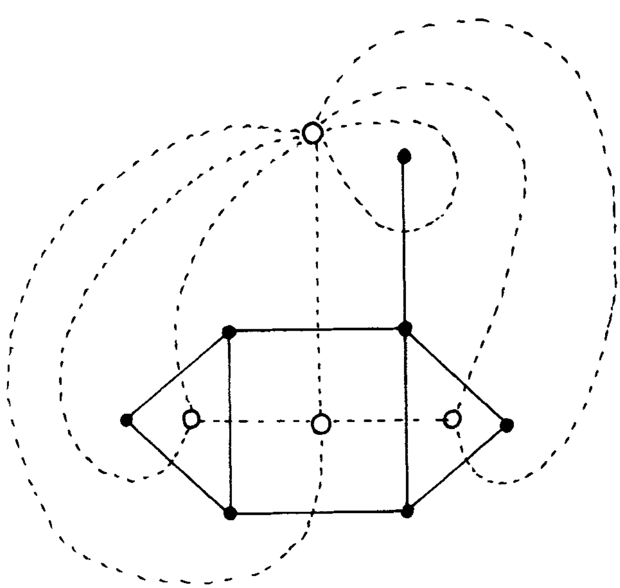


Fig. 1.14. A plane graph G and its geometric dual G^*.

Clearly the dual of the dual of a plane graph G is the original graph G. However a planar graph may give rise to two or more geometric duals since the plane embedding is not necessarily unique. If G is a subdivision of a 3-connected planar graph, then by Theorem 1.1 the plane embedding of G is essentially unique and hence the dual is unique.

The following observation is often useful in designing an efficient algorithm for planar graphs.

Lemma 1.4. *Let G be a planar graph and G^* be a geometric dual of G, then a set of edges in G forms a cycle (or cutset) in G if and only if the corresponding set of edges of G^* forms a cutset (res. cycle) in G^*.*

Proof. One may assume without loss of generality that G is a connected plane graph. Let C be a cycle in G, and let S be the set of vertices of G^* lying in the interior of C. Clearly deleting the edges of G^* corresponding to the edges of C disconnects G^* into two connected subgraphs inside and outside of C: $G^* - S$ and $G^* - (V - S)$. Thus those edges in G^* forms a cutset of G. The remaining claims are similar, and are left to the reader. Q.E.D.

The property in Lemma 1.4 motivates the following abstract definition of duality. We say that a graph G^* is a *combinatorial dual* of a graph G if there is a one–one correspondence between the edges of G and those of G^* with the property that a set of edges of G forms a cycle if and only if the corresponding set of edges of G^* forms a cutset in G^*. Fig. 1.15 depicts a graph G and its combinatorial dual G^*, with corresponding edges sharing the same letter. If G is a planar graph and G^* is a geometric dual of G, then Lemma 1.4 implies that

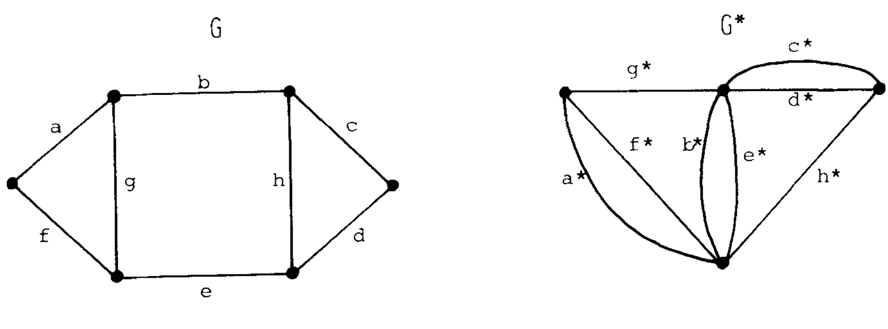


Fig. 1.15. (a) A graph G and (b) its combinatorial dual G^*.

G^* is a combinatorial dual. Thus a planar graph necessarily has a combinatorial dual. Whitney [Whi33a] proved that the converse is also true, giving the following criterion for planarity. From now on in this section we assume for simplicity that G is 2-connected.

Theorem 1.4. *A 2-connected graph G is planar if and only if G has a combinatorial dual.*

We have seen the necessity of Theorem 1.4. Before proving the sufficiency, we present and prove another criterion due to MacLane [Mac37], from which the sufficiency of Theorem 1.4 follows rather immediately.

Clearly the set of subsets of a given set E forms a vector space over the field of order 2, where the modulo 2 sum $A \oplus B$ of two subsets A and B of E is defined as the set of all edges that lie either in A or in B but not both, that is, $A \oplus B = (A - B) \cup (B - A)$. If E is the set of edges in a graph G, we call the subspace generated by the cycles of G, the *cycle space* of G. It is easy to verify that if Z and Z' are cycles in G, the sum $Z \oplus Z'$ is an edge-disjoint union of cycles in G. Thus the cycle space of G consists of the cycles of G and the edge-disjoint unions of cycles. A *cycle basis* of G is defined as a basis for the cycle space of G which consists entirely of cycles. A cycle basis B of G is called a *2-basis* of G if every edge of G is contained in at most two cycles of B. The cutset space of G is similarly defined.

It is easily shown that every planar 2-connected graph has a 2-basis. Let B be the collection of all facial cycles except the boundary of the outer face. Every cycle Z is generated by cycles in B: Z is a sum of exactly the facial cycles in the interior of Z. Furthermore the cycles in B are independent: any sum of cycles in B is nonzero. Thus B is a cycle basis of G. An edge in the boundary of the outer face is contained in precisely one cycle in B, while each of the other edges is in two of the facial cycles in B. Hence B is a 2-basis of G. Surprisingly the converse is also true; the result is known as the *McLane's criterion* for planarity. In order to prove the converse we need the following lemma.

Lemma 1.5. *Neither K_5 nor $K_{3,3}$ has a 2-basis.*

Proof. Suppose that K_5 has a 2-basis B. Since the rank of the cycle space of K_5 is 6, B consists of 6 cycles $Z_1, Z_2, \ldots, Z_6$. Since each Z_i contains 3 or more edges, the number of edges in Z_i totals at least 18. On the other hand K_5 has 10 edges, each of which is contained in at most two cycles of B. Therefore at least 8 edges of K_5 appear twice in these cycles. Then the sum $Z_1 \oplus Z_2 \oplus \cdots \oplus Z_6$ consists of at most two edges. Since $Z_1, \ldots, Z_6$ are independent, the sum must be nonzero and hence consist of at least three edges, a contradiction. Similarly one can show that $K_{3,3}$ has no 2-basis. Q.E.D.

Lemma 1.6. *No subdivision of K_5 or $K_{3,3}$ has a 2-basis.*

Proof. If a subdivison of K_5 or $K_{3,3}$ contained a 2-basis B, then the collection of cycles in K_5 or $K_{3,3}$ which naturally correspond to the cycles in B would be a 2-basis of K_5 or $K_{3,3}$, contradicting Lemma 1.5.

Lemma 1.7. *Let G be a graph, and let e be an edge of G. If G has a 2-basis B then $G' = G - e$ also has a 2-basis.*

Proof. Note that the collection of cycles in G' consists of all the cycles in G not containing e. Since B is a 2-basis of G, e is contained in one or two cycles of B. Thus there are two cases to consider.

Case 1: e is contained in only one cycle Z of B. Clearly the cycles in $B - \{Z\}$ are independent since the cycles in B are so. Since B is a cycle basis of G, any cycle C of G is a sum of cycles in B. Therefore, if C does not contain e, that is, C is a cycle of G', then C is a sum of cycles in $B - \{Z\}$. Thus $B - \{Z\}$ is a cycle basis of G', and hence clearly it is a 2-basis of G'.

Case 2: e is contained in two cycles Z and Z' of B. In this case the sum $Z \oplus Z'$ is decomposed into edge-disjoint cycles $Z_1, Z_2, \ldots, Z_r$. Similarly as in Case 1 one can easily show that $B' = (B - \{Z, Z'\}) \cup \{Z_1, \ldots, Z_r\}$ is a 2-basis of G'. Q.E.D.

Lemmas 1.6 and 1.7 above imply the following MacLane's criterion.

Theorem 1.5. *A 2-connected graph G is planar if and only if G has a 2-basis.*

Proof. Since we have already seen the necessity, we shall establish the sufficiency. Suppose that a 2-connected graph G has a 2-basis, but is nonplanar. Then Kuratowski's theorem (Theorem 1.3) implies that G has a subdivision G' of K_5 or $K_{3,3}$. Repeated application of Lemma 1.7 implies that G' has a 2-basis. However by Lemma 1.6 G' has no 2-basis, a contradiction. Q.E.D.

We are now ready to prove the sufficiency of Theorem 1.4 (Whitney's criterion): if a 2-connected graph G has a combinatorial dual G^* then G is planar.

Proof of sufficiency of Theorem 1.4. Suppose that a 2-connected graph G has a combinatorial dual $G^* = (V^*, E^*)$. Since G is 2-connected, every two edges of G lie on a common cycle, and hence every two edges of G^* lie on a common cutset. Therefore G^* must be 2-connected. Let S_v, $v \in V^*$, be the set of edges of G^* incident to vertex v, then S_v is a cutset of G^* since G^* is 2-connected. Any cutset S of a graph G^* is a sum of S_v over all vertices v in a component of $G^* - S$. Thus the cutset space of G^* is generated by all these S_v. Let u be an arbitrary vertex of G^*, then S_u is a sum of S_v over all v in $V^* - u$. Thus $B^* = \{S_v : v \in V^* - u\}$ is a basis of the cutset space of G^*. Obviously every edge of G^* is contained in at most two S_v of B^*. Thus the collection of cycles in G corresponding to cutsets of B^* is a 2-basis of G. Therefore by Theorem 1.5 G is planar. Q.E.D.

1.7. Bounds for planar graphs

In Section 1.4 we showed that nontrivial bounds on the number m of edges in a planar graph follow from Euler's formula. In this section we present several interesting bounds on other graph invariants for planar graphs, which will be used by algorithms in later chapters.

An *independent set* of a graph G is a set of nonadjacent vertices, and a *maximum independent set* is one of maximum cardinality. For example the graph in Fig. 1.1(b) has a maximum independent set $\{v_5, v_6\}$. The problem of finding a maximum independent set is "NP-hard" even for planar graphs, and no algorithm is known for finding a maximum independent set in polynomial-time. There may exist a large independent set in a planar graph if the minimum degree is 1 or 2. For example the complete bipartite graph $K_{2,n-2}$, $n \geqslant 4$, which is a planar graph with minimum degree 2, contains a maximum independent set of $n - 2$ vertices. This is not the case if the minimum degree is 3 or more. Corollary 1.2 together with an easy observation provide nontrivial upper bounds on the number of vertices in an independent set for such planar graphs. Note that the minimum degree of a planar graph is necessarily 5 or less as shown in Corollary 1.4.

Theorem 1.6. *If G is a planar connected graph with minimum degree* 3, 4, *or* 5, *then a maximum independent set of G contains less than* $2n/3$, $2n/4$, *or* $2n/5$ *vertices, respectively.*

Proof. Let $I^* \subset V$ be a maximum independent set of G. Let G' be the spanning subgraph obtained from G by deleting all the edges having both ends in $V - I^*$, and assume that G' has m' edges. Since G' is a planar bipartite graph with n ($\geqslant 4$) vertices, Corollary 1.2 implies $m' \leqslant 2n - 4$. On the other hand, since $d(G, v) = d(G', v)$ for every vertex v in I^*, we have $m' \geqslant 3|I^*|$, $4|I^*|$, or $5|I^*|$ according to the minimum degree. The two inequalities above immediately imply the claim. Q.E.D.

A *matching* of a graph G is a set of nonadjacent edges, and a *maximum matching*, denoted by $M(G)$, is one of maximum cardinality. For example for G in Fig. 1.1(b) both $\{(v_1, v_2), (v_3, v_4)\}$ and $\{(v_1, v_5), (v_2, v_3), (v_4, v_6)\}$ are matchings, and the latter is a maximum. Unlike the maximum independent set, efficient polynomial-time algorithms have been obtained for finding a maximum matching in general graphs [EK75, MV80]. Similarly as in Theorem 1.6 we can obtain a lower bound on $|M(G)|$ for planar graphs with minimum degree 3 or more. To prove the bound we need the following sophisticated characterization on $|M(G)|$, due to Tutte [Tut47] and Berge [Ber57], where $q(G)$ denotes the number of *odd components* of G, that is, ones having an odd number of vertices.

Theorem 1.7. *Let G be a graph of n vertices. If*

$$u = \max_{S \subset V} \{q(G - S) - |S|\}, \tag{1.1}$$

then $|M(G)| = (n - u)/2$, that is, there are u vertices with which no edge of $M(G)$ is incident.

Proof. See [Bol79 or Beg76].

Theorem 1.7 together with Corollary 1.2 yield the following lower bound on $|M(G)|$ for planar graphs [Nis79, NB79]. Throughout this book $\lfloor x \rfloor$ denotes the largest integer $\leqslant x$, while $\lceil x \rceil$ is the smallest integer $\geqslant x$.

Theorem 1.8. *If a planar connected graph G has minimum degree 3 or more, then*

$$|M(G)| \geqslant \min\{\lfloor n/2 \rfloor, \lceil (n + 2)/3 \rceil\}.$$

Proof. By Theorem 1.7 it suffices to show that if $u \geqslant 2$ then $u \leqslant \lfloor (n - 4)/3 \rfloor$. Suppose that $u \geqslant 2$, and let S be the subset of V attaining the maximum in (1.1), then clearly $|S| \geqslant 1$. Assume that $G - S$ has q_1 odd components containing exactly one vertex. There are two cases to consider.

Case 1: $q_1 = 0$. In this case $n \geqslant 3q(G - S) + |S|$ since every odd component contains three or more vertices. Then the desired inequality follows immediately:

$$u = q(G - S) - |S| \leqslant (n - |S|)/3 - |S| \leqslant (n - 4)/3.$$

Case 2: $q_1 \geqslant 1$. Remove from G all the vertices in the component of $G - S$ having two or more vertices, and delete all the edges with two ends in S. Let G' be the resulting graph, and let us denote by n' and m' the numbers of vertices and edges in G', respectively. Clearly

$$n' = q_1 + |S|, \tag{1.2}$$

and

$$n' \leqslant n - 3(q(G - S) - q_1),$$

which implies

$$q(G - S) - q_1 \leqslant (n - n')/3. \tag{1.3}$$

Since G' is a bipartite planar graph with n' ($\geqslant 4$) vertices and all the q_1 vertices in one partite set have degree 3 or more, Corollary 1.2 and an easy counting yield

$$3q_1 \leqslant m' \leqslant 2n' - 4. \tag{1.4}$$

We have from (1.2) and (1.4)

$$6q_1 \leqslant 4n' - 8 = n' - 8 + 3(q_1 + |S|),$$

which implies

$$q_1 - |S| \leqslant (n' - 8)/3. \tag{1.5}$$

The desired inequality immediately follows from (1.3) and (1.5):

$$u = (q_1 - |S|) + (q(G - S) - q_1) \leqslant (n - 8)/3.$$

Q.E.D.

A *vertex cover* of a graph G is a set of vertices such that every edge of G is incident on at least one of the vertices. A *minimum vertex cover* is one of minimum cardinality. For example $\{v_1, v_2, v_3, v_4\}$ is a minimum vertex cover of G in Fig. 1.1(b). Every vertex cover must contain either of the two ends of each edge in a matching. Therefore the cardinality of the minimum vertex cover of G is no less than $|M(G)|$. Thus the following corollary is an immediate consequence of Theorem 1.8.

Corollary 1.5. *If a planar connected graph G has minimum degree* 3 *or more, then the cardinality of a minimum vertex cover is at least* $\min\{\lfloor n/2 \rfloor, \lceil (n+2)/3 \rceil\}$.